APPROXIMATION OF ANALYTIC FUNCTIONS ON COMPACT SETS AND BERNSTEIN'S INEQUALITY

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ABSTRACT. The characterization of analytic functions defined on a compact set K in $\mathbf{R}_N$ by their polynomial approximation is possible if and only if K satisfies some "Bernstein type inequality", estimating any polynomial P in some neighborhood of K using the supremum of P on K. Some criterions and examples are given. Approximation by more general sets of analytic functions is also discussed.

0. Introduction. Let K be a compact set in $\mathbf{R}^N$. It is known [1] that the distance of any analytic function defined on K to the space of polynomials of degree n is exponentially decreasing as n becomes large.

We say that K satisfies (A) if, conversely, any continuous function, whose distance to the space of polynomials of degree n is exponentially decreasing, is necessarily analytic on K.

We discuss here the question: What are the compact sets in $\mathbf{R}^N$ which satisfy (A)?

S. Bernstein proved in 1912 [2] that the interval $[-1, +1]$ satisfies (A). (See also [5] and [6] for the one dimensional complex case.) He used the so-called Bernstein's inequality, estimating any polynomial P in the complex plane by means of the supremum of $|P|$ on $[-1, +1]$.

We prove here that (A) holds for K if and only if K satisfies a "Bernstein (or Markov) type inequality", which gives a precise estimate of any polynomial P on some neighborhood of K using the supremum of P on K.

This characterization yields to some positive criterions (in order that K satisfy (A)), and also some negative examples.

We show in particular that some compact sets, for which the characterization of C^∞ functions by their polynomial approximation is not possible, do satisfy (A). (See [7] for the C^∞ case.)

In part I, we discuss the approximation of analytic functions defined on compact sets by more general sets of analytic functions than polynomials. This may be used, for example, in the case of the approximation by the eigenvectors of some differential operators. We do not give here this application.

In part II, we restrict ourselves to the polynomial case and we give some criterions, positive and negative examples.

I. APPROXIMATION BY A SET OF ANALYTIC FUNCTIONS

We introduce first some notations and definitions.

Let K be a compact set in $\mathbf{R}^N$. We denote by $C(K)$ the space of complex valued continuous functions defined on K, equipped with the uniform convergence norm.

The real space $\mathbf{R}^N$ is embedded in the complex space $\mathbf{C}^N = \mathbf{R}^N + i\mathbf{R}^N$, as usual. We consider K as a compact set in $\mathbf{C}^N$.

Let Ω be an open set in $\mathbf{C}^N$ and $\mathcal{O}(\Omega)$ the Banach space of holomorphic and bounded functions defined in Ω, provided with the uniform convergence norm.

Let $\mathcal{O}(K)$ be the space of holomorphic germs defined on K. We equip $\mathcal{O}(K)$ with the inductive limit topology of $\mathcal{O}(\Omega)$ for all Ω, where Ω is an open neighborhood of K in $\mathbf{C}^N$. (We may restrict ourselves to the sets Ω such that any component of Ω intersects K.)

We denote by $\mathcal{N}_K(\Omega)$ the kernel of the natural restriction $r_\Omega \colon \mathcal{O}(\Omega) \to C(K)$, and by $\mathcal{A}_\Omega(K)$ its range provided with the quotient topology $\mathcal{O}(\Omega)/\mathcal{N}_K(\Omega)$.

The mappings r_Ω define the restriction r, $r \colon \mathcal{O}(K) \to C(K)$. The kernel of r is

$$(\text{I.1}) \qquad \mathcal{N}(K) = \operatorname*{ind\,lim}_{K \subset \Omega} \mathcal{N}_K(\Omega)$$

and its range is the space of analytic functions defined on K

$$(\text{I.2}) \qquad \mathcal{A}(K) = \operatorname*{ind\,lim}_{K \subset \Omega} \mathcal{A}_\Omega(K).$$

Let us observe that if $\mathcal{N}(K) = \mathcal{N}_\Omega(K) = 0$, the mapping r is injective and $\mathcal{A}(K)$ is isomorphic to $\mathcal{O}(K)$. This condition is satisfied if and only if no component of K is contained in an analytic set of $\mathbf{R}^N$.

Now, let us consider a sequence (H_n) of finite dimensional vector spaces satisfying

$$(\text{I.3}) \qquad H_0 \subset H_1 \subset \cdots \subset H_n \subset H_{n+1} \subset \cdots \subset \mathcal{O}(K).$$

Let (μ_n) be an increasing sequence of strictly positive real numbers such that

$$(\text{I.4}) \qquad \text{for any } a \in \,]0, 1[\,, \quad \sum_{n=0}^{\infty} a^{\mu_n} < +\infty.$$

We introduce the following properties related to the compact K and the sequences (H_n) and (μ_n).

Property (A): analytic approximation. *Any function $f \in C(K)$, for which there exists $C > 0$ and $a \in \,]0, 1[$ such that*[2] *for each $n \in \mathbf{N}$*

$$(\text{I.5}) \qquad d_K(f, r(H_n)) \leq C a^{\mu_n}$$

belongs to $\mathcal{A}(K)$.

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[2] If $f \in C(K)$ and F is a subspace of $C(K)$ we denote $d_K(f, F) = \inf_{g \in F} \sup_{x \in K} |f(x) - g(x)|$.

Property (B): Bernstein's inequality. *For any real number $b > 1$, there exists an open set Ω, neighborhood of K in $\mathbf{C}^N$, such that, for any $P \in H_n$, there exists $Q \in \mathcal{N}(K)$ satisfying*

$$(\text{I.6}) \qquad P + Q \in \mathcal{O}(\Omega) \quad and \quad \sup_{x \in \Omega}|P(x) + Q(x)| \leq b^{\mu_n} \sup_{x \in K}|P(x)|.$$

We have the following result:

Theorem (I.1). *Let K be a compact set in $\mathbf{R}^N$, and (H_n) and (μ_n) satisfy (I.3) and (I. 4). The properties (A) and (B) are equivalent.*

Proof. 1. (B) *implies* (A). We assume (B). Let $f \in C(K)$ satisfy (I.5). We shall prove that $f \in \mathcal{A}(K)$.

Let $P_n \in H_n$ such that $d_K(f, r(H_n)) = \sup_{x \in K}|f(x) - P_n(x)|$. Since we have

$$\sup_{x \in K}|P_n(x) - P_{n-1}(x)| \leq 2d_K(f, r(H_{n-1})) \leq 2Ca^{\mu_{n-1}},$$

and using (I.4), the series $P_0 + \sum_{n=1}^{\infty}(P_n - P_{n-1})$ is convergent in $C(K)$, its sum is f.

We choose $b > 1$ such that $ab < 1$, and let Q_n be the element of $\mathcal{N}(K)$ given by (B) and associated to $P_n - P_{n-1}$. The series $P_0 + Q_0 + \sum_{n=1}^{\infty}(P_n - P_{n-1} + Q_n)$ coincides with f on K and is convergent in $\mathcal{O}(\Omega)$, since by virtue of (B) we have

$$\sup_{x \in \Omega}|P_n(x) - P_{n-1}(x) + Q_n(x)| \leq 2Ca^{\mu_{n-1}}b^{\mu_{n-1}}.$$

We have shown that f is in $\mathcal{A}(K)$ and that property (A) holds.

2. (A) *implies* (B). First of all, it is easy to see that property (B) is equivalent to the following two properties:

(I.7) There exists an open set U, neighborhood of K in $\mathbf{C}^N$, such that for any $n \in \mathbf{N}$, $r(H_n) \subset \mathcal{A}_U(K)$ (i.e. $H_n \subset \mathcal{O}(U) + \mathcal{N}(K)$).

(I.8) For any real number $b > 1$, there exists an open set Ω, neighborhood of K in $\mathbf{C}^N$, $\Omega \subset U$ (U is given in (I.7)) such that for any $P \in H_n$

$$\|r(P)\|_{\mathcal{A}_\Omega(K)} \leq b^{\mu_n}\|r(P)\|_{C(K)}.$$

We prove first that (A) implies (I.7). For $a \in]0, 1[$, let us consider the space

$$E_a = \left\{ f \in C(K); \sup_n(a^{-\mu_n}d_K(f, r(H_n))) < +\infty \right\}.$$

It is a Banach space with the norm

$$\|f\|_{E_a} = \|f\|_{C(K)} + \sup_n(a^{-\mu_n}d_K(f, r(H_n))).$$

The space $E = \text{ind}\lim_{0 < a < 1} E_a$ is continuously embedded in $C(K)$.

We observe that property (A) is equivalent to the following one:

$$(\text{I.9}) \qquad\qquad\qquad E \subset \mathcal{A}(K).$$

It follows from the closed graph theorem and property (A) that the injection (I.9) is continuous. Using a well-known theorem about continuous linear mappings in inductive limits of Fréchet spaces (see [3] for example), we can state that for any $a \in]0, 1[$, there exists an open set U, neighborhood of K in $\mathbf{C}^N$, such that

$$(\text{I.10}) \qquad\qquad E_a \subset \mathscr{A}_U(K)$$

with continuous injection.

Property (I.7) follows readily from (I.10) since for any $n \in \mathbf{N}$ and any $a \in]0, 1[$ we have $r(H_n) \subset E_a$.

Now we have to show that (A) implies (I.8). We prove it by contradiction. We assume (A) (and therefore (I.7)) and we suppose that (I.8) is not true.

Let (Ω_k), $k \in \mathbf{N}$, be a fundamental system of open neighborhoods of K in $\mathbf{C}^N$, decreasing and contained in U (U being given in (I.7)). Since (I.8) is false, there exists $b > 1$ such that, for any $k \in \mathbf{N}$, there exists n_k and $P_{n_k} \in H_{n_k}$ satisfying

$$(\text{I.11}) \qquad\qquad \|r(P_{n_k})\|_{\mathscr{A}_{\Omega_k}(K)} > b^{\mu_{n_k}} \|P_{n_k}\|_{C(K)}.$$

Without loss of generality we may assume $\|P_{n_k}\|_{C(K)} = 1$ and the sequence n_k is strictly increasing. [Indeed, it is easy to see, since H_n is a finite dimensional space, that (I.8) would be true if we restrict it to a fixed H_{n_0}.]

For $a \in]0, 1[$, we denote

$$\mathscr{E}_a = \left\{ (c_n) \in \mathbf{C}^{\mathbf{N}}; \ \|c_n\|_a = \sup_n |c_n a^{-\mu_n}| < \infty \right\},$$

$$\mathscr{E} = \operatorname*{ind\,lim}_{0 < a < 1} \mathscr{E}_a,$$

provided with their natural topology. For $(c_n) \in \mathscr{E}_a$ we have

$$\left\| \sum_k c_{n_k} P_{n_k} \right\|_{C(K)} \leq \|(c_n)\|_a \sum_k a^{\mu_{n_k}}.$$

Therefore the linear mapping

$$(c_n) \mapsto \sum_k c_{n_k} P_{n_k} = \mathcal{U}((c_n))$$

is continuous from $\mathscr{E}$ into $C(K)$. Let us prove that its range is in $\mathscr{A}(K)$, which implies, using the closed graph theorem, that $\mathcal{U}$ is a continuous mapping from $\mathscr{E}$ into $\mathscr{A}(K)$.

Let $(c_n) \in \mathscr{E}_a$. In order to prove that $\mathcal{U}((c_n))$ is in $\mathscr{A}(K)$, it is sufficient to prove (using (A)) that there exist $a_1 \in]0, 1[$ and $C > 0$ such that

(I.12) For any $l \in \mathbf{N}$, $d_K(\mathcal{U}((c_n)), r(H_l)) \leq C a_1^{\mu_l}$.

Indeed, we have

$$d_K(\mathcal{U}((c_n)), r(H_{n_j})) \leq \left\| \sum_{k=j+1}^{\infty} c_{n_k} P_{n_k} \right\| \leq \|(c_n)\|_a \sum_{k=j+1}^{\infty} a^{\mu_{n_k}}.$$

If $a < a_1 < 1$, we obtain

$$d_K(\mathcal{U}((c_n)), r(H_{n_j})) \leq Ca_1^{\mu_{n_j+1}} \leq Ca_1^{\mu_{n_j}}$$

with $C = \|(c_n)\|_a \cdot \sum_n (a/a_1)^{\mu_n}$. On the other hand, we have for $n_j < q < n_{j+1}$

$$d_K(\mathcal{U}((c_n)), r(H_q)) \leq d_K(\mathcal{U}((c_n)), r(H_{n_j}))$$
$$\leq Ca_1^{\mu_{n_j+1}} \leq Ca_1^{\mu_q},$$

which ends the proof of (I.12).

We are going to prove now that the continuity of $\mathcal{U}$ from $\mathcal{E}$ into $\mathcal{A}(K)$ implies a contradiction with (I.11).

Using a well-known theorem (see the first part of the present proof), we can say that for any $a \in \,]0, 1[$, there exist $p \in \mathbf{N}$ and $L > 0$ such that, for each $n \in \mathbf{N}$,

$$(I.13) \qquad\qquad \|\mathcal{U}((c_n))\|_{\mathcal{A}_{\Omega_p}(K)} \leq L\|(c_n)\|_a.$$

We choose a such that $ab > 1$, and k large enough in order to get

$$(I.14) \qquad\qquad L \leq (ab)^{\mu_{n_k}} \quad \text{and} \quad \Omega_k \subset \Omega_p$$

(L and Ω_p being given in (I.13) with the choice we have done for a).

Let us put, in (I.13), $c_n = 0$ for $n \neq n_k$ and $c_{n_k} = 1$; we obtain

$$\|r(P_{n_k})\|_{\mathcal{A}_{\Omega_p}(K)} \leq La^{-\mu_{n_k}},$$

and using (I.14)

$$\|r(P_{n_k})\|_{\mathcal{A}_{\Omega_k}(K)} \leq \|r(P_{n_k})\|_{\mathcal{A}_{\Omega_p}(K)} \leq La^{-\mu_{n_k}} \leq b^{\mu_{n_k}}.$$

This gives us a contradiction with (I.11) and ends the proof of Theorem (I.1).

We introduce now another property :

Property (M): Markov's inequality. *For any $b > 1$ there exists $C > 0$ such that for any $P \in H_n$ there exists $Q \in \mathcal{N}(K)$ satisfying for each $\alpha \in \mathbf{N}^N$*(3)

$$(I.15) \qquad\qquad \sup_{x \in K}|D^\alpha(P + Q)(x)| < C^{|\alpha|}(\alpha!)b^{\mu_n} \sup_{x \in K}|P(x)|.$$

We have the following result:

Theorem (I.2). *Let K be a compact set in $\mathbf{R}^N$ and (H_n) and (μ_n) satisfy (I.3) and (I.4). The properties (A), (B) and (M) are equivalent.*

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(3) If $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{N}^N$, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = (\alpha_1!) \cdots (\alpha_N!)$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_N)^{\alpha_N}$.

By the virtue of Theorem (I.1), it is sufficient to show that (M) and (B) are equivalent. We leave this proof to the reader; we just observe that "(B) implies (M)" follows from the Cauchy integral formula, and "(M) implies (B)" uses a suitable Taylor expansion.

Let us point out that if $\mathcal{N}(K) = 0$ (i.e. no component of K is contained in an analytic set in $\mathbf{R}^N$), the properties (B) and (M) may be simplified since we have $Q = 0$ in (I.6) and (I.15). This case deserves more attention.

K, (H_n) and (μ_n) being given we have:

Theorem (I.3). *Let K satisfy $\mathcal{N}(K) = 0$. Property* (B) *holds for K if and only if, for any $x \in K$ there exists a compact set L_x in $\mathbf{R}^N$ satisfying*
(I.16) $x \in L_x$, $L_x \subset K$, $\mathcal{N}(L_x) = 0$ *and* (B) *holds for L_x.*

Proof. It is clear that (B) (for K) implies the property asserted in the theorem. Let us prove the converse.

The assumption is that for any $x \in K$, there exists L_x which satisfies (I.16). Therefore, for $b > 1$, there exists Ω_x, open set in $\mathbf{C}^N$, neighborhood of L_x, such that for any $P \in H_n$

$$\sup_{y \in \Omega_x} |P(y)| \leq b^{\mu_n} \sup_{y \in L_x} |P(y)|.$$

Let us denote $\Omega = \bigcup_{x \in K} \Omega_x$. Ω is a neighborhood of K in $\mathbf{C}^N$, and we have for any $P \in H_n$

$$\sup_{y \in \Omega} |P(y)| \leq b^{\mu_n} \sup_{y \in K} |P(y)|,$$

which shows that (B) holds for K.

Remark (I.1). Without the assumption $\mathcal{N}(L_x) = 0$, the preceding theorem would be false (see Example (II.2) below).

In property (B), is it possible to consider only the open set Ω, neighborhood of K in $\mathbf{R}^N$, instead of $\mathbf{C}^N$? Let us first introduce

Property (B'): Bernstein's inequality in the real domain. *For any real number $b > 1$, there exists an open set V, neighborhood of K in $\mathbf{R}^N$, such that any $P \in H_n$ is analytic in V and satisfies*

(I.17)
$$\sup_{x \in V} |P(x)| \leq b^{\mu_n} \sup_{x \in K} |P(x)|.$$

Now we have

Theorem (I.4). *Let K satisfy $\mathcal{N}(K) = 0$, and for any point x in K, there exists a fundamental system of compact neighborhoods of x in $\mathbf{R}^N$ which satisfy* (B). *Then the properties* (B) *and* (B') *are equivalent.*

Proof. Since $\mathcal{N}(K) = 0$, it is obvious that (B) implies (B'). We shall prove the inverse implication.

Let us assume (B'). We must show that for any $b > 1$, there exists Ω, open neighborhood of K in $\mathbf{C}^N$, such that any $P \in H_n$ is holomorphic in Ω and

$$(\text{I.18}) \qquad \sup_{x \in \Omega} |P(x)| \leq b^{\mu_n} \sup_{x \in K} |P(x)|.$$

Let V be the open set in $\mathbf{R}^N$ associated to K by (B') with $b^{1/2}$ (instead of b). For any point $x \in K$, there exists a compact neighborhood L_x contained in V which satisfies (B). Therefore there exists Ω_x, open neighborhood of L_x in $\mathbf{C}^N$, such that any $P \in H_n$ is holomorphic in Ω_x and

$$(\text{I.19}) \qquad \sup_{y \in \Omega_x} |P(y)| \leq b^{\mu_n/2} \sup_{y \in L_x} |P(y)|.$$

(I.19) together with

$$\sup_{y \in L_x} |P(y)| \leq \sup_{y \in V} |P(y)| \leq b^{\mu_n/2} \sup_{y \in K} |P(y)|$$

gives us (I.18) with $\Omega = \bigcup_{x \in K} \Omega_x$, and ends the proof of Theorem (I.4).

II. POLYNOMIAL APPROXIMATION OF ANALYTIC FUNCTIONS

1. Application of part I and examples. Let $\mathscr{P}_n$ be the space of polynomials with complex coefficients in N variables and with total degree $\leq n$.

We choose henceforth $H_n = \mathscr{P}_n$ and $\mu_n = n$.

Let us recall the following result (see [1]):

Let K be a compact set in $\mathbf{R}^N$ and $f \in \mathscr{A}(K)$; there exists an $a \in \,]0, 1[$ and $C > 0$ such that, for any $n \in \mathbf{N}$,

$$(\text{II.1}) \qquad d_k(f; \mathscr{P}_n) \leq Ca^n.$$

Hence, *property* (A), in this case ($\mathscr{P}_n = H_n$ and $\mu_n = n$) may be formulated in the following way:

In order that a continuous function defined on K be in $\mathscr{A}(K)$, it is necessary and sufficient that it be possible to find an $a \in \,]0, 1[$ and $C > 0$ such that, for any $n \in \mathbf{N}$, $d_k(f, \mathscr{P}_n) \leq Ca^n$.

We also recall the following Bernstein result (see [2] and also [1]).

(II.2) *Any closed, bounded parallelepiped in $\mathbf{R}^N$ whose interior is not empty satisfies* (A).

We must point out that there exist some compact sets in $\mathbf{R}^N$ which do not satisfy (A). Here are two examples.

Example (II.1). *For any strictly decreasing sequence (x_n) of positive real numbers convergent to 0 the compact $K = \{x_1, \ldots, x_n, \ldots\} \cup \{0\}$ does not satisfy* (A) *in* $\mathbf{R}$.

Proof. We use Theorem (I.1) and we show that K does not satisfy (B). We prove this fact by contradiction.

Assume that (B) holds for K. Without loss of generality we may suppose $x_1 \leq 1$. Let $b > 1$. Since (B) is true, there exists $j \in \mathbf{N}$ such that, for any $P \in \mathscr{P}_n$

(II.3)
$$\sup_{x \in [0,(x_j + x_{j-1})/2]} |P(x)| \leq b^n \sup_{x \in K} |P(x)| \,(^4).$$

Let us set

$$P_n(x) = (x - x_1) \cdots (x - x_n), \quad a = (x_{j-1} - x_j)/2.$$

We have, for each $n \in \mathbf{N}$,

$$\sup_{x \in [0,(x_j + x_{j-1})/2]} |P_n(x)| \geq |P_n((x_j + x_{j-1})/2)| \geq a^n,$$

$$\sup_{x \in K} |P_n(x)| = x_1 \cdots x_n.$$

Finally, we get from (II.3)

$$a^n \leq b^n(x_1 \cdots x_n) \quad \text{or} \quad d^n \leq x_1 \cdots x_n$$

with $d = a/b$, which is impossible since (x_n) is strictly decreasing, convergent to 0 and $d \in]0, 1[$.

Example (II.2). *In $\mathbf{R}^2$, we denote*

$$\Pi = \{(x_1, x_2), -1 \leq x_1 \leq +1, -1 \leq x_2 \leq +1\},$$

$$S = \{(x_1, x_2), 1 \leq x_1 \leq 2, x_2 = 0\},$$

$$K = \Pi \cup S.$$

Property (A) *does not hold for K in $\mathbf{R}^2$.*

Proof. Let f be defined on K by

$$f(x) = x_2/(x_1 - 2), \quad \text{for } x \in \Pi,$$

$$= 0, \quad \text{for } x \in S.$$

It is obvious that $f \in C(K)$ and $f \notin \mathcal{A}(K)$. However we can show that f satisfies (II.1) with suitable $C > 0$ and $a \in]0, 1[$. Indeed we have

$$d_K(f, \mathcal{P}_n) \leq d_\Pi(f, y\mathcal{P}_{n-1}) \leq d_\Pi(1/(x - 2), \mathcal{P}_{n-1}).$$

The rest of the proof is easy.

Remark (II.1). It is false to say that the union or the intersection of two compact sets in $\mathbf{R}^N$ satisfying (A) must satisfy (A). This can be shown using the preceding example.

(^{4}) In fact, there exists Ω, neighborhood of K in $\mathbf{C}$, such that, for any $P \in \mathcal{P}_n$ there exists $Q \in \mathcal{N}_K(\Omega)$ satisfying $\sup_{x \in \Omega} |P(x) + Q(x)| < b^n \sup_{x \in K} |P(x)|$. Since Q vanishes on K, and is analytic in Ω, Q must vanish on the component of 0 in Ω and hence vanishes on $[0, (x_j + x_{j-1})/2]$ for some $j \in \mathbf{N}$.

We give now some examples of compact sets in $\mathbf{R}^N$ for which property (A) holds.

The following case is a simple application of Theorem (I.3) together with (II.2).

Example (II.3). *Let K be a compact set in $\mathbf{R}^N$ such that, for any $x \in K$, there exists a closed parallelepiped Π_x with nonempty interior satisfying $x \in \Pi_x$ and $\Pi_x \subset K$. Then property (A) holds for K.*

Here is another example.

Example (II.4). *Let K be the compact set given in example (II.1). We define $K_1 = K \cup [-1, 0]$. The compact set K_1 satisfies (A) in $\mathbf{R}$.*

Proof. We prove, actually, that K_1 satisfies (B). Let $b > 1$. There exists Ω_1 open set in C such that for any $P \in \mathscr{P}_n$

$$\sup_{x \in \Omega_1} |P(x)| \leq b^n \sup_{x \in [-1,0]} |P(x)|.$$

Let $x_1, \ldots, x_{n_0}$ be the elements of the sequence (x_n) which are not contained in Ω. We choose $B_1, \ldots, B_{n_0}$, open balls in $\mathbf{C}$, pairwise disjoint and centered at $x_1, \ldots, x_{n_0}$ respectively. We set $\Omega = \Omega_1 \cup B_1 \cup \cdots \cup B_{n_0}$, and for $P \in \mathscr{P}_n$,

$$Q(x) = 0, \qquad\qquad \text{for } x \in \Omega_1,$$

$$\qquad = P(x_i) - P(x), \qquad \text{for } x \in B_i, i = 1, \ldots, n_0.$$

We have

$$\sup_{x \in \Omega} |P(x) + Q(x)| \leq b^n \sup_{x \in K_1} |P(x)|.$$

2. A sufficient criterion. We shall give here a sufficient criterion for a compact set K in $\mathbf{R}^N$ in order to satisfy property (A). (This criterion is more general than the one given in Example (II.3).)

We introduce first some notations. Let $I = [A, B]$ be an interval in $\mathbf{R}^N$, (I is the set of points $tA + (1 - t)B$ with $0 \leq t \leq 1$). For any $h > 1$, we denote by $I(h)$ the homothetic interval obtained from I by the homothety centered at $(A + B)/2$ and whose ratio of similitude is h.

If K is a compact set in $\mathbf{R}^N$, we denote by $K(h)$ the union of $I(h)$ where I is any interval contained in K.

$$K(h) = \bigcup_{I \subset K} I(h).$$

We say that K satisfies the *homothety criterion* if, for any $h > 1$, $K(h)$ is a neighborhood of K in $\mathbf{R}^N$.

We have

Theorem (II.1). *Property (A) holds for any compact set K in $\mathbf{R}^N$ which satisfies the homothety criterion.*

Proof. It is easy to see that if K satisfies the homothety criterion, we have $\mathscr{N}(K) = 0$. Therefore, using Theorem (I.4), it is sufficient to prove that property (B′) holds for such K.

From the classical Bernstein inequality on an interval (one variable, see [2] and [4]), we get for any $P \in \mathscr{P}_n$ and any $h > 1$

$$\sup_{x \in K(h)} |P(x)| \le (h + \sqrt{(h^2 - 1)})^n \sup_{x \in K} |P(x)|.$$

Let $b > 1$ be fixed. We choose h such that

$$h + \sqrt{(h^2 - 1)} = b.$$

Since $K(h)$ is a neighborhood of K in $\mathbf{R}^N$, property (B′) holds for K.

Example (II.5). *Let ϕ be a strictly increasing continuous function defined on $[0,1]$ with $\phi(0) = 0$.*

The compact set K in $\mathbf{R}^2$ defined by $K = \{(x_1, x_2), 0 \le x_1 \le 1, 0 \le x_2 \le \phi(x)\}$ satisfies the homothety criterion (and hence satisfies (A)).

Actually it is sufficient to prove that $(0,0)$ is an interior point of $K(h)$ for any $h > 1$. We leave the proof for the reader.

Remark (II.2). With the notations of the preceding example, we choose

$$\phi(x) = e^{-1/x} \quad \text{for } x \in [0, 1].$$

Let us consider the function $f(x_1, x_2) = x_2/x_1$ defined on K. f does not have a C^∞ extension to any neighborhood of K in $\mathbf{R}^2$ (since $\partial f/\partial x_2$ is not bounded!). However it is easy to see that there exist an $a \in]0, 1[$ and $C > 0$ such that for any $n \in \mathbf{N}$

$$d_K(f, \mathscr{P}_n) \le C a^{n^{1/2}}.$$

This example is given in [7] to show that the rapid decrease of $d_K(f, \mathscr{P}_n)$ is not sufficient to conclude that $f \in C^\infty(K)$, in this case. It is interesting to note that this same compact does satisfy (A).

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