TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 188, Issue 2, 1974

WAVE EQUATIONS WITH FINITE VELOCITY OF PROPAGATION[1]

BY

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ABSTRACT. If B is a selfadjoint translation-invariant operator on the space L^2 of complex-valued functions on n-dimensional Euclidean space which are square-summable with respect to Lebesgue measure, then the wave equation $d^2F/dt^2 + B^2F = 0$ has the solution $F(t) = (\cos tB)f + ((\sin tB)/B)g$, for f and g in L^2. In the classical case in which $-B^2$ is the Laplacian, this solution has finite velocity of propagation in the sense that (letting supp denote support of a function) supp $F(t) \subset (\text{supp } f \cup \text{supp } g) + K_t$ for all f and g and some compact set K_t independent of f and g. We show that a converse holds, namely, if $\cos tB$ has finite velocity of propagation (that is, if $\text{supp}(\cos tB)f \subset \text{supp } f + K_t$ for all f and some compact K_t) for three values of t whose reciprocals are independent over the rationals, then B^2 must be a second order differential operator.

If Euclidean space is replaced by a locally compact abelian group which does not contain the real line as a subgroup, then $\cos tB$ has finite velocity of propagation for all t if and only if it is convolution with a distribution T_t such that all T_t are supported on a compact open subgroup.

Problems of a similar nature are discussed for compact connected abelian groups and for the nonabelian group SL(2, **R**).

Introduction. Let $\mathbf{R}^n$ be the set of all n-tuples of real numbers with the usual group structure. The wave equation in n dimensions

$$\frac{\partial^2 F}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2 F}{\partial x_j^2} = 0 \tag{0.1}$$

has finite velocity of propagation in the following sense: Suppose that we have a solution to (0.1) satisfying the initial conditions $F(0, x_1, x_2, \cdots, x_n) = f(x_1, x_2, \cdots, x_n)$ and $(\partial/\partial t)F(0, x_1, \cdots, x_n) = g(x_1, \cdots, x_n)$, where f and g are sufficiently smooth functions. Then if f and g have compact support, so does F as a function of $(x_1, \cdots, x_n) \in \mathbf{R}^n$ for each fixed $t \geq 0$, and in fact if f and g are both supported on $\{x \in \mathbf{R}^n\colon |x_j| \leq a_j \ (j = 1, \cdots, n)\}$ for some positive a_j, then

Received by the editors October 3, 1972 and, in revised form, February 2, 1973.

AMS (MOS) subject classifications (1970). Primary 22B99, 34G05, 35B30, 35L05, 47B25.

Key words and phrases. Wave equation, locally compact abelian group, translation-invariant operators, Laplacian, velocity of propagation, distributions on a LCA group.

(1) Research done while the author held a National Science Foundation Graduate Fellowship.

$F(t, \cdot)$ is supported on $\{x \in \mathbf{R}^n: |x_j| \leq a_j + t \ (j = 1, \cdots, n)\}$ (see Petrovsky [9]). The problem we are concerned with is: to what extent does this property characterize $-\sum_{j=1}^n \partial^2/\partial x_j^2$ among the translation-invariant operators on $L^2(\mathbf{R}^n)$? And what is true if $\mathbf{R}^n$ is replaced by an arbitrary locally compact abelian group?

Let us first consider translation-invariant operators on groups. Let G be a locally compact abelian group with dual group Γ; if $x \in G$ and $\gamma \in \Gamma$ we write $\langle x, \gamma \rangle$ or $\gamma(x)$ for the value of γ at x. Fix a Haar measure on G. If $y \in G$ and $f \in L^2(G)$, we define $(T_y f)(x) = f(x - y)$. Then T_y is clearly an isometry on $L^2(G)$. We recall that the Fourier transform, defined by $\hat{f}(\gamma) = \int_G \langle -x, \gamma \rangle f(x)\, dx$ for $f \in L^1(G)$, is an isometry of $L^2(G)$ onto $L^2(\Gamma)$. If B is an operator defined on a subspace D of $L^2(G)$ with range in $L^2(G)$ and D is closed under every translation T_y, then we say that B is translation-invariant if $BT_y = T_y B$ for all $y \in G$. Now if β is any measurable function on Γ, then we define B on the domain $D = \{f \in L^2(G): \beta\hat{f} \in L^2(\Gamma)\}$ by $(Bf)^\wedge = \beta\hat{f}$. It is not hard to show that D is dense and closed under translations, and B is a closed operator that commutes with all translations. Conversely, if B is a closed translation-invariant operator defined on a translation-invariant subspace D of $L^2(G)$, then there is a measurable function β on Γ such that $\beta\hat{f} \in L^2(\Gamma)$ and $(Bf)^\wedge = \beta\hat{f}$ for all $f \in D$; see Segal [13, p. 454]. If B and β are related in this way we shall sometimes say that B is the operator determined by β or that β corresponds to B. If β corresponds to B, then clearly the complex conjugate function $\bar{\beta}$ corresponds to the adjoint B^*, and B is selfadjoint if and only if β is real. Note that if β is positive, then B has a selfadjoint square root, namely the operator determined by $\beta^{1/2}$. Furthermore, B is a bounded operator if and only if the corresponding function β belongs to $L^\infty(\Gamma)$.

Now consider the differential equation

$$d^2F/dt^2 + B^2F = 0 \tag{0.2}$$

where F is a function of the real variable t with values in a Hilbert space H and B is a (not necessarily bounded) selfadjoint operator on H. Given any f, $g \in H$, the function

$$F(t) = (\cos(tB))f + ((\sin(tB))/B)g \tag{0.3}$$

satisfies (0.2) in the sense that

$$(d^2/dt^2)\langle F(t), h\rangle + \langle F, B^2h\rangle = 0$$

for all h in the domain of B^2. Furthermore, F satisfies the initial conditions $F(0) = f$ and $F'(0) = g$ in the sense that

$$(d/dt)\langle F(t), h\rangle|_{t=0} = \langle g, h\rangle$$

for all h in the domain in B.

If we take $H = L^2(\mathbf{R}^n)$ and B is the square root of $-\partial^2/\partial x_1^2 - \cdots - \partial^2/\partial x_n^2$ then the solution (0.3) has finite velocity of propagation in the sense mentioned above. In Chapter 1 we prove that if (0.3) has finite velocity of propagation and if B is a closed translation-invariant operator with real corresponding β, then B^2 must be a second order differential operator; in fact, it will suffice to consider the operator $\cos(tB)$ alone. In later chapters we consider the case $H = L^2(G)$, G a locally compact abelian group.

These results are contained in a thesis submitted to the Mathematics Department of M.I.T. in partial fulfillment of the requirements for the Ph.D. degree. The author would like to thank Professor I. E. Segal for suggesting this line of investigation and for many helpful conversations.

1. The Euclidean case. Let B be a bounded translation-invariant operator on $L^2(\mathbf{R}^n)$. As discussed above, there is a bounded measurable function β on $\mathbf{R}^n$ such that $(Bf)^\wedge = \beta\hat{f}$ for all $f \in L^2(\mathbf{R}^n)$. The domain of dependence of B is the smallest closed convex set such that if $f \in L^2(\mathbf{R}^n)$ has compact support then $\operatorname{supp}(Bf) \subset N + \operatorname{supp}(f)$. The existence of such a set follows from the theory of distributions (Schwartz [12]), for we may take as N the closed convex hull of the support of the Fourier transform of the distribution β; see Fourès and Segal [4, p. 402].

Note that if B is unbounded it may not have a domain of dependence in any reasonable sense. For example, in the one-dimensional case, if $\beta(x) = \exp(a|x|)$, $a > 0$, then $(Bf)^\wedge = \beta\hat{f}$ defines a closed translation-invariant operator on $D = \{f \in L^2(\mathbf{R})\colon \beta\hat{f} \in L^2(\mathbf{R})\}$. But if $f \in D$, then f has an analytic extension to a strip about $\mathbf{R}$ in $\mathbf{C}$, so $\operatorname{supp}(f) = \mathbf{R}$. Hence $\operatorname{supp}(Bf) \subset N + \operatorname{supp}(f)$ is true for all $f \in D$ and all nonempty subsets N of $\mathbf{R}$.

The main theorem of this section is the following:

Theorem 1. *Let β be a real-valued measurable function on $\mathbf{R}^n$ whose square is equal almost everywhere on $\mathbf{R}^n$ to a second degree polynomial, that is,*

$$(\beta(x))^2 = \sum_{k,l=1;k\geq l}^{n} a_{kl}x_kx_l + \sum_{k=1}^{n} b_kx_k + c. \tag{1.1}$$

Then for any $t \geq 0$, the map from $L^2(\mathbf{R}^n)$ into itself given by

$$f \mapsto f_t, \quad \text{where } \hat{f}_t(x) = \hat{f}(x)\cos(t\beta(x)) \tag{1.2}$$

is of norm ≤ 1 and its domain of dependence is contained in $\{y \in \mathbf{R}^n\colon |y_k| \leq t\sqrt{a_{kk}},\ k = 1, \cdots, n\}$.

Conversely, let β be a real-valued measurable function on $\mathbf{R}^n$ and suppose that there are three positive numbers t_j $(j = 1, 2, 3)$ such that $\{t_j^{-1}\colon j = 1, 2, 3\}$

is independent over the rational numbers and such that the map $f \to f_{t_j}$ *defined in* (1.2) *has compact domain of dependence for* $j = 1, 2, 3$. *Then* $(\beta(x))^2$ *is equal a.e. to a second degree polynomial.*

Remark 1. Theorem 1 solves the problem posed in the introduction, that is, if $\cos(tB)$ has finite velocity of propagation then B^2 is a second order differential operator.

We break the proof of Theorem 1 into a series of lemmas.

Recall that if b_1 and b_2 are complex numbers, then $\cos(b_1) = \cos(b_2)$ if and only if $b_1 = \pm b_2 + 2\pi n$ for some integer n. Also recall that an entire function is one which is defined and analytic everywhere in the complex plane $\mathbf{C}^n$.

Lemma 1. *Let* t_1 *and* t_2 *be two real numbers independent over the rationals and let* β *be a complex-valued measurable function on* $\mathbf{R}^n$ *such that* $\cos(t_1\beta)$ *is constant almost everywhere* (a.e.) *on* $\mathbf{R}^n$ *and* $\cos(t_2\beta)$ *is equal to an entire function a.e. on* $\mathbf{R}^n$. *Then either* β^2 *is equal to a constant a.e. on* $\mathbf{R}^n$ *or else there exist integers* n_1 *and* n_2 *such that* β *assumes only the four values* $\pi(\pm n_1/t_1 \pm n_2/t_2)$ *a.e. on* $\mathbf{R}^n$.

Proof. Suppose that $\cos(t_1\beta(x)) = \cos(t_1\beta(x_0))$ for a.e. x. Then $\beta(x)$ differs from $\pm\beta(x_0)$ by an integral multiple of $2\pi/t_1$ for a.e. x, so that β (and hence $\cos(t_2\beta)$) maps almost all of $\mathbf{R}^n$ onto a countable set. Hence $\cos(t_2\beta)$ is also constant a.e.

Now suppose $\cos(t_j\beta(x)) = \cos(t_j\beta(x_0))$ for a.e. x and $j = 1,2$. Then for a.e. $x \in \mathbf{R}^n$ there exist integers $n_j(x)$ such that

$$\beta(x) = \pm\beta(x_0) + 2\pi n_j(x)/t_j \qquad (j = 1, 2). \tag{1.3}$$

If for some x the signs in (1.3) are the same for $j = 1$ and $j = 2$, then subtraction gives $0 = 2\pi(n_1(x)/t_1 - n_2(x)/t_2)$, so that $n_1(x) = n_2(x) = 0$ and $\beta(x) = \pm\beta(x_0)$. Therefore, if the set on which the signs differ has measure 0, then β^2 is a constant a.e. Otherwise, there is a set A of positive measure on which the signs in (1.3) differ for $j = 1$ and $j = 2$ and addition gives

$$\beta(x) = \pi(n_1(x)/t_1 + n_2(x)/t_2) \qquad (x \in A). \tag{1.4}$$

We may assume $x_0 \in A$. Hence (1.4) is valid when $x = x_0$, and from (1.3) we see that (1.4) must hold for a.e. x and some (perhaps different) integers $n_1(x), n_2(x)$. But now if (1.4) holds for x, y and $\cos(t_1\beta(x)) = \cos(t_1\beta(y))$, then $\pi(n_1(x) + n_2(x)t_1/t_2)$ differs from $\pm\pi(n_1(y) + n_2(y)t_1/t_2)$ by a multiple of 2π, and the independence of t_1 and t_2 implies $n_j(x) = \pm n_j(y)$. Hence, from (1.4), β has the desired form.

Lemma 2. *Let* t_1 *and* t_2 *be two real numbers independent over the rationals* *and let β be a complex-valued measurable function on* R *such that* $\cos(t_j\beta)$ *is equal a.e. on* R *to an entire function* C_j $(j = 1, 2)$. *Then either* β^2 *is equal a.e. on* R *to an entire function or else there exist integers* n_1 *and* n_2 *such that* β *assumes only the four values* $\pi(\pm n_1/t_1 \pm n_2/t_2)$ *a.e. on* R.

Proof. Let $W = \{z \in \mathbf{C}: C_j(z) \neq \pm 1 \text{ for } j = 1 \text{ and } 2\}$. Fix $x_0 \in \mathbf{R} \cap W$ such that $\cos(t_j\beta(x_0)) = C_j(x_0)$, $j = 1, 2$. Define holomorphic functions γ_j in a neighborhood of x_0 by $\gamma_j(z) = t_j^{-1} \arccos(C_j(z))$, where any branch of arc cos is chosen. Then $\cos(t_j\gamma_j) = C_j(z) = \cos(t_j\beta)$ a.e. near x_0, so

$$\gamma_j = \pm\beta + 2\pi n_j/t_j \quad \text{a.e. } x \qquad (j = 1, 2) \tag{1.5}$$

where n_j is an integer-valued function. Let A be the set of x for which the signs in (1.5) differ, B the set where they are the same. Then $\gamma_1 + \gamma_2 = 2\pi(n_1/t_1 + n_2/t_2)$ on A, so that if A has positive measure then $\gamma_1 + \gamma_2$ is constant, and we see easily that the n_j are constant on A. Similarly, $\gamma_1 - \gamma_2 = 2\pi(n_1/t_1 - n_2/t_2)$ on B, so that if B has positive measure, then $\gamma_1 - \gamma_2$ is constant and the n_j are constant on B. So if both A and B have positive measure, then the γ_j are constant, so that the C_j are constant and Lemma 1 applies. So we may assume that one of A, B has measure 0, in which case the n_j are constant on the other and (1.5) holds a.e. for fixed integers n_j. By changing the branches of arc cos used to define γ_j, we may assume $\beta = \pm\gamma_j$ a.e. Furthermore $\gamma_1^2 = \gamma_2^2$.

Now each γ_j determines a (multiple-valued) analytic function in W in the sense of Saks and Zygmund [11, p. 247]. Using the fact that arc cos can be defined locally on $C_j(W)$, it is easy to see that γ_j is arbitrarily continuable along any curve in W beginning at x_0, that $\cos(t_j\gamma_j) = C_j$, and that $\gamma_1^2 = \gamma_2^2$. Hence γ_1^2 is also arbitrarily continuable. Furthermore, γ_1^2 is single-valued. For, if γ_1 has two values $f(z)$ and $g(z)$ in a nbhd of some z_0, then $\cos(t_jf(z)) = \cos(t_jg(z))$, so that $g(z) = \pm f(z) + 2\pi m_j(z)/t_j$, and the usual reasoning shows that g is constant or else $f = \pm g$ and $f^2 = g^2$. So γ_1^2 is in fact a single-valued analytic function in W. Now if $z_0 \notin W$, then γ_1^2 is continuous in an annular nbhd of z_0, hence for any $\eta > 0$, γ_1^2 must map a sufficiently small annular neighborhood of z_0 into a connected component of $\{w \in \mathbf{C}: |\cos(t_1\sqrt{w}) - C_1(z_0)| < \eta\}$, and these components are all bounded for small η. So γ_1^2 has a removable singularity at z_0. Thus γ_1^2 is entire. Finally note that $\gamma_1^2 = \beta^2$ a.e. on R.

Lemma 3. *Let* t_1 *and* t_2 *be two real numbers independent over the rationals and let* β *be a complex-valued measurable function on* $\mathbf{R}^n$ *such that* $\cos(t_j\beta)$ *is equal a.e. on* $\mathbf{R}^n$ *to an entire function* C_j $(j = 1, 2)$. *Then either* β^2 *is equal a.e. on* $\mathbf{R}^n$ *to a continuous function or else there exist integers* n_1 *and* n_2 *such that*

β assumes only the four values $\pi(\pm n_1/t_1 \pm n_2/t_2)$ a.e. on $\mathbf{R}^n$.

Proof. Let $W = \{x \in \mathbf{R}^n: C_j(x) \neq \pm 1$ for $j = 1$ and $2\}$. For each $x_0 \in W$, define γ_j in a neighborhood of x_0 by $\gamma_j = t_j^{-1} \operatorname{arc}\cos(C_j)$. By the usual argument, $\gamma_j(x) = \pm\beta(x) + 2\pi m_j(x)/t_j$, so either C_j is constant or else $\beta = \pm\gamma_j$ a.e. Therefore, either Lemma 1 applies or else we may change β on a set of measure 0 so that β^2 is continuous on W. From now on, we assume the latter.

Now let $x_0 \notin W$. Suppose $C_1(x_0) = -1$, the other possibilities being handled similarly. Let $\epsilon > 0$ and choose a nbhd U of x_0 such that $|C_1(x) + 1| < \epsilon$ for all $x \in U$, hence $|\cos(t_1\beta(x)) + 1| < \epsilon$ for all $x \in U \cap W$. If ϵ is small enough, then $\beta(x)^2 \in \{(z/t_1)^2: |z - k\pi| < 1$ for some odd integer $k\}$ if $x \in U \cap W$, and each connected component of this set has the form $P_k = \{(z/t_1)^2: |z - k\pi| < 1\}$ for some odd integer k. We wish to show that β^2 maps all of $U \cap W$ into one of the P_k. First we show that each line in a certain direction is mapped into a single P_k, as follows: The complement of W is the zero set of the entire function $(C_1 - 1)\cdot(C_1 + 1)(C_2 - 1)(C_2 + 1)$. By the Weierstrass preparation theorem (see Gunning and Rossi [5, p. 68]), we can choose coordinates $x_1, x_2, \cdots, x_n$ for $\mathbf{R}^n$ so that in some nbhd of x_0, for every fixed $x_1, \cdots, x_{n-1}$ there exist only finitely many x_n so that $(x_1, \cdots, x_{n-1}, x_n) \notin W$. If $x_0 = (X_1, \cdots, X_n)$, we may assume that $U = \{x = (x_1, \cdots, x_n): |x_j - X_j| < \eta\}$ for some $\eta > 0$. Now fix $x_1, \cdots, x_{n-1}$ such that $|x_j - X_j| < \eta$, so that there exist $y_1 < y_2 < \cdots < y_l$ such that $(x_1, \cdots, x_{n-1}, y) \in W$ if $|y - X_n| < \eta$, unless $y = y_j$ for some j. Now each interval $\{(x_1, \cdots, x_{n-1}, y): y_j < y < y_{j+1}\}$ is connected and contained in $U \cap W$, hence is mapped by β^2 into some P_k. Fix a particular y_j, and suppose that β^2 maps $\{(x_1, \cdots, x_{n-1}, y): y_{j-1} < y < y_j\}$ into P_k and $\{(x_1, \cdots, x_{n-1}, y): y_j < y < y_{j+1}\}$ into P_m. Let a_i $(i = 1, 2, 3)$ be a sequence converging to y_j from above (that is, $a_i > y_j$) and b_i a sequence converging to y_j from below. Then $\beta(x_1, \cdots, x_{n-1}, a_i)$ and $\beta(x_1, \cdots, x_{n-1}, b_i)$ are bounded sequences, hence have convergent subsequences, and we may assume $\beta(x_1, \cdots, x_{n-1}, a_i) \to a$, $\beta(x_1, \cdots, x_{n-1}, b_i) \to b$. For $p = 1, 2$, $C_p(x_1, \cdots, x_{n-1}, y_j) = \lim_i C_p(x_1, \cdots, x_{n-1}, a_i) = \lim_i \cos(t_p\beta(x_1, \cdots, x_{n-1}, a_i)) = \cos(t_p a)$, and this also equals $\cos(t_p b)$ by the same equations. Since $(x_1, \cdots, x_{n-1}, y_j) \notin W$, we have $C_p = \pm 1$ at this point, for $p = 1$ or 2. Assuming $p = 1$ for definiteness, we have $\cos(t_1 a) = \cos(t_1 b) = \pm 1$, hence $t_1 a = k_1\pi$ and $t_1 b = k_2\pi$ for integers k_1 and k_2. But also $\cos(t_2 a) = \cos(t_2 b)$, so $t_2 a = \pm t_2 b + 2\pi k$ for some integer k, or $(t_2/t_1)k_1 = \pm(t_2/t_1)k_2 + 2k$. By the independence of t_1 and t_2, we have $k_1 = \pm k_2$ and $a^2 = b^2$. It follows that $k = m$, and that β^2 maps all of $\{(x_1, \cdots, x_{n-1}, y): |y - X_n| < \eta\} \cap W$ into a single P_k.

Suppose that β^2 maps $\{(X_1, \cdots, X_{n-1}, y): |y - X_n| < \eta\}$ into P_k. Let

$A = \{(x_1, \cdots, x_{n-1}) \in \mathbf{R}^{n-1}$: β^2 maps $\{(x_1, \cdots, x_{n-1}, y): |y - X_n| < \eta\} \cap W$ into $P_k\}$. Then A is nonempty, since $(X_1, \cdots, X_{n-1}) \in A$. Also A is open, for if $(x_1, \cdots, x_{n-1}) \in A$ then there exists y such that $(x_1, \cdots, x_{n-1}, y) \in W$ is mapped by β^2 into P_k, hence there exists an open connected set B in $\mathbf{R}^n$ containing $(x_1, \cdots, x_{n-1}, y)$, and β^2 must map B into P_k. By the previous paragraph, A contains the projection of B on $\mathbf{R}^{n-1}$. It follows similarly that A is closed in $\{(x_1, \cdots, x_{n-1}): |x_j - X_j| < \eta, j = 1, \cdots, n-1\}$, hence $A = \{(x_1, \cdots, x_{n-1}): |x_j - X_j| < \eta, j = 1, \cdots, n-1\}$, so that β^2 maps all of $U \cap W$ into P_k.

If we define $\beta(x_0)^2 = (k\pi/t_1)^2$ and define β^2 on the rest of the complement of W in a similar fashion, then it follows easily that β^2 is a continuous function.

Lemma 4. *Let β be a complex-valued function on* $\mathbf{C}$ *such that β^2 is an entire function. If the entire function* $\cos(\beta)$ *is of exponential type, then β^2 is a second degree polynomial.*

Proof. By hypothesis there exist constants k and s such that

$$|\cos(\beta(z))| \le k e^{s|z|} \quad \text{for all } z \in \mathbf{C}. \tag{1.6}$$

Write $\beta(z) = u(z) + iv(z)$ where u and v are real-valued. We claim that for every $\epsilon > 0$ there exists a constant N such that

$$|v(z)| \le (s + \epsilon)|z| \quad \text{for } |z| \ge N. \tag{1.7}$$

For, if not, there exist $\epsilon > 0$ and z_n $(n = 1, 2, 3, \cdots)$ such that $|v(z_n)| > (s + \epsilon)|z_n|$ and $|z_n| \to \infty$ as $n \to \infty$. Hence $|v(z_n)| \to \infty$ and by (1.6) we have for all n sufficiently large:

$$\begin{aligned} 2|\cos(\beta(z_n))| &= |e^{iu(z_n)} e^{-v(z_n)} + e^{-iu(z_n)} e^{v(z_n)}| \\ &> e^{|v(z_n)|} - 1 > e^{(s+\epsilon)|z_n|} - 1 \\ &\ge e^{\epsilon |z_n|} k^{-1} |\cos(\beta(z_n))| - 1. \end{aligned}$$

But this inequality obviously cannot hold as $n \to \infty$. We have a contradiction and (1.7) is established.

Now $\beta^2 = u^2 - v^2 + 2iuv$, and so $\operatorname{Re}(-\beta^2) = v^2 - u^2 \le v^2 \le (s + \epsilon)^2 |z|^2$ for large $|z|$. It follows from the real part Liouville theorem (Titchmarsh [14, p. 87]) that β^2 is a second degree polynomial.

Proof of Theorem 1. If $f \in L^2(\mathbf{R}^n)$ and f_t is defined as in (1.2), then $\|f_t\|_2 = \|\hat{f}_t\|_2 \le \|\hat{f}\|_2 = \|f\|_2$. If β is as in (1.1), we may assume that β is defined everywhere in $\mathbf{C}^n$ and satisfies (1.1) everywhere. Then $\cos(t\beta(z))$ is an entire function. Now for each fixed k, l, $k \ge l$, we know that $a_{kk}x_k^2 + a_{kl}x_k x_l + a_{ll}x_l^2 + b_l x_l + b_k x_k + c$ is positive for all x_k, x_l in $\mathbf{R}$, hence the quadratic part is positive and $a_{kl}^2 \leq 4a_{kk}a_{ll}$. It follows that, for any fixed $\epsilon > 0$, we have $|\beta(z)|^2 \leq (\Sigma_{k=1}^n (\sqrt{a_{kk}} + \epsilon)|z_k|)^2$ for $|z|$ sufficiently large, so that

$$|\cos(t\beta(z))| \leq \exp(t|\beta(z)|) \leq \exp\left(t\sum(\sqrt{a_{kk}} + \epsilon)|z_k|\right).$$

If μ_t is the distribution on $\mathbf{R}^n$ whose Fourier transform is $\cos(t\beta)$, then by the Paley-Wiener-Schwartz theorem (Schwartz [12, p. 271]) the support of μ_t is contained in $\{y \in \mathbf{R}^n: |y_k| \leq t\sqrt{a_{kk}}, k = 1, \cdots, n\}$. This proves the first part of the theorem.

Conversely, suppose that β and t_j are as stated. Then $\hat{f}$ and $\hat{f}_{t_j}$ have entire extensions which are of exponential type. Hence $\hat{f}_{t_j}/\hat{f}$ is a meromorphic function on $\mathbf{C}^n$ equal to $\cos(t_j\beta)$ a.e. on $\mathbf{R}^n$. But $\hat{f}$ can be chosen to be nonzero at any prescribed point (for example, if f is identically 1 on the interval $[0, a]$ and 0 elsewhere, then $\hat{f}(z) = -(e^{-iaz} - 1)/iz$ has zeros at $z = \pm 2\pi/a$, $\pm 4\pi/a$, $\pm 6\pi/a, \cdots$), so that $\cos(t_j\beta)$ has an entire extension which we denote C_j. By Lemma 3, we may assume β^2 is continuous and $\cos t_j\beta = C_j$ everywhere on $\mathbf{R}^n$. By a theorem of Lindelöf (see Malgrange [8, p. 306]), an entire function of one variable which is the quotient of entire functions of exponential type must itself be of exponential type. Hence if we fix any $n - 1$ of $x_1, \cdots, x_n$, then C_j as a function of the remaining variable is of exponential type. By Lemma 4, β^2 is a second degree polynomial in any one variable when the others are fixed. Hence β^2 is a polynomial of some degree, say p: $\beta^2(x) = \Sigma_{|\alpha|=p} c_\alpha x^\alpha + Q(x)$, where $\deg Q \leq p - 1$ and at least one c_α is nonzero. Let $w = (w_1, \cdots, w_n) \in \mathbf{R}^n$ be such that $\Sigma c_\alpha w^\alpha \neq 0$. Then if ρ: $\mathbf{R} \to \mathbf{R}^n$, $\rho(r) = rw$, we have $\beta^2(\rho(r)) = \Sigma c_\alpha r^{|\alpha|} w^\alpha + Q(\rho(r)) = r^p \Sigma c_\alpha w^\alpha + Q(\rho(r))$. But $\beta \cdot \rho$ satisfies the hypotheses of Lemmas 2 and 4, so $(\beta \cdot \rho)^2$ is a second degree polynomial in r, implying $p \leq 2$, as wanted.

Remark 3. Note that if we assume that both $\cos(tB)$ and $(\sin(tB))/B$ have finite velocity of propagation, then it follows more easily that β^2 has an entire extension.

Note also that if β^2 is as in (1.1), then $(\sin(tB))/B$ has finite velocity of propagation.

2. The locally compact abelian case. In order to extend Theorem 1 to an arbitrary locally compact abelian (LCA) group we use the theory of distributions on such groups developed by F. Bruhat in [2]. We will briefly review some of the important facts below.

An elementary group is a group of the form $G = \mathbf{R}^n \times \mathbf{T}^p \times \mathbf{Z}^q \times \Phi$, where p, q, and n are nonnegative integers and $\mathbf{R}$, $\mathbf{T}$, $\mathbf{Z}$, and Φ represent the real numbers, the circle group, the integers, and a finite abelian group, respectively. We let $\mathfrak{D}(G)$ be the set of complex-valued functions on G which are infinitely differentiable (with respect to the $\mathbf{R}$ and $\mathbf{T}$ variables) and have compact support. We define $\mathcal{S}(G)$ to be the set of complex-valued functions on G which are infinitely differentiable and which have the property that the product of any derivative of the function and any polynomial (in the $\mathbf{R}$ and $\mathbf{Z}$ variables) is bounded on G.

Let G be a locally compact abelian group. A pair of subgroups H, H' of G will be called a good pair if the following three conditions are satisfied: H is a subgroup generated by a compact neighborhood of the identity in G (such a subgroup must be open); H' is a compact subgroup of H; H/H' is an elementary group. If H, H' is a good pair of subgroups of G, then $\mathfrak{D}(H, H')$ is defined to be the set of continuous complex-valued functions on G which are supported on H and constant on the cosets of H' such that the function induced on H/H' is in $\mathfrak{D}(H/H')$. We define $\mathfrak{D}(G)$ to be the union of the $\mathfrak{D}(H/H')$ for all good pairs of subgroups H, H'. The spaces $\mathcal{S}(H/H')$ and $\mathcal{S}(G)$ are defined similarly. We give $\mathfrak{D}(H, H')$ the topology of $\mathfrak{D}(H/H')$, and $\mathfrak{D}(G)$ the inductive limit topology; similarly for $\mathcal{S}$. Finally, we define $\mathcal{E}(G)$ to be the set of all functions locally in $\mathfrak{D}(G)$. Then $\mathfrak{D}(G) \subset \mathcal{S}(G) \subset \mathcal{E}(G)$ and each of these inclusions is continuous with dense image. A continuous linear functional on $\mathfrak{D}(G)$ is called a distribution on G; if the functional can be extended continuously to $\mathcal{S}(G)$ it is called a tempered distribution; with the appropriate definition of support, a distribution extends continuously to $\mathcal{E}(G)$ if and only if it has compact support.

If Γ is the dual of G and $f \in L^1(G)$, the Fourier transform $\hat{f}$ of f is defined on Γ by $\hat{f}(\gamma) = \int f(x)\gamma(-x)\,dx$, where dx is a fixed Haar measure on G. The Fourier transform is an isomorphism of $\mathcal{S}(G)$ onto $\mathcal{S}(\Gamma)$. We define the Fourier transform of a distribution by transposition, that is, $\hat{T}(f) = T(\hat{f})$ for $T \in \mathcal{S}'(G)$, $f \in \mathcal{S}(\Gamma)$. The Fourier transform is an isomorphism between $\mathcal{S}'(G)$ and $\mathcal{S}'(\Gamma)$.

In order to extend Theorem 1 to LCA groups we need to know that the standard facts about distributions on Euclidean spaces are true in general. These facts are stated in the Appendix. In what follows, a reference to Theorem A.*n* refers to a theorem in the Appendix.

The following is the main result of this section.

Theorem 2. *Let G be a locally compact abelian group with dual group Γ. Suppose that Γ does not contain $\mathbf{R}$ as a subgroup. Let β be a measurable real-valued function on Γ. Suppose that for every $t \geq 0$ there exists a compact subset K_t of G such that if $f \in L^2(G)$ has compact support and if $f_t \in L^2(G)$ is defined by $\hat{f}_t(\gamma) = \hat{f}(\gamma)\cos(t\beta(\gamma))$, then f_t also has compact support and in fact*

$$\operatorname{supp}(f_t) \subset \operatorname{supp}(f) + K_t. \tag{2.1}$$

Suppose further that there exists $t_0 > 0$ such that $\bigcup\{K_t\colon 0 \leq t \leq t_0\}$ is contained in a compact set. Then for every $t \geq 0$, $\cos(t\beta)$ is the Fourier transform of a *distribution* μ_t *on* G, *and there is a compact open subgroup* H *of* G *such that* $\operatorname{supp}(\mu_t) \subset H$ *for all* $t \geq 0$. *If* f_t *is defined as above, then* $f_t = f * \mu_t$. *The function* $|\beta|$ *is equal almost everywhere to a function which is constant on the cosets of the annihilator of* H.

Conversely, given any real-valued measurable function β *on* Γ *such that* $|\beta|$ *is constant on the cosets of a compact open subgroup* Λ *of* Γ, *for every* $t \geq 0$ *there exists* $\mu_t \in \mathcal{E}'(G)$ *such that* $\hat{\mu}_t = \cos(t\beta)$ *and such that, if* $f \in L^2(G)$ *has compact support and* f_t *is defined as above, then* (2.1) *holds with* K_t *equal to the annihilator of* Λ *for all* $t \geq 0$.

Proof. We prove the converse first. Suppose $|\beta|$ is constant on the cosets of compact open $\Lambda \subset \Gamma$. For any $t \geq 0$, $\cos(t\beta)$ is in $L^\infty(\Gamma)$, so by Theorem A.1 there exists $\mu_t \in \mathcal{E}'(G)$ with $\hat{\mu}_t = \cos(t\beta)$. By Theorem A.2, $\operatorname{supp}(\mu_t) \subset H$ for all $t \geq 0$, where H is the annihilator of Λ. Now H is compact open because Λ is compact open. If $f \in L^2(G)$ has compact support and $f_t \in L^2(G)$ is defined by $\hat{f}_t = \hat{f}\cos(t\beta)$, then, by Theorem A.6, $\hat{f}_t = (f * \mu_t)\hat{}$ and since the Fourier transform is an isomorphism we have $f_t = f * \mu_t$. $\operatorname{supp}(f_t) \subset \operatorname{supp}(f) + \operatorname{supp}(\mu_t) \subset \operatorname{supp}(f) + H$.

We now turn to the first part of the theorem. We start by proving β^2 continuous.

Lemma 5. *Under the hypotheses of the theorem,* β^2 *is equal a.e. to a continuous function, even without the assumption that* Γ *does not contain* $\mathbf{R}$ *as a subgroup. If* Γ *is metrizable we need not assume that* $\bigcup\{K_t\colon 0 \leq t \leq t_0\}$ *is contained in a compact set for some* t_0.

Proof. For fixed t, for all $f \in L^2(G)$ with compact support we have $\hat{f}(\gamma)\cos(t\beta(\gamma)) = \hat{f}_t(\gamma)$ for almost every $\gamma \in \Gamma$, and both $\hat{f}$ and $\hat{f}_t$ are continuous. It follows that for every t there exists a continuous function C_t on Γ such that $\cos(t\beta(\gamma)) = C_t(\gamma)$ for almost every $\gamma \in \Gamma$. Hence there is a subset A of Γ whose complement has Haar measure 0 such that if $\gamma \in A$ then $\cos(t\beta(\gamma)) = C_t(\gamma)$ for almost every $t \geq 0$.

Now suppose that $\gamma_n \to \gamma$, where $\gamma_n \in A$ for all $n = 1, 2, \cdots$. Then there is a subset B of $\mathbf{R}$ which is almost all of the positive axis such that if $t \in B$ then

$$\cos(t\beta(\gamma_n)) = C_t(\gamma_n) \to C_t(\gamma) \quad \text{as } n \to \infty. \tag{2.2}$$

Since these functions are bounded by 1, we see that for all $s > 0$

$$\frac{\sin(s\beta(\gamma_n))}{\beta(\gamma_n)} = \int_0^s \cos(t\beta(\gamma_n))\,dt \to \int_0^s C_t(\gamma)\,dt$$

where we use the convention $(\sin(s \cdot 0))/0 = s$. We claim that there must exist

$s \in B$ such that $\int_0^s C_t(\gamma)dt \neq 0$. For, otherwise $C_t(\gamma) = 0$ for a.e. $t \geq 0$, and from (2.2), $\cos(t\beta(\gamma_n)) \to 0$ for a.e. $t \geq 0$ as $n \to \infty$. Hence $\cos(2t\beta(\gamma_n)) = 2\cos^2(t\beta(\gamma_n)) - 1 \to -1$ for a.e. t, a contradiction. Therefore there exists $s \in B$ as wanted, and

$$\beta(\gamma_n)^2 = \frac{1 - [\cos s\beta(\gamma_n)]^2}{[(\sin s\beta(\gamma_n))/\beta(\gamma_n)]^2} \to \frac{1 - C_s(\gamma)^2}{(\int_0^s C_t(\gamma)\,dt)^2}.$$

If $\gamma \in A$, then $C_s(\gamma) = \cos(s\beta(\gamma))$ and $(\beta(\gamma_n))^2 \to (\beta(\gamma))^2$. Hence β^2 is sequentially continuous on A. Furthermore, if $\gamma \notin A$, then $(\beta(\gamma_n))^2$ has a limit as $n \to \infty$ for any sequence γ_n in A converging to γ (such sequences exist since A is almost all of Γ), and this limit is independent of the choice of sequence. It is easy to see that if β is redefined on the complement of A to be this limit then β^2 is sequentially continuous, hence continuous if Γ is metrizable.

In the general case, in which Γ is not necessarily metrizable, let K be a symmetric compact nbhd of 0 in G such that $K_t \subset K$ if $0 \leq t \leq t_0$. Then K generates a σ-compact open subgroup S of G. Now the dual of S is metrizable, and this dual is Γ/Σ, where Σ is the annihilator of S. Suppose that γ_1 and γ_2 lie in the same coset of Σ in Γ. Define $f \in L^2(G)$ by

$$f(x) = \langle x, \gamma_1\rangle \quad \text{if } x \in K,$$
$$= 0 \qquad \text{otherwise.}$$

Then $\hat{f}(\gamma_1) =$ measure of $K \neq 0$. If $\hat{f}_t = \hat{f}\cos(t\beta)$, then it is clear that both f and f_t are supported on S if $t \leq t_0$, hence $\hat{f}$ and $\hat{f}_t$ are constant on the cosets of Σ, that is, for all $t \leq t_0$, $C_t(\gamma_1)\hat{f}(\gamma_1) = \hat{f}_t(\gamma_1) = \hat{f}_t(\gamma_2) = C_t(\gamma_2)\hat{f}(\gamma_2)$ and $\hat{f}(\gamma_1) = \hat{f}(\gamma_2) \neq 0$, hence $C_t(\gamma_1) = C_t(\gamma_2)$. Now there exists $A \subset \Gamma$ whose complement has measure 0 such that, for $\gamma \in A$, $C_t(\gamma) = \cos(t\beta(\gamma))$ for a.e. $t \geq 0$; hence if γ_1 and γ_2 are in A then $\cos(t\beta(\gamma_1)) = \cos(t\beta(\gamma_2))$ for a.e. t, hence $(\beta(\gamma_1))^2 = (\beta(\gamma_2))^2$. Define $\tilde{\beta}$ on Γ/Σ as follows. Since A is almost all of Γ, by Lemma 6 (below), almost every coset of Γ/Σ must be of the form $\gamma + \Sigma$, $\gamma \in A$, and if $\tilde{\beta}(\gamma + \Sigma) = \beta(\gamma)$, then $\tilde{\beta}$ is well defined a.e. by the above. Suppose $f \in L^2(S)$ has compact support. Define $f_t \in L^2(S)$ by $\hat{f}_t = \hat{f}\cos(t\tilde{\beta})$. Now the Haar measure of S is the restriction of the Haar measure of G since S is open, and f and f_t can be considered functions on G. Then $\hat{f}_t(\gamma) = \hat{f}(\gamma)\cos(t\beta(\gamma))$ a.e. $\gamma \in \Gamma$. Hence $\operatorname{supp}(f_t) \subset \operatorname{supp}(f) + K_t$ $(t \leq t_0)$. By the metrizable case proven above, $\tilde{\beta}$ can be altered on a set of Γ/Σ-measure 0 so that $|\tilde{\beta}|$ is continuous. Hence by Lemma 6 (below), β can be altered on a set of Γ-measure 0 so that $|\beta| = |\tilde{\beta} \circ \pi|$ is continuous.

Lemma 6. *Let Γ be a locally compact abelian group with compact subgroup Σ. Let $\pi\colon \Gamma \to \Gamma/\Sigma$ be the natural projection and let μ be a Haar measure on Γ.*

If we define ν *on* Γ/Σ *by* $\nu(E) = \mu(\pi^{-1}(E))$, *then* ν *is a Haar measure on* Γ/Σ. *Furthermore, if* A *is a subset of* Γ *whose complement has Haar measure* 0, *then the complement of* $\pi(A)$ *has Haar measure* 0 *as a subset of* Γ/Σ.

Proof. It is clear that ν is a translation-invariant measure, and it suffices to show that it is nonzero on open sets and finite on compact sets. The first is clear because $\pi^{-1}(E)$ is open if E is open, and so it only remains to show that if E is compact in Γ/Σ then $\pi^{-1}(E)$ is compact in Γ. Given compact E in Γ/Σ, let $(U_j)_{j\in J}$ be an open cover of $\pi^{-1}(E)$ by sets with compact closure. Then $(\pi(U_j))_{j\in J}$ is an open cover of E, and by compactness E is contained in a finite subcover, $E \subset \pi(U_{j_1}) \cup \cdots \cup \pi(U_{j_n})$. Clearly, $\pi^{-1}(E) \subset (U_{j_1} \cup \cdots \cup U_{j_n}) + H$. So $\pi^{-1}(E)$ is contained in a compact set and since it is closed it must be compact.

As for the second statement, let $\tilde{A}$ and $\sim\pi(A)$ denote the complements of A and $\pi(A)$ in G and G/H. Then $\nu(\sim\pi(A)) = \mu(\pi^{-1}(\sim\pi(A))) \leq \mu(\tilde{A}) = 0$.

We now return to the theorem. Let β have the property of the Theorem. We may assume by Lemma 5 that β^2 is continuous. Let $f \in L^2(G)$ have compact support and define f_t by $\hat{f}_t = \hat{f}\cos(t\beta)$. By Theorem A.1 there exists $\mu_t \in \mathcal{S}'(G)$ such that $\hat{\mu}_t$ is the function $\cos(t\beta)$. By Theorem A.6, $f_t = f * \mu_t$. By hypothesis, $\operatorname{supp}(f_t) \subset \operatorname{supp}(f) + K_t$. Given any compact nbhd J of 0, there is a net of functions f_α all having support in J such that $f_\alpha \to \delta_0$ in $\mathfrak{D}'(G)$, δ_0 being the point mass at $0 \in G$. If $f_{\alpha,t} = f_\alpha * \mu_t$, then $f_{\alpha,t} \to \mu_t$ in $\mathfrak{D}'(G)$ (see Bruhat [2, p. 57]) and $\operatorname{supp}(f_{\alpha,t}) \subset \operatorname{supp}(f_\alpha) + K_t \subset J + K_t$. Hence $\operatorname{supp}(\mu_t) \subset J + K_t$, and since J is any compact nbhd of $0 \in G$ we have $\operatorname{supp}(\mu_t) \subset K_t$.

We suppose from now on that $K_t = \operatorname{supp}(\mu_t)$.

Consider any one-parameter subgroup of Γ, that is a continuous homomorphism ρ: $\mathbf{R} \to \Gamma$. For fixed $\gamma_0 \in \Gamma$, we have

$$\hat{\mu}_t(\gamma_0 + \rho(r)) = \cos(t\beta(\gamma_0 + \rho(r))).$$

In Theorem A.7, take $G_2 = G$ and $G_1 = \mathbf{R}$, and let $\nu_t = \rho_0''((-\gamma_0)\mu_t)$. Then $\nu_t \in \mathcal{E}'(\mathbf{R})$ and for all $r \in \mathbf{R}$

$$\begin{aligned}\langle \nu_t, -r\rangle = \langle(-\gamma_0)\mu_t, -\rho(r)\rangle &= ((-\gamma_0)\mu_t)^\wedge(\rho(r)) \\ &= \hat{\mu}_t(\gamma_0 + \rho(r)) = \cos(t\beta(\gamma_0 + \rho(r))).\end{aligned}$$

Hence if $f_t = f * \nu_t$, $f \in L^2(\mathbf{R})$ with compact support, then $\hat{f}_t(r) = \hat{f}(r)\cos(t\beta(\gamma_0 + \rho(r)))$, and by Theorem 1 there exist real a_0, a_1, and a_2 such that

$$(\beta(\gamma_0 + \rho(r)))^2 = a_2 r^2 + a_1 r + a_0.$$

Now the image of ρ lies in the connected component Γ_0 of the identity in Γ.

Since a connected group is the product of a Euclidean space and a compact connected group, Γ_0 must be compact, hence so are its cosets. But $\{\gamma_0 + \rho(r): r \in \mathbf{R}\}$ is in such a coset and β^2 is continuous, hence bounded, hence constant, that is,

$$(\beta(\gamma_0 + \rho(r)))^2 = (\beta(\gamma_0))^2 \quad \text{for all } r \in \mathbf{R}.$$

Since the one-parameter subgroups of Γ generate Γ_0 (Hewitt and Ross [6, p. 410]), it follows that β^2 is constant on the cosets of Γ_0. Hence also $\cos(t\beta)$ is constant on the cosets of Γ_0, and, by Theorem A.2, $\operatorname{supp}(\mu_t) \subset H_0$ for all $t \geq 0$, where H_0 is the annihilator of Γ_0, hence open.

Since Γ/Γ_0 is totally disconnected, every compact subset of H_0 is contained in a compact open subgroup of H_0, hence of G. Suppose $K_t \subset K$ for $t \leq t_0$. Then $K \subset H$, where H is a compact open subgroup. It follows from Theorem A.6 and the identity $\cos(2\theta) = 2(\cos(\theta))^2 - 1$ that $\mu_{2t} = 2\mu_t * \mu_t - \delta_0$, and now it is easy to see that $K_t \subset H$ for all $t \geq 0$.

A well-known structure theorem for locally compact abelian groups states that any such group is isomorphic to the product of a Euclidean space and a group that does not contain $\mathbf{R}$ as a subgroup (Weil [15, p. 110]). By applying Theorems 1 and 2 to each factor we get the following result, whose proof we omit (see [1]).

Theorem 3. *Let G be a locally compact abelian group with dual group Γ. Suppose $\Gamma = \mathbf{R}^n \times \Gamma_1$ where Γ_1 does not contain $\mathbf{R}$ as a subgroup. Let β be a measurable real-valued function on Γ. Suppose that for every $t \geq 0$ there exists a compact subset K_t of G such that if $f \in L^2(G)$ has compact support and if $f_t \in L^2(G)$ is defined by $\hat{f}_t = \hat{f}\cos(t\beta)$, then f_t also has compact support and in fact $\operatorname{supp}(f_t) \subset \operatorname{supp}(f) + K_t$. Suppose further that there exists $t_0 > 0$ such that $\bigcup\{K_t\colon 0 \leq t \leq t_0\}$ is contained in a compact set. Then after changing β on a set of measure 0 we have, for all $(y, \gamma) \in \mathbf{R}^n \times \Gamma_1$,*

$$(\beta(y, \gamma))^2 = \sum_{1 \leq j \leq k \leq n} a_{jk}(\gamma) y_j y_k + \sum_{j=1}^{n} b_j(\gamma) y_j + c(\gamma)$$

where all of the coefficients a_{jk}, b_j, and c are constant on the cosets of some compact open subgroup Λ_1 of Γ_1 and $|a_{jj}(\gamma)| \leq M$ for all j $(=1, \cdots, n)$ and $\gamma \in \Gamma$ and some constant M.

And conversely.

3. **Compact groups.** The distributions discussed in Chapter 2 do not in general have the property that $\operatorname{supp}(\mu_t)$ is small if t is small. The following theorem characterizes such μ_t in the case of the circle group $\mathbf{T}$.

We represent the circle as the interval $[-\pi, \pi]$ with the endpoints identified and addition modulo 2π. For $f \in L^1(T)$, we define its Fourier transform to be

$\hat{f}(n) = \int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$, $n \in \mathbf{Z}$. Any $f \in L^1(-\pi, \pi)$ can be considered as a function on $\mathbf{R}$ merely by defining it to be 0 off $[-\pi, \pi]$; the extended function is in $L^1(\mathbf{R})$ and its Fourier transform on $\mathbf{R}$ is an extension of $\hat{f}$ on $\mathbf{Z}$. Note that if $f \in \mathfrak{D}(\mathbf{T})$ is 0 in a nbhd of π then its extension to $\mathbf{R}$ is in $\mathfrak{D}(\mathbf{R})$, and conversely if $f \in \mathfrak{D}(\mathbf{R})$ has its support in $(-\pi, \pi)$ then f can be considered a function on $\mathbf{T}$. Similarly one can identify distributions on $\mathbf{T}$ which are 0 in a nbhd of π and distributions on $\mathbf{R}$ with support in $(-\pi, \pi)$.

Theorem 4. *Let β be a nonnegative function on $\mathbf{Z}$ and define $\mu_t \in \mathfrak{D}'(\mathbf{T})$ by $\hat{\mu}_t = \cos(t\beta)$. Suppose that there exists $t_0 > 0$ such that $\operatorname{supp}(\mu_t) \subset (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ if $0 \le t \le t_0$. Then there exist a_0, a_1, and a_2 in $\mathbf{R}$ such that*

$$(\beta(n))^2 = a_2 n^2 + a_1 n + a_0 \quad \textit{for all } n \in \mathbf{Z}, \tag{3.1}$$

and

$$a_2 x^2 + a_1 x + a_0 \ge 0 \quad \textit{for all } x \in \mathbf{R}. \tag{3.2}$$

Conversely, given β satisfying (3.1) *and* (3.2), *and letting $\hat{\mu}_t = \cos(t\beta)$, we have*

$$\operatorname{supp}(\mu_t) \subset [-t\sqrt{a_2}, t\sqrt{a_2}] \quad \textit{if } t\sqrt{a_2} < \pi.$$

Proof. To establish the converse, let β satisfy (3.1) and (3.2). By Theorem 1, there exists $\nu_t \in \mathfrak{D}'(\mathbf{R})$ such that $\hat{\nu}_t(x) = \cos(t\beta(x))$ for all $x \in \mathbf{R}$, and $\operatorname{supp}(\nu_t) \subset [-t\sqrt{a_2}, t\sqrt{a_2}]$. If $t\sqrt{a_2} < \pi$ then ν_t can be considered a distribution on $\mathbf{T}$, and $\hat{\nu}_t(n) = \hat{\mu}_t(n)$ so that $\nu_t = \mu_t$.

Now suppose we have $\mu_t \in \mathfrak{D}'(\mathbf{T})$ with $\hat{\mu}_t = \cos(t\beta)$ and $\operatorname{supp}(\mu_t) \subset (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ if $0 \le t \le t_0$. Let $s(t)$ be the smallest number such that $\operatorname{supp}(\mu_t) \subset [-s(t), s(t)]$, so that $s(t) < \frac{1}{2}\pi$ if $0 \le t \le t_0$. Now $\mu_t * \mu_t = \frac{1}{2}(\mu_{2t} + \delta_0)$, δ_0 being the point mass at 0, and by the theorem of Titchmarsh-Lions (see [7]) $s(2t) = 2s(t)$ if $2t \le t_0$. By induction, $s(t) = 2^{-n}s(2^n t)$ if $2^n t \le t_0$, $n \in \mathbf{N}$. Now given any nonzero $t \le t_0$, there exists $n \in \mathbf{N}$ such that $2^n t \le t_0 < 2^{n+1}t$. Then $s(t) = 2^{-n}s(2^n t) < 2^{-n}(\frac{1}{2}\pi) < \pi t/t_0$, that is, $s(t) < s_0 t$, where $s_0 = \pi/t_0$.

If $f \in L^2(\mathbf{R})$, $\operatorname{supp}(f) \subset (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and $t \le t_0$, then $\operatorname{supp}(f * \mu_t) \subset (-\pi, \pi)$, and both f and $f * \mu_t$ can be considered as functions on $\mathbf{R}$. Thus we have a map $f \mapsto f * \mu_t$ from $\{f \in L^2(\mathbf{R}) \colon \operatorname{supp}(f) \subset (-\frac{1}{2}\pi, \frac{1}{2}\pi)\}$ into $L^2(\mathbf{R})$ which decreases norms. We wish to show that $f \mapsto f * \mu_t$ is a norm-decreasing map on all of $L^2(\mathbf{R})$ into $L^2(\mathbf{R})$. Now if $f \in L^2(\mathbf{R})$ and $\operatorname{supp}(f) \subset (k - \frac{1}{2}\pi, k + \frac{1}{2}\pi)$ for some k, then letting τ_k denote translation by k we have

$$f * \mu_t = \tau_k(\tau_{-k} f * \mu_t) \in L^2(\mathbf{R}) \quad \text{and} \quad \|f * \mu_t\|_2 \le \|f\|_2.$$

If $f \in L^2(\mathbf{R})$ has compact support, then f is a finite sum $f = \Sigma_j f_j$, each f_j having support in an interval of length $< \pi$ and $\operatorname{supp}(f_j)$ pairwise disjoint, and so $f * \mu_t = \Sigma_j (f_j * \mu_t) \in L^2(\mathbf{R})$ and

$$\|f * \mu_t\|_2 \leq \sum_j \|f_j * \mu_t\|_2 \leq \sum_j \|f_j\|_2 = \left\|\sum_j f_j\right\|_2 = \|f\|_2.$$

Hence $f \mapsto f * \mu_t$ is a norm-decreasing map of $\{f \in L^2(\mathbf{R})\colon \operatorname{supp}(f) \text{ is compact}\}$ into $L^2(\mathbf{R})$. Hence it has an extension Φ_t defined on all of $L^2(\mathbf{R})$, and this extension is also norm-decreasing. Now the map $f \mapsto f * \mu_t$ commutes with translations, so Φ_t must also. Hence by the theorem quoted in the introduction there exists $\phi_t \in L^\infty(\mathbf{R})$ such that $\|\phi_t\|_\infty \leq 1$ and $(\Phi_t f)\hat{}(x) = \hat{f}(x)\phi_t(x)$ for $f \in L^2(\mathbf{R})$ and $x \in \mathbf{R}$. But if f has compact support then $(\Phi_t f)\hat{} = \hat{f}\hat{\mu}_t$, and so $\phi_t = \hat{\mu}_t$. Now by the Paley-Wiener theorem, $\hat{\mu}_t$ has an entire extension F_t of exponential type $\leq s_0 t$ if $t \leq t_0$, and we have that $F_t = \phi_t$ is bounded by 1 on $\mathbf{R}$.

Now β is defined only on $\mathbf{Z}$, but $\cos(t\beta(n)) = F_t(n)$ for all $t \leq t_0$ and F_t is defined on all of $\mathbf{C}$. We must extend β to $\mathbf{R}$ so that we can apply Theorem 1.

First we get a bound on β. By a theorem of Bernstein (Zygmund [16, p. 276]), we see that $|F_t'(x)| \leq s_0 t$ for all $x \in \mathbf{R}$. Hence for any $n \in \mathbf{N}$ and $t \leq t_0$, we have

$$\left|\int_{\beta(n)}^{\beta(n+1)} \sin(ts)\, ds\right| = t^{-1}|\cos(t\beta(n+1)) - \cos(t\beta(n))|$$

(3.3)

$$= t^{-1}|F_t(n+1) - F_t(n)| = t^{-1}\left|\int_n^{n+1} F_t'(x)\, dx\right| \leq s_0.$$

We claim

(3.4) $$|\beta(n+1) - \beta(n)| \leq 2\sqrt{2}\, s_0 \quad \text{for all } n \in \mathbf{Z}.$$

For, if not, there exists $n \in \mathbf{Z}$ such that $|\beta(n+1) - \beta(n)| > 2\sqrt{2}\, s_0$, hence if $t_n = \pi(\beta(n+1) + \beta(n))^{-1}$ then $t_n \leq t_0$. Let $m = \frac{1}{2}(\beta(n+1) + \beta(n))$, and consider two cases:

(i) Suppose $\beta(n) \geq \frac{1}{2}m$. Then $3(\beta(n+1) + \beta(n)) \geq 4\beta(n+1)$, and if $\beta(n) \leq s \leq \beta(n+1)$, we have

$$\frac{\pi}{4} \leq \frac{\beta(n)\pi}{\beta(n+1) + \beta(n)} = t_n\beta(n) \leq t_n s \leq t_n\beta(n+1) = \frac{\pi\beta(n+1)}{\beta(n+1) + \beta(n)} \leq \frac{3\pi}{4}$$

and so

$$\left|\int_{\beta(n)}^{\beta(n+1)} \sin t_n s\, ds\right| \geq |\beta(n+1) - \beta(n)| \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}|\beta(n+1) - \beta(n)|$$

$$> (1/\sqrt{2})2\sqrt{2}\, s_0 > s_0$$

which contradicts (3.3).

(ii) Suppose $\beta(n) < \frac{1}{2}m$. Then

$$\left|\int_{\beta(n)}^{\beta(n+1)} \sin t_n s\,ds\right| > \left|\int_{m/2}^{3m/2} \sin t_n s\,ds\right|$$

$$> \left(\sin\frac{\pi}{4}\right)m = \frac{1}{\sqrt{2}}\,\frac{\beta(n+1)+\beta(n)}{2} > s_0$$

again contradicting (3.3). Hence (3.4) must hold. It follows easily that

$$|\beta(n)| \le 2\sqrt{2}\, s_0|n| + |\beta(0)| \quad \text{for all } n \in \mathbf{Z}. \tag{3.5}$$

Recall that the Fourier coefficients of a function in $\mathfrak{D}(\mathrm{T})$ go to 0 faster than any polynomial, and such a function is the uniform limit of its Fourier series.

Let f be any function in $\mathfrak{D}(\mathrm{T})$ with support in $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$. Defining f_t, $t \le \frac{1}{2}t_0$, by $\hat{f}_t(n) = \hat{f}(n)\cos(t\beta(n))$, we see that $\operatorname{supp}(f_t) \subset (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. As functions on $\mathbf{R}$, f and f_t have Fourier transforms which extend to entire functions on $\mathbf{C}$. For $z \in \mathbf{C}$,

$$\hat{f}(z)F_t(z) = \hat{f}_t(z) = \int_{\mathbf{R}} f_t(s)e^{-isz}\,ds$$

$$= \int_{-\pi/2}^{\pi/2}\left(\sum_{n=-\infty}^{\infty} \hat{f}_t(n)e^{ins}\right)e^{-isz}\,ds$$

$$= \sum_{n=-\infty}^{\infty} \hat{f}(n)\cos(t\beta(n))\int_{-\pi/2}^{\pi/2} e^{-is(z-n)}\,ds.$$

Considering these as functions of t for fixed z, we get by formal termwise differentiation

$$\frac{d}{dt}[\hat{f}(z)F_t(z)] = -\sum_{-\infty}^{\infty} \hat{f}(n)\beta(n)\sin(t\beta(n))\int_{-\pi/2}^{\pi/2} e^{-is(z-n)}\,ds,$$

$$\frac{d^2}{dt^2}[\hat{f}(z)F_t(z)] = -\sum_{-\infty}^{\infty} \hat{f}(n)(\beta(n))^2\cos(t\beta(n))\int_{-\pi/2}^{\pi/2} e^{-is(z-n)}\,ds,$$

and this is justified because for any $z \in \mathbf{C}$, the series are uniformly summable in t since $f \in \mathfrak{D}(\mathrm{T})$ and (3.5) holds. Also

$$\frac{d^2}{dt^2}[\hat{f}(z)F_t(z)] = -\int_{-\pi/2}^{\pi/2}\left(\sum_{n=-\infty}^{\infty} \hat{f}(n)(\beta(n))^2\cos(t\beta(n))e^{ins}\right)e^{-isz}\,ds$$

is the Fourier transform of a function in $L^2(\mathrm{R})$ supported on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

Now $\hat{f}(m)F_t(m) = \hat{f}(m)\cos(t\beta(m))$ if $m \in \mathbf{Z}$ and

$$(d^2/dt^2)[\hat{f}(m)F_t(m)] = -(\beta(m))^2\hat{f}(m)\cos(t\beta(m)) = -(\beta(m))^2\hat{f}(m)F_t(m)$$

so that if we let

$$(\beta_{f,t}(z))^2 = \frac{(d^2/dt^2)[\hat{f}(z)F_t(z)]}{F_t(z)\hat{f}(z)}$$

where the denominator is $\neq 0$ then $\beta^2_{f,t}$ agrees with β^2 whenever they are both defined. Also, for every $z \in \mathbf{C}$ there exists $t \leq \frac{1}{2}t_0$ such that $F_t(z) \neq 0$, since $2F_t^2 - F_{2t} = 1$. Now if $t_1 \neq t_2$, $f_1 \neq f_2$,

$$\frac{d^2}{dt^2}[\hat{f}_1(z)F_t(z)]\Big|_{t=t_1}[F_{t_2}(z)\hat{f}_2(z)] = \frac{d^2}{dt^2}[\hat{f}_2(z)F_t(z)]\Big|_{t=t_2}[\hat{f}_1(z)F_{t_1}(z)]$$

since these are the Fourier transforms of functions with compact support $\subset(-\pi, \pi)$ and agree on $\mathbf{Z}$, that is, the Fourier coefficients are the same. Hence we get a single entire function β^2 extending the given β^2. Also $\hat{f}\cos(t\beta) = \hat{f}F_t'$ on $\mathbf{Z}$ and both are Fourier transforms of functions in $\mathfrak{D}(\mathbf{R})$ with compact support, so they are equal on $\mathbf{C}$. Hence $F_t(x) = \cos(t\beta(x))$ for all $x \in \mathbf{R}$.

So we have a family μ_t $(t \leq \frac{1}{2}t_0)$ of distributions on $\mathbf{R}$ with compact support such that $\hat{\mu}_t = \cos(t\beta)$ for some function β on $\mathbf{R}$. Define μ_t for $\frac{1}{2}t_0 \leq t \leq t_0$ by $\mu_{2t} = 2\mu_t^2 - \delta_0$. Then $\hat{\mu}_t = \cos(t\beta)$ for these t also. Continue, defining $\mu_t \in \mathfrak{S}'(\mathbf{R})$ for all $t \geq 0$, so that $\hat{\mu}_t = \cos(t\beta)$. Now β satisfies the hypotheses of Theorem 1 and so must have the desired form.

Theorem 4 can be generalized to arbitrary compact connected abelian groups. Recall that a group is compact and connected if and only if its dual is discrete and torsion-free; see [10, p. 47].

Theorem 5. *Let G be a compact connected abelian group with dual group Γ. Let β be a real function on Γ and let $\mu_t \in \mathfrak{D}'(G)$ satisfy $\hat{\mu}_t = \cos(t\beta)$. The following are equivalent:*

(i) *For every nbhd U of 0 in G there exists $t_0 > 0$ such that $\operatorname{supp}(\mu_t) \subset U$ for all $t \leq t_0$.*

(ii) *For every homomorphism $\rho: \mathbf{Z} \to \Gamma$ and every $\lambda \in \Gamma$, there exist $a(\rho, \lambda)$, $b(\rho, \lambda)$, and $c(\rho, \lambda)$ in $\mathbf{R}$ such that*

$$(\beta(\lambda + \rho(n)))^2 = a(\rho, \lambda)n^2 + b(\rho, \lambda)n + c(\rho, \lambda) \quad \textit{for all } n \in \mathbf{Z},$$

$$\sup_\lambda a(\rho, \lambda) < \infty \quad \textit{for each such } \rho, \quad \textit{and}$$

$$a(\rho, \lambda)r^2 + b(\rho, \lambda)r + c(\rho, \lambda) \geq 0 \quad \textit{for all } r \in \mathbf{R}.$$

Proof. See [1].

4. **A nonabelian group.** In this section we take a very brief look at what can happen on nonabelian groups. We consider only a particular group where there is an analogue of the Paley-Wiener theorem.

Consider the group $\mathcal{G}$ of complex matrices $\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$ such that $|\alpha|^2 - |\beta|^2 = 1$. This group is isomorphic to SL(2, R) under the map

$$\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \mapsto \begin{pmatrix} a_1 + b_1 & b_2 - a_2\\ b_2 + a_2 & a_1 - b_1\end{pmatrix},$$

where $\alpha = a_1 + ia_2$, $\beta = b_1 + ib_2$. Each element of the group defines a conformal mapping of the interior of the unit circle onto itself by $z \mapsto (\alpha z + \beta)/(\bar{\beta} z + \bar{\alpha})$, and two matrices define the same conformal mapping if and only if they are equal or one is the negative of the other. The group we shall consider is the group G of all conformal mappings of the unit circle, or, equivalently, the quotient of $\mathcal{G}$ by $\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\}$.

We let K be the compact subgroup of G consisting of all rotations, or the subgroup

$$\left\{\begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta}\end{pmatrix}\right\} \quad \text{of } \mathcal{G}$$

(this will cause no confusion). A function f on G (or $\mathcal{G}$) is called a spherical function if $f(kgk') = f(g)$ for all $k, k' \in K$ and $g \in G$ (or $\mathcal{G}$).

For each $s \in \mathbf{C}$ on the line $\mathrm{Re}(s) = \frac{1}{2}$, define a unitary representation $U(\cdot, s)$ of G on $L^2(\mathrm{T})$ by

$$[U(g, s)f](z) = |d(gz)/dz|^s f(gz) \qquad (g \in G,\ f \in L^2(\mathrm{T}),\ |z| = 1).$$

(For details of this and what follows, see Ehrenpreis and Mautner [3].) We call $\{s \in \mathbf{C}\colon \mathrm{Re}(s) = \frac{1}{2}\}$ the dual of G. Let $a_n = e^{in\theta}$, $n = 0, \pm 1, \pm 2, \cdots$ be the usual orthonormal basis of $L^2(\mathrm{T})$. The Fourier transform of a function $f \in L^1(G)$ is in general an operator-valued function; for each s, $\hat{f}(s)$ is a bounded operator on $L^2(\mathrm{T})$. But if f is a spherical function, then $\langle \hat{f}(s)a_n, a_m\rangle = 0$ unless $n = m = 0$, and so $\hat{f}(s)$ is determined by the scalar $\langle \hat{f}(s)a_0, a_0\rangle = \int_G f(g)\langle U(g, s)a_0, a_0\rangle\, dg$. Hence the Fourier transform of a spherical L^1 function can be considered a scalar-valued function $\hat{f}(s)$ on $\{\mathrm{Re}(s) = \frac{1}{2}\}$.

A form of the Plancherel theorem holds for spherical functions, the measure on $\{\mathrm{Re}(s) = \frac{1}{2}\}$ being

$$-i(s-\tfrac{1}{2})\tan(-\pi(s-\tfrac{1}{2}))ds,$$

or, letting $s = \tfrac{1}{2} + it$,

$$(it)\tan(-i\pi t)dt = t\frac{e^{\pi t} - e^{-\pi t}}{e^{\pi t} + e^{-\pi t}}dt.$$

That is, if f is a spherical function in $L^1 \cap L^2(G)$, then

$$\int_G |f(g)|^2 dg = \int |f(\tfrac{1}{2} + it)|^2 t \frac{e^{\pi t} - e^{-\pi t}}{e^{\pi t} + e^{-\pi t}} dt.$$

Now G has a bi-variant Laplace-Beltrami operator Δ, which on the spherical functions corresponds to multiplication by $s(1-s)$ on the dual. That is, if f is in the space SC_0^∞ of infinitely differentiable spherical functions with compact support, then $(\Delta f)\hat{}(s) = s(1-s)\hat{f}(s)$. If $a \geq 0$ and $c \geq -\tfrac{1}{4}a$, then $a\Delta + c$ corresponds to multiplication by $as(1-s) + c = a(\tfrac{1}{2}+it)(\tfrac{1}{2}-it) + c = a(\tfrac{1}{4} + t^2) + c \geq 0$. If B is defined on the spherical L^2 functions by $(Bf)\hat{}(s) = |as(1-s)+c|^{1/2}\hat{f}(s)$, then B is densely defined and both $\cos(tB)$ and $(\sin(tB))/B$ are bounded operators on the spherical L^2 functions by the Plancherel theorem. The Paley-Wiener theorem for G (Ehrenpreis [3, p. 13]) states that if $f \in SC_0^\infty$, then $\hat{f}$ is in $\mathcal{S}(\mathbf{R})$, $\hat{f}$ has an entire extension of exponential type (the type depending on the size of the support), and $\hat{f}(s) = \hat{f}(1-s)$; and conversely. From this it follows easily that $\cos(tB)$ maps SC_0^∞ onto itself, and in fact $\operatorname{supp}(\cos(tB)) \subset \operatorname{supp}(f) + K_t$ independent of f; and the same is true of $(\sin(tB))/B$.

Note that if does not make sense to say that B is a translation-invariant operator on the spherical functions because the set of spherical functions is not invariant under translation.

A converse holds in the following sense: Let β be a real-valued measurable function on $\{\operatorname{Re}(s) = \tfrac{1}{2}\}$. Then by the Plancherel theorem β determines a densely defined operator B on the spherical L^2 functions by $(Bf)\hat{}(s) = \hat{f}(s)\beta(s)$. If $\operatorname{supp}(\cos(tB)f) \subset \operatorname{supp}(f) + K_t$ or all $t \geq 0$ and all $f \in SC_0^\infty$ and some K_t independent of f, then $B^2 = a\Delta + c$, $a \geq 0$, $c \geq -\tfrac{1}{4}a$. For, $\cos(t\beta)$ is entire of exponential type for the usual reasons and $\cos(t\beta(s)) = \cos(t\beta(1-s))$, hence $(\beta(s))^2 = bs^2 + as + c$ and $(\beta(s))^2 = (\beta(1-s))^2$. Therefore $a + b = 0$, $(\beta(s))^2 = as(1-s) + c$.

APPENDIX

We state here those facts about distributions on LCA groups which are needed in the proof of Theorem 2. They are probably well known to the experts, but do not seem to have found their way into print. The proofs invariably depend upon reduction to the case of elementary groups. Complete proofs can be found in [1].

Theorem A.1. *If $f \in L^p(G)$, $1 \le p \le \infty$, then the map $\Phi \mapsto \int_G \phi(x)f(x)\,dx$ is a tempered distribution on G. If $f \in L^2(G)$, then the Fourier transform of the distribution determined by f is the distribution determined by $\hat{f} \in L^2(\Gamma)$.*

For any $x \in G$ we define τ_x on $\mathcal{S}(G)$ by $(\tau_x\phi)(y) = \phi(y - x)$. Now τ_x is a continuous map of $\mathcal{S}(G)$ into itself, and we define τ_x on $\mathcal{S}'(G)$ by transposition, that is, $(\tau_x T)(\phi) = T(\tau_x \phi)$ for all $\phi \in \mathcal{S}(G)$.

Theorem A.2. *Let H be an open subgroup of G and $\Lambda \subset \Gamma$ the annihilator of H. Let $T \in \mathcal{S}'(G)$. Then the support of T is contained in H if and only if $\hat{T}$ is invariant under translation by Λ, that is, if and only if $\tau_\gamma \hat{T} = \hat{T}$ for all $\gamma \in \Lambda$.*

We recall the notion of tensor product and convolution of distributions from Bruhat [2]. Given two locally compact groups G_1 and G_2, and $S \in \mathfrak{D}'(G_1)$ and $T \in \mathfrak{D}'(G_2)$, there is a unique distribution $S \otimes T \in \mathfrak{D}'(G_1 \times G_2)$ such that $(S \otimes T)(\phi_1 \otimes \phi_2) = S(\phi_1)T(\phi_2)$ for all $\phi_j \in \mathfrak{D}(G_j)$, $j = 1, 2$. Then Fubini's theorem is true, that is, for any $\phi \in \mathfrak{D}(G_1 \times G_2)$, $(S \otimes T)(\phi) = S_x T_y \phi(x, y) = T_y S_x \phi(x, y)$.

For any closed sets A and B of G, the following two conditions are equivalent:

(i) For every compact subset K of G, $(A \times B) \cap K^+$ is compact in $G \times G$, where $K^+ = \{(x, y): x + y \in K\}$.

(ii) For every compact $K \subset G$, $A \cap (K - B)$ is compact in G.

If A and B satisfy these conditions, we say that they are compatible.

If S, $T \in \mathfrak{D}'(G)$ and $\operatorname{supp}(S)$ and $\operatorname{supp}(T)$ are compatible, then we define $S * T(\phi) = (S_x \otimes T_y)\phi(x + y)$ for $\phi \in \mathfrak{D}(G)$, and then $S * T \in \mathfrak{D}'(G)$.

Theorem A.3. *Let $\alpha \in \mathfrak{S}(G)$, $T \in \mathfrak{D}'(G)$, and suppose that* $\operatorname{supp}(\alpha)$ *and* $\operatorname{supp}(T)$ *are compatible. Then $\alpha * T$ is the function*

$$(\alpha * T)(x) = \langle T_y, \alpha(x - y)\rangle,$$

and this function is in $\mathfrak{S}(G)$.

Definition. Let $G = \mathbf{R}^n \times \mathbf{T}^p \times \mathbf{Z}^q \times \Phi$. Define $\mathcal{O}_M(G)$ to be the set of all complex-valued functions f on G which are infinitely differentiable with respect to the $\mathbf{R}$ and $\mathbf{T}$ variables, and such that for every $\sigma \in \mathbf{N}^n$ and $\tau \in \mathbf{N}^p$ there exists $k \in \mathbf{Z}$ such that

$$(1 + |x|^2)^k (1 + |m|^2)^k D_{x,t}^{\sigma,\tau} f(x, t, m, \phi)$$

is bounded on G, where

$$D_{x,t}^{\sigma,\tau} = \frac{\partial^{\sigma_1}}{\partial x_1^{\sigma_1}} \cdots \frac{\partial^{\sigma_n}}{\partial x_n^{\sigma_n}} \frac{\partial^{\tau_1}}{\partial t_1^{\tau_1}} \cdots \frac{\partial^{\tau_n}}{\partial t_n^{\tau_n}}.$$

Definition. For any locally compact abelian group G, define $\mathcal{O}_M(G)$ to be the set of all functions α in $\mathcal{E}(G)$ such that for every good pair H, H' in G there exists a subgroup H'' of G such that H, H'' is a good pair and the restriction of α to H is constant on the cosets of H'', inducing a function $\tilde{\alpha}$ in $\mathcal{O}_M(H/H'')$.

Theorem A.4. *For any locally compact abelian group G, we have* $\mathcal{O}_M(G) = \{\alpha \in \mathcal{E}(G): \alpha\phi \in \mathcal{S}(G)$ *for all* $\phi \in \mathcal{S}(G)\}$.

Theorem A.5. *If $T \in \mathcal{E}'(G)$, where G is a locally compact abelian group, then $\hat{T} \in \mathcal{O}_M(\Gamma)$, and in fact $\hat{T}$ is given by the function $\hat{T}(\gamma) = \langle T, \bar{\gamma}\rangle$.*

Theorem A.6. *If G is a locally compact abelian group, $S \in \mathcal{S}'(G)$, and $T \in \mathcal{E}'(G)$, then $S * T \in \mathcal{S}'(G)$ and $(S*T)\hat{} = \hat{S}\hat{T}$.*

Theorem A.7. *Let $\rho: \Gamma_1 \to \Gamma_2$ be a continuous homomorphism between locally compact abelian groups, $\rho_0: G_2 \to G_1$ its adjoint. Then $\rho_0'(\phi) = \phi \circ \rho_0$ defines a map $\rho_0': \mathcal{E}(G_1) \to \mathcal{E}(G_2)$ according to Bruhat* [2, p. 57, *Proposition* 8]. *Let $\rho_0'': \mathcal{E}'(G_2) \to \mathcal{E}'(G_1)$ be its adjoint, that is, if $\mu \in \mathcal{E}'(G_2)$* then $\langle \rho_0''(\mu), \phi\rangle = \langle \mu, \rho_0'(\phi)\rangle$ *for all $\phi \in \mathcal{E}(G_1)$. Then for every $\gamma_1 \in \Gamma_1$ and $\mu \in \mathcal{E}'(G_2)$, we have $\langle \mu, \rho(\gamma_1)\rangle = \langle \rho_0''(\mu), \gamma_1\rangle$. Also* $\operatorname{supp}(\rho_0''\mu) \subset \rho_0(\operatorname{supp}(\mu))$.

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