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FUNDAMENTAL CONSTANTS FOR RATIONAL FUNCTIONS

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ABSTRACT. Suppose R is a rational function with n poles all of which lie inside Γ, a closed Jordan curve. Lower bounds for the uniform norm of the difference $R - p$ on Γ, where p is any polynomial, are obtained (in terms of the norm of R on Γ). In some cases these bounds are independent of Γ as well as R and p. Some related results are also given.

1. **Introduction and preliminary definitions.** For a complex valued function f defined on a compact set E in the plane, let $\|f\|_E = \sup_{z\in E}|f(z)|$.

If Γ is a closed Jordan curve and $R(z)$ is a rational function having at least one pole inside Γ, then one can easily show that there exists a $\delta > 0$ such that $\|R - p\|_\Gamma \geq \delta$ for all polynomials p. Obviously the same δ will not work for all Γ and all R since $\|R\|_\Gamma$ can be arbitrarily small. However, if we normalize the problem by considering functions of the form $R_n(z) = q_{n-1}(z)/\prod_{i=1}^n (z - a_i)$, where q_{n-1} is a polynomial of degree $n - 1$ (or less), all the a_i's are inside Γ and $\|R_n\|_\Gamma = 1$, then one might inquire as to the existence of a $\delta_n > 0$, independent of Γ and R_n, with the property that $\|R_n - p\|_\Gamma \geq \delta_n$ for all polynomials p.

Some of the results contained herein were presented by S. J. Poreda at the International Conference on Padé Approximation held at the University of Colorado in June 1972. A note outlining these results will appear in the proceedings of that conference [1].

2. **Some partial answers.** A weaker question than the one just stated pertains to the existence of a $\delta_n(\Gamma) > 0$, independent of R but not of Γ, such that $\|R - p\|_\Gamma \geq \delta_n(\Gamma)$ for all polynomials p. The following theorem establishes the existence of a $\delta_n(\Gamma) > 0$ for all analytic Jordan curves Γ.

Theorem 1. *Let Γ be an analytic Jordan curve with interior Ω. There then exists for each $n \in N = \{1,2,\ldots\}$ a constant $\delta_n(\Gamma) > 0$ such that if*

$$R_n(z) = q_{n-1}(z)\Big/\prod_{i=1}^{n} (z - a_i),$$

where q_{n-1} is a polynomial of degree $n - 1$, $a_i \in \Omega$ for $i = 1, 2, \ldots, n$, and $\|R_n\|_\Gamma = 1$, then $\|R_n - p\|_\Gamma \geq \delta_n(\Gamma)$ for all polynomials p.

Proof. We shall begin by proving this theorem in the case where $\Gamma = U$ is the unit circle. In fact, we shall show that one can choose

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$$\delta_n(U) = \delta_{n-1}(U)/(3 + 2\delta_{n-1}(U)),$$

for $n = 2, 3, \ldots$ and $\delta_1(U) = 1/2$.

In the case where $n = 1$, we have

$$\left\|\frac{1-|a_1|}{z-a_1} - p(z)\right\|_U = \left\|\frac{1-|a_1|}{1-\bar{a}_1 z} - \left(\frac{z-a_1}{1-\bar{a}_1 z}\right)p(z)\right\|_U$$
$$\geq \frac{1}{1+|a_1|} > \frac{1}{2}.$$

Proceeding inductively we can write

$$\left\|\left(q_{n-1}(z)\Big/\prod_1^n (z-a_k)\right)\left(\prod_2^n\left(\frac{z-a_j}{1-\bar{a}_j z}\right)\right)\right\|_U \tag{1}$$
$$= \left\|q_{n-1}(z)\Big/(z-a_1)\prod_2^n (1-\bar{a}_j z)\right\|_U = 1$$

and

$$q_{n-1}(z)/(z-a_1)\prod_2^n (1-\bar{a}_j z) = A/(z-a_1) + q_{n-2}(z)\Big/\prod_2^n (1-\bar{a}_j z), \tag{2}$$

where $A = q_{n-1}(a_1)/\prod_2^n (1-\bar{a}_j a_1)$ and q_{n-2} is a polynomial of degree $n-2$. We have two cases.

Case 1. Suppose $|A| \geq 2\delta_n(U)(1-|a_1|)$. In this case we see that if $p(z)$ is a polynomial then

$$\left\|q_{n-1}(z)\Big/\prod_1^n (z-a_k) - p(z)\right\|_U = \left\|q_{n-1}(z)\Big/\prod_1^n (1-\bar{a}_k z) - \prod_1^n\left(\frac{z-a_k}{1-\bar{a}_k z}\right)p(z)\right\|_U$$
$$\geq \left|q_{n-1}(a_1)\Big/\prod_1^n (1-\bar{a}_k a_1)\right|$$
$$\geq |A|/(1-|a_1|^2) > \delta_n(U).$$

Case 2. Now suppose $|A| < 2\delta_n(U)(1-|a_1|)$. In this case we see that $\|A/(z-a_1)\|_U < 2\delta_n(U)$ and so by (1) and (2) we have that

$$\left\|q_{n-2}(z)\Big/\prod_2^n (1-\bar{a}_j z)\right\|_U \geq 1 - 2\delta_n(U).$$

Now if $p(z)$ is a polynomial then

$$\left\|q_{n-1}(z)\Big/\prod_1^n (z-a_k) - p(z)\right\|_U$$
$$= \left\|q_{n-2}(z)\Big/\prod_2^n (z-a_k) - p(z)\right\|_U - \left\|\frac{A}{z-a_1}\prod_2^n\left(\frac{1-\bar{a}_j z}{z-a_j}\right)\right\|_U$$
$$\geq \delta_{n-1}(U)(1-2\delta_n(U)) - 2\delta_n(U) = \delta_n(U),$$

and so our theorem in the case where $\Gamma = U$ follows.

In order to prove our theorem in the general case we make use of the following lemma.

Lemma 1. *Let* Γ, Ω *and* $R_n(z)$ *be as in the statement of our theorem and* $w = \phi(z)$ *be a univalent mapping of* Ω *onto the unit disc* $D = \{|w| < 1\}$. *Write*

$$R_n(\phi^{-1}(w)) = R_n^*(w) + r(w), \tag{3}$$

where $R^*(w) = q_{n-1}^*(w)/\prod_1^n (w - \alpha_j)$, q_{n-1}^* *is a polynomial of degree* $n-1$, $\alpha_j \in D$ *for* $j = 1, 2, \ldots, n$ *and* $r(w)$ *is analytic for* $|w| \le 1$. *Then there exists a constant* $\tau_n(\Gamma, \phi) > 0$, *independent of* R_n, *such that* $\|R_n^*\|_U \ge \tau_n(\Gamma, \phi)$.

Proof. Suppose such a constant does not exist and there exists a sequence of rational functions $\{R_{n,k}\}_{k=1}^\infty$ of the type described in our theorem and that $\|R_{n,k}^*\|_U \to 0$ as $k \to \infty$ where for each k, $R_{n,k}^*$ is as in (3) the rational function with poles in D, corresponding to $R_{n,k}$. That is,

$$R_{n,k}(\phi^{-1}(w)) = R_{n,k}^*(w) + r_k(w),$$

where $R_{n,k}^*(w) = q_{n-1,k}^*(w)/\prod_{j=1}^n (w - \alpha_{j,k})$. Since each $q_{n-1,k}^*$ is a polynomial of degree $n-1$, it follows that $q_{n-1,k}^*(w) \to 0$ as $k \to \infty$ locally uniformly in the entire plane. Since $\alpha_{j,k} \in D$ for each j and k we may assume, taking a suitable subsequence if necessary, that $R_{n,k}^*(w) \to 0$ as $k \to \infty$ locally uniformly in the entire plane with the exception of at most n points in $\{|w| \le 1\}$.

Since the sequence $\{R_{n,k}(z)\}_{k=1}^\infty$ is by hypothesis uniformly bounded by one on Γ, it similarly follows that we may assume that it converges locally uniformly in the entire plane minus at most n points in $\Gamma \cup \Omega$, to a function of the form

$$S_n(z) = Q_{n-1}(z)\Big/\prod_{j=1}^n (z - \lambda_j),$$

where Q_{n-1} is a polynomial of degree $n-1$ and $\lambda_j \in \Gamma \cup \Omega$ for $j = 1, 2, \ldots, n$. Now choose $\varepsilon > 0$ such that $\phi^{-1}(w)$ is univalent on $\{|w| \le 1 + \varepsilon\}$. The sequence $R_{n,k}(\phi^{-1}(w))$ will then converge uniformly on $\{|w| = 1 + \varepsilon\}$ to $S_n(\phi^{-1}(w))$ as $k \to \infty$.

Claim. $S_n(\phi^{-1}(w)) \equiv 0$.

Since $r_k(w) = R_{n,k}(\phi^{-1}(w)) - R_{n,k}^*(w)$, and since both of these functions on the right converge uniformly on $\{|w| = 1 + \varepsilon\}$, each of the functions $r_k(w)$ is analytic in $\{|w| \le 1 + \varepsilon\}$ and thus the sequence $\{r_k\}_{k=1}^\infty$ converges uniformly on that set to a function $f(w)$ that is analytic in $\{|w| < 1 + \varepsilon\}$ and continuous on $\{|w| \le 1 + \varepsilon\}$.

Now for $|w| = 1 + \varepsilon$,

$$\lim_{k\to\infty} r_k(w) = \lim_{k\to\infty} [R_{n,k}(\phi^{-1}(w)) - R_{n,k}^*(w)] = S_n(\phi^{-1}(w)).$$

Thus $S_n(\phi^{-1}(w)) = f(w)$ on $\{|w| = 1 + \varepsilon\}$ which implies that $S_n(z)$ is analytic in $\Omega + \Gamma$ (except for at most n removable singularities). This is only possible if

$S_n(z) \equiv 0$ and so our claim follows.

Finally, since $r_k(z) \to f(w) \equiv 0$, as $k \to \infty$ for $|w| = 1 + \varepsilon$ and since $R^*_{n,k}(w) \to 0$ as $k \to 0$ uniformly on $|w| = 1$, we have that for k sufficiently large,

$$1 = \|R_{n,k}(\phi^{-1}(w))\|_U \leq \|R^*_{n,k}(w)\|_U + \|r_k(w)\|_U < \tfrac{1}{2}$$

which is a contraction and so our lemma is proven.

To conclude the proof of Theorem 1, let $\phi(z) = w$ and R^*_n be as in Lemma 1. We can then write

$$\begin{aligned} \|R_n(z) - p(z)\|_\Gamma &= \|R_n(\phi^{-1}(w)) - p(\phi^{-1}(w))\|_U \\ &= \|R^*_n(w) + r(w) - p(\phi^{-1}(w))\|_U \\ &\geq \|R^*_n(w)\|_U \cdot \delta_n(U) \\ &\geq \tau_n(\Gamma, \phi)\delta_n(U), \end{aligned}$$

since $r(w) - p(\phi^{-1}(w))$ can be uniformly approximated by polynomials on U. Thus we can let $\delta_n(\Gamma) = \tau_n(\Gamma, \phi)\delta_n(U)$, and this proves our theorem.

Let us now weaken our original problem in another way, by considering only those rational functions whose poles have a common locus. We then have the following theorem.

Theorem 2. *For each $n = 1, 2, \ldots,$ there exists $\delta^*_n > 0$ such that if Γ is any closed Jordan curve and if $R(z) = q_{n-1}(z)/(z-a)^n$, where q_{n-1} is a polynomial of degree $n-1$ (or less), a lies in the interior of Γ and $\|R\|_\Gamma = 1$ then $\|R - p\|_\Gamma \geq \delta^*_n$ for all polynomials p.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \delta^*_n$ be the positive constants determined by the system of linear equations:

$$\begin{aligned} \lambda_1 + \lambda_2 + \cdots + \lambda_n &= 1, \\ \lambda_n &= 4^n \delta^*_n, \\ \lambda_{n-1} &= 4^{n-1}(\delta^*_n + \lambda_n), \\ \lambda_{n-2} &= 4^{n-2}(\delta^*_n + \lambda_n + \lambda_{n-1}), \\ &\vdots \\ \lambda_1 &= 4(\delta^*_n + \lambda_n + \lambda_{n-1} + \cdots + \lambda_2). \end{aligned} \tag{4}$$

This system can be solved directly since we can write:

$$\lambda_n = 4^n \delta_n^*,$$

$$\lambda_{n-1} = 4^{n-1}(1 + 4^n)\delta_n^*,$$

$$\lambda_{n-2} = 4^{n-2}(1 + 4^n + 4^{n-1}(1 + 4^n))\delta_n^*,$$

$$\vdots$$

$$\lambda_1 = 4[1 + 4^n + 4^{n-1}(1 + 4^n) + \cdots + 4^2(1 + 4^n + 4^{n-1}(1 + 4^n) + \cdots)]\delta_n^*,$$

and so in particular δ_n^* is given by

$$\begin{aligned} \delta_n^* = \{4^n + 4^{n-1}(1 + 4^n) + 4^{n-2}(1 + 4^n + 4^{n-1}(1 + 4^n)) + \cdots \\ + 4[1 + 4^n + 4^{n-1}(1 + 4^n) + \cdots + 4^2(1 + 4^n + 4^{n-1}(1 + 4^n) + \cdots)]\}^{-1}. \end{aligned} \tag{5}$$

For example, $\delta_1^* = 1/4$, $\delta_2^* = 1/84$, $\delta_3^* = 1/5524$, etc.

We now claim that the number δ_n defined by (5) satisfies the requirements of Theorem 1. To this end choose any Γ and $R(z)$ as in the statement of our theorem. We can then write

$$R(z) = \sum_{j=1}^{n} \frac{A_j}{(z-a)^j}.$$

Since $\|R\|_\Gamma = 1$ we have that $\|A_k/(z-a)^k\|_\Gamma \geq \lambda_k$ for some $k = 1, 2, \ldots, n$ where the λ_k's are as before. Let k_0 be in fact the largest k for which this is true, and let p by any polynomial. We then have

$$\begin{aligned} \|R - p\|_\Gamma &\geq \left\| \sum_{j=1}^{k_0} \frac{A_j}{(z-a)^j} - p(z) \right\|_\Gamma - \left\| \sum_{j=k_0+1}^{n} \frac{A_j}{(z-a)^j} \right\|_\Gamma \\ &\geq \left\| \sum_{j=1}^{k_0} \frac{A_j}{(z-a)^j} - p(z) \right\|_\Gamma - \sum_{j=k_0+1}^{n} \lambda_j. \end{aligned} \tag{6}$$

Now let $\phi(z)$ be a function that conformally maps the interior of Γ onto the unit disc in such a way that $\phi(a) = 0$, and let d denote the distance from a to Γ. As a consequence of the one-quarter theorem [2, p. 17], $d|\phi'(a)| \geq 1/4$, and so we can write

$$\begin{aligned} \left\| \sum_{j=1}^{k_0} \frac{A_j}{(z-a)^j} - p(z) \right\|_\Gamma &= \left\| [\phi(z)]^{k_0} \left[\sum_{j=1}^{k_0} \frac{A_j}{(z-a)^j} - p(z) \right] \right\|_\Gamma \\ &\geq |[\phi'(a)]^{k_0} A_{k_0}| \geq |A_{k_0}/d^{k_0}| \cdot 1/4^{k_0} \\ &\geq \lambda_{k_0}/4^{k_0}. \end{aligned} \tag{7}$$

This follows since the function $[\phi(z)]^{k_0}[\sum_{j=1}^{k_0}(A_j/(z-a)^j)-p(z)]$ is analytic in the interior of Γ, has a continuous extension to Γ and assumes the value $[\phi'(a)]^{k_0}A_{k_0}$ at $z=a$. Also we note that $\|A_{k_0}/(z-a)^{k_0}\|_\Gamma=|A_{k_0}|/d^{k_0}$.

Now by (6) and (7) we have $\|R-p\|_\Gamma\geq\lambda_{k_0}/4^{k_0}-\sum_{j=k_0+1}^{n}\lambda_j=\delta_n^*$, by virtue of the definition of k_0 and (4). Our theorem is now proven.

Remark. In the special case where $\Gamma=U$ is the unit circle, one can show that the δ_n^* defined by (5) can be replaced by the number defined by the same expression except where all of the 4's are replaced by 2's. Furthermore, if in this special case $a=0$ one can also show that δ_n^* can be replaced by $1/n$.

3. A related question. We now turn our attention to the following question. By Theorem 2 we have that if Γ is a closed curve and the point $z=a$ lies in the interior of Γ then there exists a $\delta_n^*>0$ such that $\|R-p\|_\Gamma\geq\delta_n^*$ for all corresponding functions R and p as in the statement of Theorem 2. Now let $\delta_n^*(\Gamma,a)$ be the largest such constant that works for a particular curve Γ and a particular point a inside Γ. Since the lower bounds we establish for these constants in the proof of Theorem 2 tend to zero as n increases, we are naturally led to the question of whether the $\lim_{n\to\infty}\delta_n^*(\Gamma,a)=0$ for each Γ and each a. The next theorem answers this question affirmatively in the case where $\Gamma=U$ is the unit circle.

Theorem 3. *Let $\delta_n^*(\Gamma,a)$ be as defined above; then*

$$\lim_{n\to\infty}\delta_n^*(U,a)=0\ (\textit{for all}\quad |a|<1).$$

Proof. We show first that $\lim_{n\to\infty}\delta_n^*(U,0)=0$. Consider the sequence of rational functions $\{S_n\}_{n=2}^{\infty}$ given by

$$S_n(z)=\sum_{k=2}^{n}\frac{z^{-k}}{k\log k}-\sum_{k=2}^{n}\frac{z^k}{k\log k}.$$

We can write

$$\begin{aligned}S_n(e^{i\theta})&=\sum_{k=2}^{n}\frac{\cos k\theta-i\sin k\theta}{k\log k}-\sum_{k=2}^{n}\frac{\cos k\theta+i\sin k\theta}{k\log k}\\&=-2i\sum_{k=2}^{n}\frac{\sin k\theta}{k\log k}.\end{aligned}$$

Now by [3, p. 253] we have that

(i) $\sum_{k=2}^{\infty}(\sin kx)/(k\log k)$ converges uniformly for all x, and

(ii) $\sum_{k=2}^{\infty}(\cos kx)/(k\log k)$ diverges at $x=0$.

In particular this implies that $\lim_{n\to\infty}\|S_n\|_U<\infty$ and that

$$\lim_{n\to\infty}\left\|\sum_{k=2}^{n}\frac{z^{-k}}{k\log^k}\right\|_U=\infty.$$

Our theorem in the case where $a = 0$ now follows by virtue of this example.

We now claim that if $|a| < 1$ then $\delta_n^*(U,a) \leq ((1 + |a|)/(1 - |a|))\delta_n^*(U,0)$. To this end let $\beta > \delta_n^*(U,0)$. There then exists a polynomial q_{n-1} of degree $n - 1$ such that $\|q_{n-1}(z)/z^n\|_U = 1$ and a polynomial p such that $\|q_{n-1}(z)/z^n - p(z)\|_U < \beta$.

Write $q_{n-1}(z) = A_{n-1}z^{n-1} + \cdots + A_1 z + A_0$, and set $z = (w - a)/(1 - \bar{a}w)$ where $|a| < 1$. Then

$$\frac{q_{n-1}(z)}{z^n} = q_{n-1}\left(\frac{w-a}{1-\bar{a}w}\right)\Big/\left(\frac{w-a}{1-\bar{a}w}\right)^n$$

$$= \frac{(1-\bar{a}w)[A_{n-1}(w-a)^{n-1} + \cdots + A_1(w-a)(1-\bar{a}w)^{n-2} + A_0(1-\bar{a}w)^{n-1}]}{(w-a)^n}$$

$$= \frac{(1-\bar{a}w)p_{n-1}(w)}{(w-a)^n},$$

where p_{n-1} is a polynomial of degree $n - 1$.

Now $\|(1 - \bar{a}w)p_{n-1}(w)/(w - a)^n\|_U = 1$ implies that $\|p_{n-1}(w)/(w - a)^n\|_U \geq 1/(1 + |a|)$. Moreover, $f(w) = ((1 + |a|)/(1 - \bar{a}w))p((w - a)/(1 - \bar{a}w))$ is analytic in $\{|z| \leq 1\}$ and so

$$\left\|\frac{(1+|a|)p_{n-1}(w)}{(w-a)^n} - \left(\frac{1+|a|}{1-\bar{a}w}\right)p\left(\frac{w-a}{1-\bar{a}w}\right)\right\|_U$$

$$\leq \left(\frac{1+|a|}{1-|a|}\right)\left\|\frac{(1+|a|)p_{n-1}(w)}{(w-a)^n} - p\left(\frac{w-a}{1-\bar{a}w}\right)\right\|_U < \left(\frac{1+|a|}{1-|a|}\right)\beta.$$

Thus $\delta_n^*(U,a) \leq ((1 + |a|)/(1 - |a|))\delta_n(U,0)$ and so our claim and our theorem follow.

4. **An application to rational approximation.** If f is a function that is continuous on Γ, a closed Jordan curve, and $\varepsilon > 0$, there exists [2, p. 100] a rational function $S_{n,k}$ of the form

$$S_{n,k}(z) = q_{n-1}(z)\Big/\prod_{j=1}^{n}(z - a_j) + p_k(z)$$

where q_{n-1} and p_k are polynomials of respective degrees $n - 1$ and k (for some n and k), the a_j's lie inside Γ, and such that $\|f - S_{n,k}\|_\Gamma < \varepsilon$. If ε is sufficiently small, $\|S_{n,k}\|_\Gamma < 2\|f\|_\Gamma$ and so a natural question to ask is whether $\|q_{n-1}(z)/\prod_{j=1}^n (z - a_j)\|_\Gamma$ is bounded in any way. If, as described in §1, there exists a $\delta_n > 0$ (possibly independent of Γ) we would then have

$$\left\|q_{n-1}(z)\Big/\prod_1^n (z - a_j)\right\|_\Gamma < 2\frac{\|f\|_\Gamma}{\delta_n}.$$

In this way, one can immediately state corollaries to Theorems 1 and 2.

As a corollary to Theorem 3, we likewise see that it is, in general, impossible to bound $\|q_{n-1}(z)/\prod_1^n (z - a_j)\|_\Gamma$ independently of n.

5. **Remarks.** In the proof of Theorem 2, we made use of the famous one-quarter theorem for univalent functions. A close examination of that proof shows that Theorem 2 in the case where $n = 1$ is in fact an equivalent formulation of the one-quarter theorem. Consequently, Theorems 1 and 2 can both be considered as generalizations of it.

Finally, this paper leaves as many questions unanswered as it answers. The foremost question is, of course, whether one can eliminate the hypothesis that Γ be analytic in Theorem 1. Should this be possible an intriguing question is whether or not one can show if the constants $\delta_n(\Gamma)$ can be made independent of Γ as they are in the special case discussed in Theorem 2. Theorem 3 can also be strengthened by replacing U with any Jordan curve Γ.

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