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DIFFERENTIABILITY OF SOLUTIONS TO HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS

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ABSTRACT. This paper establishes conditions for the differentiability of solutions to mixed problems for first order hyperbolic systems of the form $(\partial/\partial t - \sum A_j\partial/\partial x_j - B)u = F$ on $[0,T]\times\Omega$, $Mu = g$ on $[0,T]\times\partial\Omega$, $u(0,x) = f(x)$, $x \in \Omega$. Assuming that $\mathcal{L}_2$ a priori inequalities are known for this equation, it is shown that if $F \in H^s([0,T]\times\Omega)$, $g \in H^{s+1/2}([0,T]\times\partial\Omega)$, $f \in H^s(\Omega)$ satisfy the natural compatibility conditions associated with this equation, then the solution is of class C^p from $[0,T]$ to $H^{s-p}(\Omega)$, $0 \le p \le s$. These results are applied to mixed problems with distribution initial data and to quasi-linear mixed problems.

1. **Introduction.** This paper studies the differentiability of solutions to the following mixed problem for first order hyperbolic systems.

$$
(1.1)\qquad \begin{gathered} Lu = F \quad \text{in } [0,T]\times\Omega, \qquad Mu = g \quad \text{in } [0,T]\times\partial\Omega, \\ u(0,x) = f(x) \quad \text{for } x \in \Omega. \end{gathered}
$$

The methods can also be applied to hyperbolic equations of higher order. Applications are given to mixed problems with distribution initial data and to quasilinear mixed problems.

Here $Lu = \partial_t u - Gu$ with

$$Gu = \sum_{j=1}^{m} A_j(t,x)\partial_j u + B(t,x)u.$$

The A_j and B are smooth, complex $k\times k$ matrix-valued functions defined on $R\times\overline{\Omega}$, and they are constant outside a compact subset of $R\times\overline{\Omega}$. The solution $u(t,x)$ is a vector-valued function with k components. Ω is an open subset of R^m with smooth, compact boundary $\partial\Omega$ such that Ω lies on one side of $\partial\Omega$. Here $\partial_j = \partial/\partial x_j$ and $\partial_t = \partial/\partial t$. The boundary matrix, $A_n = \sum n_jA_j$, is assumed to be nonsingular on $R\times\partial\Omega$ ($n = (n_1,\dots,n_m)$ is the exterior unit normal to $\partial\Omega$).

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$M(t,x)$ is a smooth $l \times k$ matrix-valued function defined on $R \times \partial\Omega$. M is of rank l everywhere and constant for $|t|$ large.

It is assumed that there exist constants C and β such that

$$\|u(t)\|_\Omega \le Ce^{\beta(t-s)}(\|u(s)\|_\Omega + \|Lu\|_{[s,t]\times\Omega}), \tag{1.2}$$

$$\|v(s)\|_\Omega \le Ce^{-\beta(s-t)}(\|v(t)\|_\Omega + \|L^*v\|_{[s,t]\times\Omega}), \tag{1.3}$$

$-\infty < s \le t < \infty$, for all $u, v \in C^\infty_{(0)}(R \times \Omega)$ such that $Mu = 0$, $M^*v = 0$ on $R \times \partial\Omega$.

Here L^*, the formal adjoint of L, is defined by $L^*v = -\partial_t v - G^*v$, with $G^*v = -\sum \partial_j A_j^* v + B^*v$. $A_j^*(t,x)$ is the conjugate transpose of the matrix $A_j(t,x)$ and similarly for B^*. The kernel of $M(t,x)$ is the boundary subspace and is denoted by $N(t,x)$. The adjoint boundary subspace is $N^* = (A_n[N])^\perp$. We let $M^*(t,x)$, the adjoint boundary operator, be any smooth $(k-l) \times k$ matrix-valued function on $R \times \partial\Omega$ which is constant for $|t|$ large and whose kernel is N^*.

If $V \subset R^m$, then $C^\infty_{(0)}(V)$ is the space of functions on V which are the restrictions to V of functions in $C^\infty_0(R^m)$. $\mathcal{L}_2(V)$ is the space of functions $u = (u_1, \ldots, u_k)$ which are square integrable with respect to the Lebesgue measure on V (surface measure on V, if V is a hypersurface). The norm and inner product in $\mathcal{L}_2(V)$ are denoted by $\|\cdot\|_V$ and $(\cdot\,,\cdot)_V$. $H^s(V)$ is the usual Sobolev space of functions on V whose derivatives of order up to s lie in $\mathcal{L}_2(V)$; its norm is denoted by $\|\cdot\|_{s,V}$.

If $u = u(t,x)$ is a function of t and x, then, for fixed t, $u(t)$ is the function of x obtained by freezing t; $u(t)(x) = u(t,x)$. Similarly, if G is a differential operator, as above, which has only x derivatives, then $G(t)$ is the operator with t frozen which acts on functions $f(x)$ of x alone; $G(t)f = \sum A_j(t)\partial_j f + B(t)f$.

Remark 1.1. The *a priori* inequalities (1.2) and (1.3) hold for a variety of hyperbolic mixed problems, for example, the symmetric hyperbolic problems studied by Friedrichs, Lax and Phillips [1], [6], the strictly hyperbolic problems studied by Kreiss and Rauch [5], [8], and the symmetrizable problems studied by Ikawa [3].

Remark 1.2. In case the A_j, B, and M do not depend on t, let $\mathcal{G}$ be the operator in $\mathcal{L}_2(\Omega)$ with domain

$$\mathcal{D}(\mathcal{G}) = \{u \in C^\infty_{(0)}(\Omega)\colon Mu = 0 \text{ on } \partial\Omega\},$$

and defined by $\mathcal{G}u = \sum A_j \partial_j u + Bu$, $u \in \mathcal{D}(\mathcal{G})$. The inequalities (1.2) and (1.3) hold if and only if $\overline{\mathcal{G}}$, the closure of $\mathcal{G}$ in $\mathcal{L}_2(\Omega)$, is the generator of a C_0-semigroup in $\mathcal{L}_2(\Omega)$.

Remark 1.3. The above hypotheses on A_j, B, Ω and M may be weakened somewhat; see §4.

In general, the assumption that f, F and g are smooth is not sufficient to insure that the solution of (1.1) is smooth. For example, the mixed problem

$$Lu = \partial_t u + \partial_x u = 0, \qquad x > 0,\ t \geq 0,$$

$$u(t,0) = 0, \quad t \geq 0, \qquad u(0,x) = f(x), \quad x > 0,$$

has solution u which is given by $u(t,x) = f(x - t)$ for $x \geq t$ and $u(t,x) = 0$ for $0 \leq x \leq t$. If $f(0) \neq 0$, the solution is discontinuous along the line $t = x$. This difficulty is due to the fact that the initial conditions $u = f$ at $t = 0$ and boundary conditions $u = 0$ for $x = 0$ are *incompatible* at the point $t = x = 0$. However, if f is continuous and vanishes at $x = 0$, then the solution will be continuous. The higher order compatibility conditions $f^{(j)}(0) = 0$ imply continuity of higher order derivatives of u.

The appropriate compatibility conditions on the data F, g and f in (1.1) which are necessary to insure that the solution u is s times differentiable may be obtained as follows. Suppose, for $0 \leq p \leq s$, $\partial_t^p u$ is continuous on $[0,T]$ to $H^{s-p}(\Omega)$. Since the surface $t = 0$ is noncharacteristic, the classical computation of the Cauchy-Kowalewsky theorem shows that $(\partial_t^p u)(0) = f_p$, where f_p is determined from f and F by $f_0 = f$ and

$$f_p = \sum_{i=0}^{p-1} \binom{p-1}{i} G_i(0) f_{p-1-i} + (\partial_t^{p-1} F)(0), \tag{1.4}$$

$1 \leq p \leq s$, with $G_0 = G$ and $G_i = \sum_j (\partial_t^i A_j)\partial_j + (\partial_t^i B) = [\partial_t, G_{i-1}]$, $i \geq 1$.

The restriction of u to $[0,T] \times \partial\Omega$ is of class C^p from $[0,T]$ to $H^{s-p-1}(\partial\Omega)$ for $p \leq s - 1$. Thus $\partial_t^p Mu = \partial_t^p g$ on $[0,T] \times \partial\Omega$ for $p \leq s - 1$. Application of Leibniz' rule to compute this derivative yields the compatibility condition

$$\sum_{i=0}^{p} \binom{p}{i} (\partial_t^i M)(0) f_{p-i} = \partial_t^p g(0) \quad \text{on } \partial\Omega \text{ for } 0 \leq p \leq s - 1. \tag{1.5}$$

That the data F, g and f satisfy (1.5) is therefore a necessary condition for the solution u of (1.1) to be of class C^p on $[0,T]$ to $H^{s-p}(\Omega)$ for $0 \leq p \leq s$. It will be shown that this is also a sufficient condition; see Theorem 3.1 and §4. In §4 we also indicate how these results may be extended to higher order equations (for example, the mixed problems studied by Sakamoto [10]).

In §5 these results are used to treat mixed problems with distribution initial data by means of a duality argument. A similar method was employed previously by Rauch and Taylor [9].

§6 contains an application to quasilinear mixed problems. Using the differentiability results for the linear case, one may construct solutions to quasilinear equations using an iterative technique.

In §7 it is shown that differentiability assumptions on u weaker than the above still imply the compatibility conditions (1.5) are satisfied.

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2. Preliminaries. If X is a Banach space and $I \subset R$ is an interval, then $C^s(I; X)$ is the space of s times continuously differentiable functions on I with values in X.

If $u \in C^r([0, T]; H^{s-r}(\Omega))$, $0 \leq r \leq s$, and $0 \leq t \leq T$, then we define

$$|||u(t)|||^2_{s,\Omega} = \sum_{j=0}^{s} \|\partial_t^j u(t)\|^2_{s-j,\Omega}. \tag{2.1}$$

When the boundary conditions in (1.1) are homogeneous we use the following notion of a strong solution. Given $F \in \mathcal{L}_2([0, T] \times \Omega)$, $f \in \mathcal{L}_2(\Omega)$, a function $u \in \mathcal{L}_2([0, T] \times \Omega)$ is a strong solution to

$$\begin{gathered} Lu = F \quad \text{on } [0, T] \times \Omega, \qquad Mu = 0 \quad \text{on } [0, T] \times \partial\Omega, \\ u(0) = f \quad \text{on } \Omega, \end{gathered} \tag{2.2}$$

if there exists a sequence $\{u_n\} \subset C^\infty_{(0)}([0, T] \times \Omega)$ with $u_n \to u$, $Lu_n \to F$ in $\mathcal{L}_2([0, T] \times \Omega)$, $Mu_n = 0$ on $[0, T] \times \partial\Omega$, and $u_n(0) \to f$ in $\mathcal{L}_2(\Omega)$.

Proposition 2.1. *For each $F \in \mathcal{L}_2([0, T] \times \Omega)$, $f \in \mathcal{L}_2(\Omega)$, there exists a unique strong solution to* (2.2) *which belongs to $C([0, T]; \mathcal{L}_2(\Omega))$ and satisfies*

$$\|u(t)\|_\Omega \leq Ce^{\beta t}(\|f\|_\Omega + \|F\|_{[0,t]\times\Omega}), \tag{2.3}$$

$0 \leq t \leq T$, with C and β as in (1.2).

Proof. If u is a solution to (2.2) there is an approximating sequence $u_n \in C^\infty_{(0)}([0, T] \times \Omega)$ with the properties above. Applying (1.2) to $u_n - u_m$, we see that $\{u_n\}$ converges (to u) in $C([0, T]; \mathcal{L}_2(\Omega))$ and (2.3) holds.

To prove existence, we may reduce to the case where $f \in C_0^\infty(\Omega)$ by approximating the general $f \in \mathcal{L}_2(\Omega)$ by $f_n \in C_0^\infty(\Omega)$ with $f_n \to f$ in $\mathcal{L}_2(\Omega)$. If u_n is the solution of (2.2) with f replaced by f_n, then, by applying (2.3) to $u_n - u_m$, we may conclude that $\{u_n\}$ converges in $C([0, T]; \mathcal{L}_2(\Omega))$, say $u_n \to u$. This u is the desired solution to (2.2).

Assuming $f \in C_0^\infty(\Omega)$, we may reduce to the case where $f = 0$ by letting $v(x, t) = u(x, t) - f(x)$. The equation (2.2) for u is equivalent to the equation $Lv = F - Gf$, $Mv = 0$, $v(0) = 0$, which has the same form as (2.2) except $f = 0$.

Once $f = 0$, a strong solution of (2.2) may be obtained using a standard argument; see, for example, Lax and Phillips [6]. □

3. The differentiability theorem when the boundary conditions are homogeneous.

Theorem 3.1. *Let $f \in H^s(\Omega)$, $F \in H^s([0, T] \times \Omega)$, and let f_p be defined from f and F by* (1.4). *A necessary and sufficient condition that the solution u of* (2.2) *belong to $C^r([0, T]; H^{s-r}(\Omega))$ for $0 \leq r \leq s$ is that the compatibility conditions* (1.5) *be satisfied with $g = 0$. In this case there exists a constant C_s, independent of F and f, such that*

$$|||u(t)|||_{s,\Omega} \leq C_s(\|f\|_{s,\Omega} + \|F\|_{s,[0,t]\times\Omega} + |||F(0)|||_{s-1,\Omega}) \tag{3.1}$$

for $0 \leq t \leq T$.

Remark 3.1. Note that for $t \geq \delta > 0$, the last term on the right side of (3.1) may be estimated by the second term.

Example. Suppose, as in Remark 1.2, the A_j, B and M are independent of t. If $F = 0$, $g = 0$, then the compatibility conditions (1.5) take the simple form $f \in \mathfrak{D}(\overline{\mathcal{G}}^s) \cap H^s(\Omega)$, that is, $f \in H^s(\Omega)$ and $\mathcal{G}^p f(x) \in N(x)$, $x \in \partial\Omega$, $0 \leq p \leq s - 1$. Theorem 3.1 implies that $\mathfrak{D}(\overline{\mathcal{G}}^s) \cap H^s(\Omega)$ is *admissible* (as defined by Kato [4, Definition 2.1]) with respect to $\overline{\mathcal{G}}$, i.e. the semigroup generated by $\overline{\mathcal{G}}$ leaves $\mathfrak{D}(\overline{\mathcal{G}}^s) \cap H^s(\Omega)$ invariant and forms a C_0-semigroup in this space. Here $\mathfrak{D}(\overline{\mathcal{G}}^s) \cap H^s(\Omega)$ is viewed as a Hilbert space with the norm of $H^s(\Omega)$.

Proof of Theorem 3.1. The necessity part of the theorem was established in §1; it remains to prove the sufficiency part of the theorem and the inequality (3.1). We isolate the technical details of the proof into the following four lemmas.

Lemma 3.1. *Let* $f \in H^{2s+3}(\Omega)$, $F \in H^{2s+3}([0, T] \times \Omega)$, *and suppose the compatibility conditions* (1.5) *are satisfied with* $g = 0$ *not only for* $0 \leq p \leq s - 1$, *but also for* $p = s, s + 1$ (*where* f_s, f_{s+1} *are defined by* (1.4) *with* $p = s, s + 1$). *Then there exists* $w \in H^{s+2}([0, T] \times \Omega)$ *such that* $Mw = 0$ *on* $[0, T] \times \partial\Omega$, $w(0) = f$, *and* $\partial_t^p(Lw - F)(0) = 0$ *for* $0 \leq p \leq s$.

Lemma 3.2. *Suppose* $u \in H^{s+1}([0, T] \times \Omega)$ *satisfies* (2.2). *Then the inequality* (3.1) *holds.*

Lemma 3.3. *If the data* f, F *satisfy the hypotheses of Theorem* 3.1 *and* $r \geq s + 2$, *then there exist sequences* $\{f_n\} \subset H^r(\Omega)$, $\{F_n\} \subset H^r([0, T] \times \Omega)$ *with* $f_n \to f$ *in* $H^s(\Omega)$, $F_n \to F$ *in* $H^s([0, T] \times \Omega)$ *and such that* f_n, F_n *satisfy the compatibility conditions* (1.5) *with* $g = 0$ *for* $0 \leq p \leq s + 1$ *and each* n.

Lemma 3.4. *If* $f = 0$, $F \in H^s([0, T] \times \Omega)$ *and* $\partial_t^j F(0) = 0$ *for* $0 \leq j \leq s - 1$, *then the solution of* (2.2) *lies in* $H^s([0, T] \times \Omega)$.

Assuming for the moment that these lemmas are true, we complete the proof of Theorem 3.1.

Suppose f, F satisfy the hypotheses of Lemma 3.1. We claim the solution u of (2.2) belongs to $H^{s+1}([0, T] \times \Omega)$ (hence $u \in C^r([0, T]; H^{s-r}(\Omega))$, $0 \leq r \leq s$) and the inequality (3.1) holds. To see this, let $v = u - w$, where w satisfies the conclusions of Lemma 3.1. Then v is a strong solution to $Lv = F - Lw$, $Mv = 0$, $v(0) = 0$. We have $F - Lw \in H^{s+1}([0, T] \times \Omega)$ and $\partial_t^j(F - Lw)(0) = 0$, $0 \leq j \leq s$. It follows from Lemma 3.4 that v belongs to $H^{s+1}([0, T] \times \Omega)$. Therefore u does also, and it follows immediately from Lemma 3.2 that the inequality (3.1) holds.

Now suppose f, F are general data satisfying the hypotheses of Theorem 3.1, and let $\{f_n\}$, $\{F_n\}$ be sequences which satisfy the conclusions of Lemma 3.3. If u_n is the solution to (2.2) corresponding to the data f_n, F_n, then inequality (3.1) applied to $u_n - u_m$ shows that the sequence $\{u_n\}$ converges in $C^r([0, T]; H^{s-r}(\Omega))$, $0 \leq r \leq s$. Since $u_n \to u$ in $\mathcal{L}_2([0, T] \times \Omega)$, u belongs to $C^r([0, T]; H^{s-r}(\Omega))$, $0 \leq r \leq s$. □

Proof of Lemma 3.1. We make a change of dependent variables, $v = ru$, where $r(t,x)$ is a smooth, invertible $k \times k$ matrix-valued function on $[0,T] \times \bar{\Omega}$ which maps $N(t,x)$ onto $N(0,x)$ for $(t,x) \in [0,T] \times \partial\Omega$. Such an r may be constructed using, for example, the method in [7] (see the proof of Theorem 3). It is not hard to show that $v = ru$ is a strong solution to $Kv = E$, $Pv = 0$, $v(0) = g$, where $K = rLr^{-1}$, $E = rF$, $P = Mr^{-1}$, and $g = r(0)f$.

The definition (1.4) of the f_p is such that

$$u_{ck} \equiv \sum_{p=0}^{s+1} (t^p/p!) f_p$$

is an approximate (Cauchy-Kowalewsky) solution of the problem $Lu = F$, $u(0) = f$, in the sense that $\partial_t^p(Lu_{ck} - F)(0) = 0$ on Ω for $p \leq s$, and $u_{ck}(0) = f$. Let $v_{ck} = \sum_{p=0}^{s+1} (t^p/p!) g_p$ be the corresponding approximate solution of the transformed problem $Kv = E$, $v(0) = g$. One easily sees that $r^{-1}v_{ck}$ is another approximate solution to the original problem. Hence u_{ck} is the Taylor expansion of $r^{-1}v_{ck}$ up to order $s + 1$ at $t = 0$.

The compatibility conditions on F and f assert that $\partial_t^p(Mu_{ck})(0) = 0$ on $\partial\Omega$ for $p \leq s + 1$. Therefore $\partial_t^p(Mr^{-1}v_{ck})(0) = 0$ on $\partial\Omega$ for $p \leq s + 1$. Now, since $\ker((Mr^{-1})(t,x)) = \ker(M(0,x)) = N(0,x)$, we have

$$M(x,0) = S(t,x)(Mr^{-1})(t,x)$$

for $x \in \partial\Omega$, where S is a smooth $l \times l$ matrix-valued function defined on $[0,T] \times \partial\Omega$. It follows that $\partial_t^p(M(0,x)v_{ck})(0) = 0$ on $\partial\Omega$ for $p \leq s + 1$. In other words, $g_p \in N(0,x)$ on $\partial\Omega$ for $p \leq s + 1$.

One can verify without difficulty that $w = r^{-1}v_{ck}$ satisfies the requirements of Lemma 3.1. □

Proof of Lemma 3.2. Using localization, we reduce to the case where M is constant and the relevant portion of the boundary of Ω is contained in a hyperplane. Then the tangential derivatives of u are easily estimated since they satisfy mixed problems obtained by differentiation of the original equation $Lu = F$. The normal derivative of u is estimated by expressing it in terms of the other derivatives of u using the equation $Lu = F$ and the fact that the boundary matrix is nonsingular. Since the methods are standard, we are quite brief here. In particular, the reader is referred to Ikawa [3, cf. Theorem 1 and Proposition 2.11].

Since $|||u(0)|||_{s,\Omega} \leq \text{const}(\|u(0)\|_{s,\Omega} + |||F(0)|||_{s-1,\Omega})$, it suffices to prove

$$|||u(t)|||_{s,\Omega}^2 \leq \text{const}(|||u(0)|||_{s,\Omega}^2 + \|Lu\|_{s,[0,t]\times\Omega}^2), \qquad 0 \leq t \leq T. \tag{3.2}$$

Using a partition of unity and changes of the dependent variables and the independent x variables, we may reduce to the case where $\Omega = \{x \in R^m: x_m > 0, |x| < 1\}$, the support of u is contained in $\{x \in \bar{\Omega}: |x| \leq \frac{1}{2}\}$, and the boundary conditions take the form $Mu = 0$ on $[0,T] \times \Gamma$, where $\Gamma = \{x$

$\in R^m\colon x_m = 0, |x| < 1\}$ and M is constant, independent of x and t (cf. Ikawa [3, pp. 449–452]). Furthermore, A_m (which is now the boundary matrix) is nonsingular in $[0, T] \times \bar{\Omega}$.

Since M is constant, we have $M\partial_t u = M\partial_j u = 0$ on $[0, T] \times \Gamma$, $1 \leq j \leq m - 1$. Substituting $\partial_t u$, $\partial_j u$ into (3.2) with s replaced by $s - 1$ (we are using induction on s), we obtain

$$(3.3) \qquad |||\partial_t u(t)|||^2_{s-1,\Omega} \leq \text{const}(|||u(0)|||^2_{s,\Omega} + \|Lu\|^2_{s,[0,t]\times\Omega} + \|u\|^2_{s,[0,t]\times\Omega}),$$

and similarly for $\partial_j u$, $1 \leq j \leq m - 1$.

Since A_m is nonsingular, we may express $\partial_m u$ in terms of Lu, u, $\partial_t u$, and $\partial_j u$, $1 \leq j \leq m - 1$ (cf. [3, Equation (2.33)]). Using this and (3.3), we obtain an inequality of the form (3.3) with $\partial_t u$ replaced by $\partial_m u$. Combining these inequalities, we see that $|||u(t)|||^2_{s,\Omega}$ is dominated by the right side of (3.3). Noting that $\|u\|^2_{s,[0,t]\times\Omega} = \int_0^t |||u(r)|||^2_{s,\Omega}\,dr$, we obtain (3.2) by an application of Gronwall's inequality. □

Proof of Lemma 3.3. It suffices to prove this lemma in a weaker form where the sequences $\{f_n\}$, $\{F_n\}$ satisfy all the conditions of the lemma except (1.5) for $p = s + 1$. Having done this, one can first approximate f and F by sequences $\{f_n\}$, $\{F_n\}$ satisfying (1.5) for $0 \leq p \leq s$ and then approximate the f_n, F_n by sequences satisfying (1.5) for $0 \leq p \leq s + 1$.

By making the same change of variables used in the proof of Lemma 3.1, we may reduce to the case where the boundary operator M is independent of t. Then (1.5) becomes $Mf_p = 0$ on $\partial\Omega$ for $0 \leq p \leq s - 1$, and we must approximate f and F by sequences $\{f_n\}$, $\{F_n\}$ satisfying $Mf_{n,p} = 0$ on $\partial\Omega$ for $0 \leq p \leq s$.

One sees that the definition (1.4) can be written as $f_p = B_p f + E_p F$, where

$$B_p f = G^p(0)f + \text{terms of the form } G_{i_1}(0)\cdots G_{i_q}(0)f,$$

with $q \leq p - 1$, and

$$E_p F = \text{sum of terms of the form } G_{i_1}(0)\cdots G_{i_q}(0)\partial_t^\sigma F(0),$$

with $q + \sigma \leq p - 1$.

Now choose $\{F_n\} \subset H^{r+s}([0, T] \times \Omega)$, $\{g_n\} \subset H^{r+s}(\Omega)$ with $F_n \to F$ in $H^s([0, T] \times \Omega)$, $g_n \to f$ in $H^s(\Omega)$, and write the desired sequence $\{f_n\}$ as $f_n = g_n - h_n$, where $h_n \in H^r(\Omega)$ must be chosen so that $h_n \to 0$ in $H^s(\Omega)$ and $MB_p h_n = M(B_p g_n + E_p F_n)$ on $\partial\Omega$ for $0 \leq p \leq s$. Let $T(x)$ be the inverse of $M(x)$ when it is restricted to the orthogonal complement of the kernel of $M(x)$. Then it suffices to solve the equation $B_p h_n = a_{p,n}$ on $\partial\Omega$ for $0 \leq p \leq s$, where $a_{p,n} = TM(B_p g_n + E_p F_n)$.

The operator B_p has the form

$$B_p f = A_n^p \partial_\nu^p f + \sum_{i=0}^{p-1} C_{p,p-i}\,\partial_\nu^i f,$$

where $\partial_\nu f$ is the normal derivative of f on $\partial\Omega$ and $C_{p,p-i}$ is an operator of order $p-i$ which only involves derivatives tangential to $\partial\Omega$. Thus the equation $B_p h_n = a_{p,n}$ can be written as $\partial_\nu^p h_n = b_{p,n}$, where

$$b_{p,n} = A_n^{-p}\left(a_{p,n} - \sum_{i=0}^{p-1} C_{p,p-i} b_{i,n}\right).$$

We see that $b_{p,n} \in H^{r+s-p-1/2}(\partial\Omega)$ for $0 \le p \le s$, and $b_{p,n} \to 0$ in $H^{s-p-1/2}(\partial\Omega)$ only for $0 \le p \le s-1$.

At this point we need the following results about the trace. Let $R_+^m = \{x \in R^m: x_m > 0\}$ and $R_{x'}^{m-1} = \{x \in R^m: x_m = 0\}$. For $q > 0$ there is a continuous linear operator

$$\mathcal{R}: \prod_{j=0}^{q-1} H^{q-j-1/2}(R_{x'}^{m-1}) \to H^q(R_+^m),$$

with $\partial_{x_m}^j \mathcal{R}(a_0, \ldots, a_{q-1})|_{x_m=0} = a_j$. For $s < q$, consider the operator $\mathcal{S}$ defined by

$$\mathcal{S}(a_0, \ldots, a_{s-1}) = \mathcal{R}(a_0, \ldots, a_{s-1}, 0, \ldots, 0).$$

The operator $\mathcal{R}$ can be chosen so that $\mathcal{S}$ extends to a continuous operator from $\prod_{j=0}^{s-1} H^{s-j-1/2}(R_{x'}^{m-1})$ to $H^s(R_+^m)$. For example the operator in the proof of Theorem 2.5.7 of [2] has this property.

Using a partition of unity for $\bar{\Omega}$ and local coordinate changes we can construct a continuous linear operator R: $\prod_{j=0}^{q-1} H^{q-j-1/2}(\partial\Omega) \to H^q(\Omega)$ with $\partial_\nu^j R(a_0, \ldots, a_{q-1})|_{\partial\Omega} = a_j$ and such that the operator S analogous to $\mathcal{S}$ is continuous from $\prod_{j=0}^{s-1} H^{s-j-1/2}(\partial\Omega)$ to $H^s(\Omega)$.

Using this operator S with $q = r + s$, let $v_n \in H^{r+s}(\Omega)$ be given by $v_n = S(b_{0,n}, \ldots, b_{s-1,n})$. Then $v_n \to 0$ in $H^s(\Omega)$ and $\partial_\nu^p v_n = b_{p,n}$ for $0 \le p \le s-1$. We write $h_n = v_n + w_n$, where $w_n \in H^r(\Omega)$ must be chosen so that $w_n \to 0$ in $H^s(\Omega)$, $\partial_\nu^p w_n = 0$ for $0 \le p \le s-1$, and $\partial_\nu^s w_n = b_{s,n} - \partial_\nu^s v_n \equiv c_n$.

To find such a sequence $\{w_n\}$, we may reduce to the case where $\Omega = R_+^m$ by using a partition of unity for $\bar{\Omega}$ and local coordinate changes.

Let $\psi_n(t) = (s!)^{-1} t^s \phi(nt)$, where $\phi \in C_0^\infty(R)$ with $\phi(t) = 1$ for t near 0. Then $\psi_n^{(i)}(0) = 0$ for $i \le s-1$ and $\psi_n^{(s)}(0) = 1$. Also, $\|\psi_n\|_{s,(0,\infty)} \le \text{const}\, n^{-1/2}$. By repeating the c_n, we may assume $\|c_n\|_s \le \text{const}\, n^{1/4}$. Then the desired w_n are given by $w_n(x) = \psi_n(x_m) c_n(x_1, \ldots, x_{m-1})$. □

Proof of Lemma 3.4. We first show that if $\alpha > \max(4\beta, 0)$ then

$$\|u\|_{R\times\Omega} \le 2^{1/2} C \alpha^{-1} \|(L+\alpha)u\|_{R\times\Omega} \tag{3.4}$$

for all $u \in C_{(0)}^\infty(R \times \Omega)$ such that $Mu = 0$ on $R \times \partial\Omega$ (with C as in (1.2)).

Examining the proof of Proposition 2.1, one sees that the same argument shows that if $-\infty < s < t < \infty$, $f \in \mathcal{L}_2(\Omega)$ there is a strong solution $u \in C([s,t]; \mathcal{L}_2(\Omega))$ to $Lu = 0$ on $[s,t] \times \Omega$, $Mu = 0$ on $[s,t] \times \partial\Omega$, $u(s) = f$ on Ω.

Let $U(t,s)f = u(t)$, where u is the solution to this equation. We have $\|U(t,s)\| \leq Ce^{\beta(t-s)}$ when $U(t,s)$ is considered as a bounded operator on $\mathcal{L}_2(\Omega)$.

To prove (3.4), let $F = (L + \alpha)u$ and suppose the support of u is contained in $(\sigma,\tau) \times \overline{\Omega}$. Then

$$u(s) = e^{-\alpha s} \int_\sigma^s e^{\alpha r} U(s,r) F(r)\, dr,$$

$$\|u(s)\|_\Omega \leq Ce^{(\beta-\alpha)s} \int_\sigma^s e^{(\alpha/2-\beta)r} e^{\alpha r/2} \|F(r)\|_\Omega\, dr.$$

Application of the Schwarz inequality yields

$$\|u(s)\|_\Omega^2 \leq C^2(\alpha - 2\beta)^{-1} e^{-\alpha s} \int_\sigma^s \|F(r)\|_\Omega^2 e^{\alpha r}\, dr.$$

We integrate this from σ to τ and interchange the order of integration to get

$$\|u\|_{R\times\Omega}^2 \leq C^2(\alpha - 2\beta)^{-1}\alpha^{-1} \|F\|_{R\times\Omega}^2.$$

This proves (3.4) since $\alpha - 2\beta > \alpha/2$ for $\alpha > 4\beta$.

It follows from (3.4) that if $\alpha > \max(4\beta, 0)$ and $F \in \mathcal{L}_2(R \times \Omega)$ then there is a strong solution w of

$$(3.5) \qquad (L + \alpha)w = F \quad \text{on } R \times \Omega, \qquad Mw = 0 \quad \text{on } R \times \partial\Omega,$$

which satisfies (3.4).

The results of Tartakoff [11, Theorem 1] together with (3.4) imply that, given any integer $s > 0$, there is an α_s such that if $\alpha > \alpha_s$ and $F \in H^s(R \times \Omega)$, then the solution w of (3.5) also lies in $H^s(R \times \Omega)$. It should be noted that the hypotheses on Ω in Theorem 1 of [11] can be weakened to allow for domains of the form $R \times \Omega$, where Ω has a smooth, compact boundary.

To prove the lemma, we note that the hypotheses on F imply that there exists $F' \in H^s(R \times \Omega)$ which is an extension of F to $R \times \Omega$ such that $F'(t) = 0$ for $t \leq 0$. Let $\alpha > \alpha_s$ and let w be the solution of (3.5) with F replaced by $e^{-\alpha t}F'$. Since $w \in H^s(R \times \Omega)$, it remains to show $u = e^{\alpha t}w$ in $[0,T] \times \Omega$. Since the solution to (2.2) is unique, it suffices to show $e^{\alpha t}w$ is such a solution (with $f = 0$). There exists $\{w_n\} \subset C_{(0)}^\infty(R \times \Omega)$ with $w_n \to w$, $(L + \alpha)w_n \to e^{-\alpha t}F'$ in $\mathcal{L}_2(R \times \Omega)$ and $Mw_n = 0$ on $R \times \partial\Omega$. Applying the inequality (1.2) to $e^{\alpha t}w_n$, we see that $w_n(0) \to 0$ in $\mathcal{L}_2(\Omega)$. By considering the sequence $u_n = e^{\alpha t}w_n$, we see that $e^{\alpha t}w$ (when restricted to $[0,T] \times \Omega$) is a strong solution to (2.2). $\square$

4. Extensions.

1. *Inhomogeneous boundary conditions.* When $g \neq 0$ the equation (1.1) can be reduced to (2.2) using a standard technique.

Proposition 5.1. *Let* $f \in H^s(\Omega)$, $F \in H^s([0,T] \times \Omega)$, $g \in H^{s+1/2}([0,T] \times \partial\Omega)$. *Then the solution* u *of* (1.1) *belongs to* $C^r([0,T]; H^{s-r}(\Omega))$, $0 \leq r \leq s$, *if* (*and only if*) (1.5) *holds.*

Proof. There exists $v \in H^{s+1}([0,T] \times \Omega)$ such that $Mv = g$ on $[0,T] \times \partial\Omega$. If $F' = F - Lv$ and $f' = f - v(0)$, then one can show that F' and f' satisfy (1.5) with $g = 0$. Let w be the solution to (2.2) with F and f replaced by F' and f'. By Theorem 3.1 w belongs to $C^r([0,T]; H^{s-r}(\Omega))$, $0 \leq r \leq s$. Since $u = v + w$, u also has these properties. □

It is possible to reduce the assumptions on g in this proposition to $g \in H^s([0,T] \times \partial\Omega)$ provided one can strengthen the inequality (1.2) to

$$\|u(t)\|_{\Omega} \leq Ce^{\beta(t-s)}(\|u(s)\|_{\Omega} + \|Lu\|_{[s,t]\times\Omega} + \|Mu\|_{[s,t]\times\partial\Omega}), \tag{4.1}$$

$-\infty < s < t < \infty$, $u \in C^{\infty}_{(0)}(R \times \Omega)$. The justification of this requires a number of modifications in the proof of Theorem 3.1 and is omitted.

Rauch [8] has shown that the inequality (4.1) holds for the class of mixed problems formulated by Kreiss [5]. In addition, for these problems (4.1) may be strengthened by adding the term $\|u\|_{[s,t]\times\partial\Omega}$ to the left side. This implies that in the above proposition one may also conclude that the restriction of u to $[0,T] \times \partial\Omega$ lies in $H^s([0,T] \times \partial\Omega)$.

2. *Higher order problems.* Our techniques can also be applied to mixed problems for higher order systems provided the appropriate $\mathcal{L}_2$ estimates are known. In particular, sharp differentiability theorems can be obtained for the mixed problems of Sakamoto [10]. In these problems, as with Kreiss's problems, the boundary conditions are inhomogeneous and the boundary values of the solution can be estimated. For first order systems one starts with square integrability as a continuable initial condition and proceeds to prove that "satisfies compatibility conditions up to order s" is continuable. For Sakamoto's problems, on the other hand, the differentiability theorems yield the first continuable conditions, i.e. satisfies compatibility conditions up to order s with $s + 1$ greater than or equal to the order of the equation.

3. *Assumptions on A_j, B, M and Ω.* The assumption that the A_j, B and M are constant outside a compact set can be removed by one of two devices. One can work locally relying on finite speed of propagation and get differentiability theorems in the spaces H^s_{loc}. Alternatively, one can assume that the A_j, B and M and their derivatives up to a certain order are uniformly bounded on $R \times \overline{\Omega}$ and $R \times \partial\Omega$ and then carry out the proof as before.

There are many situations in which $\partial\Omega$ is not compact but the methods of §3 still work, for example if Ω is a half space or a cylinder $\mathcal{O} \times R^m$ with $\mathcal{O}$ compact. More generally, the results extend to the case where $\overline{\Omega}$ can be covered by a finite number of coordinate patches and the patches do not grow narrow as one approaches infinity. The situation is further complicated if the A_j are not constant outside a compact set. In that case one must also assume that $\partial\Omega$ is uniformly noncharacteristic in the sense that, on the patches which meet $\partial\Omega$, A_n is invertible with $\|A_n\|$ and $\|A_n^{-1}\|$ bounded. Of course one can work locally avoiding these global problems.

Theorem 3.1 is true as stated if the A_j, B and M are only defined for $0 \leq t \leq T$. The proof proceeds as before except for Lemma 3.4 which cannot be reduced to the results of Tartakoff but must be proven directly.

5. **Application to distribution initial data.** In this section we discuss the solution of the mixed problem

$$(5.1) \qquad Lu = 0 \text{ in } [0,T] \times \Omega, \qquad u \in N \text{ in } [0,T] \times \partial\Omega, \qquad u(0) = \Phi$$

with distribution initial data Φ. In addition to the assumptions already in force we suppose that the mixed problems L, N and L^*, N^* (the latter with time running backward) have finite speed of propagation.

Definition 5.1. The mixed problem (5.1) is said to have finite speed of propagation if there is a number σ such that, for $\Phi \in \mathcal{L}_2(\Omega)$, the solution u of (5.1) satisfies

$$\operatorname{supp} u \subset \{(t,x) \in [0,T] \times \overline{\Omega}\colon (\exists\ \tilde{x} \in \operatorname{supp} \Phi)\, |x - \tilde{x}| \leq \sigma t\}.$$

It is tempting to prescribe $\Phi \in \mathcal{D}'(\Omega)$ and to construct a solution by choosing $\Phi_n \in \mathcal{D}(\Omega)$ with $\Phi_n \to \Phi$ in $\mathcal{D}'(\Omega)$ hoping that the solutions with data Φ_n will converge in a weak sense. Unfortunately, they may not, or they may converge to the "wrong" solution.

Example. Let $\Omega = \{x \in \mathbf{R}\colon x > 0\}$, $L = \partial_t + \partial_x$, $N = \{0\} \subset \mathbf{C}^1$. As in the example in §1, the solution with $u(0) = \Phi \in \mathcal{D}(\Omega)$ is given by $u(t) = \tau_t \Phi$ where τ_t is translation to the right by t units. If Φ is the distribution defined by the function $e^{1/x}$ then the above limiting procedure will lead to a "solution" whose value at $t = 1$ is a distribution Γ with $\Gamma = 0$ on $(0,1)$ and $\Gamma = e^{1/(x-1)}$ on $(1,\infty)$. No such distribution exists. For the initial data $\Phi = 0$ the correct solution is $u \equiv 0$. However for $\Phi \in \mathcal{D}(\Omega)$ with $\int \Phi = 1$, $\varepsilon^{-1}\Phi(x/\varepsilon) \to 0$ in $\mathcal{D}'(\Omega)$ while the corresponding solutions $u_\varepsilon(t)$ have $\lim_{\varepsilon\to 0} u_\varepsilon(t) = \delta_t$ (the unit mass at $x = t$). This "solution" is the correct one for $u(0) = \delta_0$ which is an extension of the zero distribution to $\overline{\Omega}$.

The source of difficulty in the above examples is that it is not enough to specify the behavior of Φ in the interior of Ω. Some information must be known about its boundary behavior.

We will use the following notation. For a topological vector space E and its dual E', $\langle\,,\rangle$ denotes the natural pairing $E' \times E \to \mathbf{C}$. If E or E' is a set of functions on V we sometimes write $\langle\,,\rangle_V$ to emphasize this fact. Smooth functions yield linear functionals via the $\mathcal{L}_2$ scalar product. For example, to $\psi \in C^\infty_{(0)}(V)$ we associate the distribution $\mathcal{D}(V) \ni \phi \mapsto (\phi,\psi)_V$. For $F \in \mathcal{D}((0,T) \times \Omega)$ let v_F be the solution of $L^* v_F = F$ in $[0,T] \times \Omega$, $v_F(T) = 0$, and $v_F \in N^*$ on $[0,T] \times \partial\Omega$.

If u is a classical solution of (5.1) then $(u,F)_{[0,T]\times\Omega} = (u(0), v_F(0))_\Omega$. Therefore, it is reasonable to say that $u \in \mathcal{D}'((0,T) \times \Omega)$ is a solution of (5.1) if

(5.2) $\langle u, F\rangle_{(0,T)\times\Omega} = \langle \Phi, v_F(0)\rangle$ for all $F \in \mathcal{D}((0,T)\times\Omega)$.

The difficulty here is that $v_F(0)$ may not be a test function, in which case the right-hand side of (5.2) may not be meaningful for every $\Phi \in \mathcal{D}'(\Omega)$. However, $v_F(0)$ is severely restricted at $\partial\Omega$ for it satisfies all the compatibility conditions associated with the adjoint problem. The desired restriction on $u(0) = \Phi$ at $\partial\Omega$ is that Φ act continuously on functions satisfying these compatibility conditions.

In order to make this precise we make the following definitions. For $f \in H^s(\Omega)$ let $f_0^* = f$ and

$$f_p^* = \sum_{i=0}^{p-1} \binom{p-1}{i} G_i^*(t) f_{p-1-i}^*$$

where $G_0^* = G^*$ and $G_i^* = [-\partial_t, G_{i-1}^*]$. We say that f satisfies the adjoint compatibility conditions up to order s at time t if (compare (1.5))

$$\sum_{i=0}^{p} \binom{p}{i} (\partial_t^i M^*)(t) f_{p-i}^* = 0 \quad \text{on } \partial\Omega,\ 0 \le p \le s-1. \tag{5.3}$$

Let $\mathcal{K} = \{\phi \in C_{(0)}^\infty(\Omega)\colon \phi$ satisfies (5.3) for $t = 0$ and all $s\}$, and let $\mathcal{K}_N = \{\phi \in \mathcal{K}\colon \phi = 0$ for $|x| \ge N\}$. $\mathcal{K}_N$ is naturally a Fréchet space (it is a closed subspace of $C^\infty(\overline{\Omega})$), and $\mathcal{K} = \bigcup_{N=1}^\infty \mathcal{K}_N$ is given the (strict) inductive limit topology. $\mathcal{K}'$, the dual of $\mathcal{K}$, is equipped with the weak star topology so $(\mathcal{K}')' = \mathcal{K}$. In this topology a net S_ν converges to S in $\mathcal{K}'$ if and only if $\langle S_\nu, \phi\rangle \to \langle S, \phi\rangle$ for all $\phi \in \mathcal{K}$.

For $\psi \in C_0^\infty(\Omega)$ we associate the element of $\mathcal{K}'$ given by $\mathcal{K} \ni \phi \mapsto (\phi, \psi)_\Omega$. Thus $C_0^\infty(\Omega)$ is imbedded in $\mathcal{K}'$.

Lemma. *$C_0^\infty(\Omega)$ is dense in $\mathcal{K}'$.*

Proof. Suppose $\phi \in (\mathcal{K}')'$ annihilates $C_0^\infty(\Omega)$. Then $\phi \in \mathcal{K}$ and $\phi(\psi) = \langle \psi, \phi\rangle$ for $\psi \in \mathcal{K}'$. Thus $0 = \langle \psi, \phi\rangle = (\phi, \psi)_\Omega$ for all $\psi \in C_0^\infty(\Omega)$. The fundamental lemma of the calculus of variations yields $\phi = 0$. □

Definition. For $\Phi \in \mathcal{K}'$ a distribution $u \in \mathcal{D}'((0,T)\times\Omega)$ is a solution of (5.1) if (5.2) holds.

Theorem 5.1. *For any $\Phi \in \mathcal{K}'$ there is a unique solution of* (5.1). *In addition, the map $\Phi \to u$ is a continuous map of $\mathcal{K}'$ into $\mathcal{D}'((0,T)\times\Omega)$ (both spaces with weak star topology).*

Proof. *Uniqueness.* If u_1 and u_2 are solutions then $0 = \langle u_1 - u_2, F\rangle_{(0,T)\times\Omega}$ for all test functions F so $u_1 = u_2$.

Existence. The map $F \overset{u}{\mapsto} \langle v_F(0), \Phi\rangle_\Omega$ defines a linear functional u which satisfies (5.2). All we need to show is the continuity of u. If $F_n \to 0$ in $\mathcal{D}((0,T)\times\Omega)$ then $v_{F_n}(0) \to 0$ in $\mathcal{K}$ by our differentiability results and finiteness of propagation speed for L^*, N^*. Therefore, $u(F_n) = \langle v_{F_n}(0), \Phi\rangle \to 0$ showing that u is a distribution.

Continuous dependence. Suppose Φ_ν is a net converging to zero in $\mathcal{K}'$ and u_ν are the corresponding solutions. For any $F \in \mathcal{D}((0,T)\times\Omega)$, $\langle u_\nu, F\rangle = \langle v_F(0), \Phi_\nu\rangle \to 0$ so $u_\nu \to 0$ in $\mathcal{D}'((0,T)\times\Omega)$. □

Since the solution u is viewed as a state evolving in time, regularity results asserting continuity in time are expected. In fact, we will show (Corollary 5.3) that u is continuous on $[0,T]$ with values in $\mathcal{D}'(\Omega)$. First we give precise regularity results when Φ is restricted to lie in certain Hilbert spaces defined by "negative norms."

For s a nonnegative integer let $K^s(t) = \{f \in H^s(\Omega): f \text{ satisfies } (5.3)\}$. $K^s(t)$ is a closed linear subspace of $H^s(\Omega)$ so it is a Hilbert space with norm $\| \|_{s,\Omega}$. Let $K^{-s}(t) = (K^s(t))'$ be the dual of $K^s(t)$. For $\phi \in C^\infty_{(0)}(\Omega)$ we associate the functional $K^s(t) \ni \chi \mapsto (\chi,\phi)_\Omega$. By this correspondence $\mathcal{D}(\Omega)$ is densely imbedded in $K^{-s}(t)$. Define $C_{[0,T]}(K^{-s}(t))$ to be the completion of $C^\infty_{(0)}([0,T]\times\overline{\Omega})$ in the norm $\sup_{0\le t\le T}\|\psi(t)\|_{K^{-s}(t)}$. We think of $C_{[0,T]}(K^{-s}(t))$ as continuous functions of t whose value at time t is in $K^{-s}(t)$ for $0 \le t \le T$. Since the norm in $C_{[0,T]}(K^{-s}(t))$ dominates that of $C([0,T]; H^{-s}(\Omega))$ we conclude that if $u_n \to u$ in $C_{[0,T]}(K^{-s}(t))$ then $u_n \to u$ in $\mathcal{D}'((0,T)\times\Omega)$.

Theorem 5.2. *For $\Phi \in K^{-s}(0)$ the solution u of* (5.1) *is in $C_{[0,T]}(K^{-s}(t))$ and*

$$\sup_{0\le t\le T}\|u(t)\|_{K^{-s}(t)} \le c\|\Phi\|_{K^{-s}(0)} \tag{5.4}$$

with c independent of Φ.

Proof. First we prove inequality (5.4) for the solutions of (5.1) with $u(0) \in \mathcal{D}(\Omega)$. For $\Psi \in K^s(t)$ let v be the solution of

$$L^*v = 0 \text{ in } [0,t]\times\Omega, \qquad v \in N^* \text{ in } [0,t]\times\partial\Omega, \qquad v(t) = \Psi.$$

Then $(u(t),v(t))_\Omega = (u(0),v(0))_\Omega$ so $|(u(t),\Psi)_\Omega| \le \|v(0)\|_{s,\Omega}\|u(0)\|_{K^{-s}(0)}$. Theorem 3.1 shows $\|v(0)\|_{s,\Omega} \le c\|v(t)\|_{s,\Omega}$, so

$$|(u(t),\Psi)_\Omega| \le c\|u(0)\|_{K^{-s}(0)}\|\Psi\|_{s,\Omega}$$

which proves that $\|u(t)\|_{K^{-s}(t)} \le c\|u(0)\|_{K^{-s}(0)}$. The proof is completed by choosing $\phi_n \in \mathcal{D}(\Omega)$ with $\phi_n \to \Phi \in K^{-s}(0)$ (in particular $\phi_n \to \Phi$ in $\mathcal{K}'$) and letting u_n be the solution of (5.1) with data ϕ_n. Then u_n is Cauchy in $C_{[0,T]}(K^{-s}(t))$ and $u_n \to u$ in $\mathcal{D}'((0,T)\times\Omega)$ so $u_n \to u$ in $C_{[0,T]}(K^{-s}(t))$. □

Definition. $u \in \mathcal{D}'((0,T)\times\Omega)$ is a *continuous function on* $[0,T]$ *with values in* $\mathcal{D}'(\Omega)$ if for all $\psi \in \mathcal{D}(\Omega)$ the distribution $[u,\psi] \in \mathcal{D}'((0,T))$ defined by $\mathcal{D}((0,T)) \ni \phi \mapsto \langle u, \phi\psi\rangle_{(0,T)\times\Omega}$ is a continuous function on $[0,T]$.

Corollary 5.3. *The solution u of Theorem* 5.1 *is a continuous function on* $[0,T]$ *with values in* $\mathcal{D}'(\Omega)$.

Proof. For $\psi \in \mathcal{D}(\Omega)$ choose R with $\operatorname{supp}\psi \subset \{x \in \Omega: |x| < R\} \equiv \Omega_R$ and let N be an integer such that $N > R + \sigma T$ with σ as in Definition 5.1. The norms

$\| \|_{s,\Omega}$ define the topology on $\mathcal{K}_N$ (Sobolev lemma), so for $\Phi \in \mathcal{K}'$ there is an integer $s \geq 0$ with $|\langle \Phi, \phi \rangle_\Omega| < c\|\phi\|_{s,\Omega}$ for all $\phi \in \mathcal{K}_N$. The Hahn-Banach theorem shows that there is $\tilde{\Phi} \in K^{-s}(0)$ with $\tilde{\Phi} = \Phi$ on $\mathcal{K}_N$. If $\tilde{u}$ is a solution of (5.1) with initial data $\tilde{\Phi}$, we have $\tilde{u} = u$ on $(0, T) \times \Omega_R$ (finite speed for L, N). Therefore

$$\langle u, \phi\psi \rangle_{(0,T)\times\Omega} = \langle \tilde{u}, \phi\psi \rangle_{(0,T)\times\Omega} = \int_0^T \phi(t) \langle \tilde{u}(t), \psi \rangle_\Omega \, dt,$$

so $[u, \psi]$ is the continuous function $\langle \tilde{u}(t), \psi \rangle_\Omega$. □

The above argument actually shows that u is C^∞ with values in $\mathcal{D}'(\Omega)$. Also if G and N do not depend on time it proves that u is C^∞ with values in $\mathcal{K}'$.

6. **Applications to quasilinear mixed problems.** We sketch the proof of a local (in time) existence theorem for quasilinear mixed problems of the form

$$\begin{aligned} \partial_t u &= \sum A_j(t,x,u)\partial_j u + B(t,x,u)u + F, \\ M(t,x,u)u &= 0 \quad \text{for } x \in \partial\Omega, \\ u(0,x) &= f(x). \end{aligned} \tag{6.1}$$

It is well known that classical solutions may not exist for all time. As in the linear case the arguments may be extended to cover inhomogeneous boundary conditions and higher order quasilinear equations.

It is assumed that if u is a smooth function with $|u(t,x) - f(x)|$ sufficiently small then the linear problem with coefficients $A_j(t,x,u(t,x))$, $B(t,x,u(t,x))$, $M(t,x,u(t,x))$ satisfies inequalities (1.2), (1.3) for $0 \leq t \leq T$. The solution to (6.1) is obtained as the limit of functions u_n where $u_0(t,x) = f(x)$ and

$$\begin{aligned} \partial_t u_n &= \sum A_j(t,x,u_{n-1})\partial_j u_n + B(t,x,u_{n-1})u_n + F, \\ M(t,x,u_{n-1})u_n &= 0 \quad \text{for } x \in \partial\Omega, \\ u_n(0,x) &= f(x). \end{aligned} \tag{6.2}$$

To carry out this iteration scheme it is necessary for u_{n-1} to be smooth for all n and for $|u_{n-1}(t,x) - f(x)|$ to be small so that the linear problem (6.2) is well set. The smoothness of u_{n-1} is assured if certain compatibility conditions are satisfied and $\|u_{n-1}(t) - f\|$ will be small provided we seek solutions in a thin strip $0 \leq t \leq \delta$.

A remark about the compatibility conditions is needed. Definition 1.4 of the f_p and the compatibility conditions (1.5) must be modified to take into account the fact that A_j, B, and M depend on u. When this is done the f_p are determined by F and the x derivatives of f, A_j, and B at $t = 0$. In problem (6.2) these values do not depend on n since $u_n(0) = f$ is independent of n. Thus, $\sum (t^p/p!)f_p$ is the

Taylor series in time of u_n for all n and also the Taylor series for the solution u of (6.1) (assuming it exists). It follows that the compatibility conditions for the problem (6.1) which assert the vanishing of the first Taylor coefficients of $M(t,x,u(t,x))u$ and for (6.2) which assert the vanishing of Taylor coefficients of $M(t,x,u_{n-1}(t,x))u_n$ are the same conditions. Consequently, if we assume that these compatibility conditions on f, F hold up to order s then for each n compatibility conditions up to order s will be satisfied for (6.2). If s is sufficiently large this assures the smoothness of u_n. A complete proof that the above scheme leads to a solution of (6.1) will be given at a later date.

7. Remarks on the necessity of the compatibility conditions. The necessity part of Theorem 3.1 may be strengthened somewhat.

Theorem 7.1. *Let* $f \in \mathcal{L}_2(\Omega)$, $F \in H^s([0,T] \times \Omega)$ *and suppose the solution u to* (2.2) *belongs to* $C((0,T]; H^s(\Omega))$. *Then either* $f \in H^s(\Omega)$ *and the compatibility conditions* (1.5) *hold* (*with* $g = 0$) *or* $\|u(t)\|_{s,\Omega} \to \infty$ *as* $t \to 0$.

Remark 7.1. In the proof we shall need the following fact. Let X and Y be Banach spaces, $\{u_k\}$ be a sequence in X which converges weakly to u, and $\{T_k\}$ be a sequence in $B(X,Y)$ which converges to T in the norm of $B(X,Y)$. Then $T_k u_k$ converges to Tu weakly in Y.

Proof of Theorem 7.1. Arguing as in the introduction, we see that the equation $\partial_t u = Gu + F$ implies $\partial_t^p u \in C((0,T]; H^{s-p}(\Omega))$ for $0 \le p \le s$ and we may write $\partial_t^p u(t) = B_p(t)u(t) + E_p(t)F$, $0 < t \le T$, where $B_p(t)$ and $E_p(t)$, respectively, are bounded operators from $H^s(\Omega)$ and $H^s([0,T] \times \Omega)$ into $H^{s-p}(\Omega)$ and the maps $t \to B_p(t)$ and $t \to E_p(t)$ are continuous on $[0,T]$ to $B(H^s(\Omega), H^{s-p}(\Omega))$ and $B(H^s([0,T] \times \Omega), H^{s-p}(\Omega))$, respectively. Differentiating the boundary conditions, $Mu = 0$, p times with respect to t, one obtains a relation of the form $C_p(t)u(t) = K_p(t)F$, $0 < t \le T$, $0 \le p \le s-1$, where $C_p(t)$ and $K_p(t)$, respectively, are bounded operators from $H^s(\Omega)$ and $H^s([0,T] \times \Omega)$ into $H^{s-p-1}(\partial\Omega)$ and the maps $t \to C_p(t)$ and $t \to K_p(t)$ are continuous on $[0,T]$ to $B(H^s(\Omega), H^{s-p-1}(\partial\Omega))$ and $B(H^s([0,T] \times \Omega), H^{s-p-1}(\partial\Omega))$, respectively. For $f \in H^s(\Omega)$ the compatibility conditions (1.5) (with $g = 0$) can be written as

$$C_p(0)f = K_p(0)F, \qquad 0 \le p \le s-1. \tag{7.1}$$

Suppose $\liminf_{t \to 0} \|u(t)\|_{s,\Omega} < \infty$. Then there is a sequence $t_k \to 0$ such that $u_k \equiv u(t_k)$ converges weakly in $H^s(\Omega)$. By Proposition 2.1, u_k converges to f in $\mathcal{L}_2(\Omega)$. So $f \in H^s(\Omega)$ and u_k converges to f weakly in $H^s(\Omega)$. According to Remark 7.1, $C_p(t_k)u_k$ converges to $C_p(0)f$ weakly in $H^{s-p-1}(\partial\Omega)$. Since $K_p(t_k)F \to K_p(0)F$ in $H^{s-p-1}(\partial\Omega)$, we conclude that (7.1) holds. □

Example. Let $\Omega = [0,\infty)$, $\phi \in C_{(0)}^{\infty}(\Omega)$ and $\phi(0) \neq 0$. Then $f(x) = x^{1/2}\phi(x)$ belongs to $\mathcal{L}_2(\Omega)$ but not to $H^1(\Omega)$. The solution to the mixed problem $\partial_t u - \partial_x u = 0$, $t \ge 0$, $x \ge 0$, u free for $x = 0$, $u(0,x) = f(x)$, is $u(t,x)$

$= f(t+x)$. This shows that it is possible for the solution u of (2.2) to belong to $C((0,T];H^1(\Omega))$ even when $f \notin H^1(\Omega)$.

Theorem 7.1 can be strengthened if the mixed problem is *reversible.* This means that inequalities of that form (1.2) and (1.3) hold not only for L, but also with L replaced by $-\partial_t - G$. This implies that $u(t_1)$ depends continuously on $u(t_0)$ for $t_1 < t_0$ as well as $t_1 > t_0$, and it is possible to solve these mixed problems with time running backward with Theorem 3.1 still holding. For simplicity we state the following theorem with $s = 1$ and $F = 0$, but it may be extended to $s > 1$ and $F \neq 0$.

Theorem 7.2. *Suppose the mixed problem* (2.2) *is reversible and* $f \in \mathcal{L}_2(\Omega)$. *Suppose the solution to* (2.2) *with* $F = 0$ *has the property that for some* $t_0 \in [0,T]$ *one has* $u(t_0) \in H^1(\Omega)$ *and* $M(t_0)u(t_0) = 0$. *Then* $f \in H^1(\Omega)$ *and* $M(0)f = 0$.

Proof. This u is also the solution to the mixed problem (with time running backward) $Lu' = 0$ on $[0,t_0] \times \Omega$, $Mu' = 0$ on $[0,t_0] \times \partial\Omega$, with "initial value" $u'(t_0) = u(t_0)$ prescribed at $t = t_0$. It follows from Theorem 3.1 that $u \in C([0,T];H^1(\Omega))$ and $M(t)u(t) = 0$ for $0 \leq t \leq t_0$. Taking $t = 0$, we obtain the conclusion of the theorem. □

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