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CLASSICAL QUOTIENT RINGS

BY

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ABSTRACT. Throughout R is a ring with right singular ideal $Z(R)$. A right ideal K of R is *rationally closed* if $x^{-1}K = \{y \in R: xy \in K\}$ is not a dense right ideal for all $x \in R - K$. A ring R is a Cl-ring if R is (Goldie) right finite dimensional, $R/Z(R)$ is semiprime, $Z(R)$ is rationally closed, and $Z(R)$ contains no closed uniform right ideals. These rings include the quasi-Frobenius rings as well as the semiprime Goldie rings. The commutative Cl-rings have Cl-classical quotient rings. The injective ones are cogenerator rings.

In what follows, R is a Cl-ring. A dense right ideal of R contains a right nonzero divisor. If R satisfies the minimum condition on rationally closed right ideals then R has a classical Artinian quotient ring. The complete right quotient ring Q (also called the Johnson-Utumi maximal quotient ring) of R is a Cl-ring. If R has the additional property that bR is dense whenever b is a right nonzero divisor, then Q is classical. If Q is injective, then Q is classical.

1. Introduction. Throughout Q denotes the complete ring of right quotients of R whereas $Z(R)$ $(Z(Q))$ denotes the right singular ideal of R (of Q). In this paper we look for various conditions on R for Q to be a classical quotient ring. We construct the classical quotient rings via Cl-rings. In a Cl-ring there is an essential direct sum of uniform right ideals $b_1R + \cdots + b_nR$ such that $b_1 + \cdots + b_n$ is a right nonzero divisor. Furthermore, every dense right ideal contains a right nonzero divisor. This leads to the following theorem:

Theorem 3.2. *Suppose that R is a Cl-ring and bR is dense whenever b is a right nonzero divisor of R. Then Q is a classical Cl-ring with Jacobson radical $Z(Q)$ and $Q/Z(Q)$ is a completely reducible (semiprime Artinian) classical quotient ring of $R/Z(R)$. The converse holds.*

Immediately, a semiprime Goldie ring (which is a Cl-ring) has a completely reducible classical quotient ring [5]. In a commutative ring, bR is dense whenever b is a nonzero divisor. Therefore, all commutative Cl-rings have classical quotient rings. The abstract states additional applications. (Also cf. Theorem 4.2, Corollary 4.7.)

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The Q of a Cl-ring is a Cl-ring but not conversely. However, if $R/Z(R)$ is semiprime and Q is a Cl-ring then R is also a Cl-ring.

For a nonempty subset A of R we set $r(A) = \{x \in R: ax = 0 \text{ for all } a \in A\}$; for $x \in R - A$ we define $x^{-1}A = \{y \in R: xy \in A\}$. Finally, for $b \in R$ we write $r(b)$ instead of $r(\{b\})$ and we equate bR with the principal right ideal generated by b.

2. **Dense right ideals of Cl-rings.** We observe a trivial fact which we use several times.

Remark 2.1. Suppose a and b belong to a uniform right ideal of a ring R. If $ab \in R - Z(R)$ then the sum $aR + r(a)$ is direct.

This portion of the paper is devoted to proving Theorem 2.2.

Theorem 2.2. *In a Cl-ring R every dense right ideal contains a right nonzero divisor. Furthermore, so does every right ideal whose image is essential in $R/Z(R)$.*

Proof. For the moment assume that there is an essential direct sum B of uniform right ideals $b_1R + \cdots + b_nR$ such that $r(b_1 + \cdots + b_n) = (0)$ and $r(b_i) \cap b_iR = (0)$ for $1 \leq i \leq n$. Suppose L is a right ideal with $L \cap b_iR \not\subset Z(R)$ for $1 \leq i \leq n$. Hence, $(L \cap b_iR)^2 \not\subset Z(R)$, and by Remark 1 we choose $x_i \in L \cap b_iR$ such that $r(x_i) \cap x_iR = (0)$. Since b_iR and x_iR are uniform with nonzero intersection, $r(x_ib_i) = r(b_i)$ for $1 \leq i \leq n$. Therefore, $x_1b_1 + \cdots + x_nb_n \in L$ and is a right nonzero divisor. If D is a dense right ideal and $Z(R)$ is rationally closed then $b_i^{-1}D \not\subset Z(R)$ for $1 \leq i \leq n$. Hence, $b_iR \cap D \not\subset Z(R)$ and D contains a right nonzero divisor. Next we show that the direct sum B does exist in R.

Lemma 2.3. *Assume that R is right finite dimensional and $Z(R)$ contains no square of a closed uniform right ideal. Then there is an essential direct sum $b_1R + \cdots + b_kR$ of uniform right ideals such that $r(b_1 + \cdots + b_k) = (0)$ and $r(b_i) \cap b_iR = (0)$ for $1 \leq i \leq k$.*

Proof. Let C denote the collection of direct sums of the form $b_1R + \cdots + b_kR + r(b)$ where $b = b_1 + \cdots + b_k$ and $r(b_i) \cap b_iR = (0)$ for $1 \leq i \leq k$. Remark 2.1 shows that C is nonempty. Let $b_1R + \cdots + b_kR + r(b)$ be a maximal element of C in that if $c_1R + \cdots + c_pR + r(c)$ is an element of C then $p \leq k$. Equate $B = b_1R + \cdots + b_kR$. If $r(b) = (0)$, then bR is essential and we are finished. Assume that $r(b) \neq (0)$ and by Zorn's lemma choose a closed uniform right ideal A subject to $A \cap r(b) \neq (0)$ and $A \cap B = (0)$. Since $A^2 - Z(R)$ is nonempty by hypothesis, we choose $a \in A$ such that $r(a) \cap aR = (0)$; see Remark 2.1. It suffices to show that $(B + aR) \cap r(a + b) = (0)$ for then $a + r(b)$ is not maximal, a contradiction. Suppose $x = b' + a'$ with $b' \in B$ and $a' \in BR$ and $bx = ax = 0$. If $a' = 0$ then $bx = bb' = 0$ and $b' = 0 = x$ since $r(b) \cap B = (0)$. If $a' \neq 0$ then $a'R \cap r(b) \neq (0)$; recall the uniform right ideal A has nonzero intersection with $r(b)$.

There is $z \in R$ such that $0 \neq xz = b'z + a'z$ with $a'z \in r(b) - (0)$. Hence, $bxz = bb'z = 0$ implies $b'z = 0$. Again $axz = aa'z$ implies $a'z = 0$. Therefore, $xz = 0$, a contradiction. We conclude that $x = 0$ which completes the lemma.

Clearly, a Cl-ring satisfies the hypothesis of Lemma 2.3 which completes the proof of Theorem 2.2.

3. **A classical quotient theorem for Cl-rings.** It is easy to see that if each dense right ideal of R contains a nonzero divisor and nonzero divisors are invertible in Q, then Q is a classical quotient ring with no proper dense right ideals; the converse holds. We set up conditions on R for this to occur.

Remark 3.1. Assume that b is a right nonzero divisor of R. Then bR is dense if and only if b is invertible in Q.

To see this, the map br to r implies that $qb = 1$ for an appropriate $q \in Q$. Since bR is essential, $r(q) = (0)$ and $bq = 1$ which proves the remark.

Theorem 3.2. *Assume that R is a Cl-ring and bR is dense whenever b is a right nonzero divisor of R. Then Q is a Cl-classical quotient ring of R and the Jacobson radical of Q is $Z(Q)$. Furthermore, $Q/Z(Q)$ is a completely reducible classical quotient ring of $R/Z(R)$. The converse holds.*

Proof. From Remark 3.1 and Theorem 2.2 we see that Q is a classical quotient ring and that $Q/Z(Q)$ contains no proper essential right ideals. Therefore, $Q/Z(Q)$ is a completely reducible ring. If $ab^{-1} \in Z(Q)$ with $r(b) = (0)$ and $a \in R$ then $(1 - ab^{-1})b$ is a right nonzero divisor of R and is invertible in Q. Hence, the Jacobson radical of Q is $Z(Q)$. For the converse it is easy to see that R is a Cl-ring. If $b \in R$ with $r(b) = (0)$ then the right nonzero divisor $b + Z(R)$ is invertible in $Q/Z(Q)$. Since $Z(Q)$ is the Jacobson radical it follows that b is invertible in Q. Remark 3.1 completes the implication.

Corollary 3.3 [5]. *If R is a semiprime Goldie ring then R has a classical completely reducible quotient ring. The converse holds.*

Proof. The proof is clear.

4. **Applications.** We recall some known relationships between R and Q. Clearly, $Z(Q) \cap R = Z(R)$. Closed right ideals are rationally closed. Recall, a right ideal K is rationally closed if $x^{-1}K = \{y \in R:\ xy \in K\}$ is not a dense right ideal. An equivalent statement for a right ideal K to be rationally closed is that the injective hull of R contains a subset S such that $K = \{y \in R:\ sy = 0$ for all $s \in S\}$ [18]. It follows that the maximum (minimum) condition on rationally closed right ideals as well as the finite dimensional property passes from R to Q and conversely. The Cl-property passes from R to Q.

Theorem 4.1. *The Q of a Cl-ring is a Cl-ring. If $R/Z(R)$ is semiprime and Q is a Cl-ring then R is a Cl-ring.*

Proof. Let R be a Cl-ring and let $q \in Q - Z(Q)$. If $q^{-1}Z(Q)$ were dense then it would contain a right nonzero divisor b by Theorem 2.2 and $qb \in Z(Q)$. However, $r(qb) = bR \cap r(q)$ and $qb \in R - Z(Q)$ because $r(q)$ is not essential. We have a contradiction and conclude that $Z(Q)$ is rationally closed. If K is a uniform closed right ideal of Q, then $K \cap R$ is closed and $K \cap R \not\subset Z(R)$; hence, $K \cap Q \not\subset Z(Q)$. For the next implication, $Z(R) = R \cap Z(Q)$ is rationally closed since $Z(Q)$ is. Let A denote a closed uniform right ideal of R whereas A' denotes a closed uniform right ideal of Q containing A. Then $A' \not\subset Z(Q)$ implies $A' \cap R = A \not\subset Z(R)$ which completes the proof.

Although Q may be a Cl-ring, R need not be. Let R denote the upper triangular 2×2 matrices over a field. Then Q is completely reducible and $Z(R) = (0)$ yet R is not semiprime.

We apply Theorem 2.2 to self-injective rings. A ring R is a *self-injective cogenerator* if R_R is injective and R_R is a cogenerator in the category of unital right R-modules [10]. These rings are finite dimensional and properly contain the quasi-Frobenius rings [13].

Suppose R is a finite dimensional self-injective ring. Then R is a cogenerator ring if and only if $Z(R)$ is rationally closed [19]. *Hence, a self-injective ring is a Cl-ring if and onlyif it is a cogenerator ring.*

A ring R has the *dense extension property* if each map from a right ideal of R into R can be extended to a dense right ideal. The complete ring of right quotients Q of R is self-injective if and only if R has the dense extension property. This is an immediate consequence of the Lambek-Gabriel characterization as in [3] and [4].

Theorem 4.2. *A Cl-ring with the dense extension property has a classical quotient ring which is a self-injective cogenerator ring. The converse holds.*

Proof. The Q of a ring R with the above properties is an injective Cl-ring. If $b \in R$ with $r(b) = (0)$ then bQ is an essential injective Q-submodule of Q and thus $bQ = Q$. Therefore, right nonzero divisors are invertible in Q. From Theorem 3.2 and Remark 3.1, Q is classical. The converse is clear.

The corollary below is a special case of Theorem 5.3 of [18].

Corollary 4.3. *If $R/Z(R)$ is semiprime and Q is a self-injective cogenerator ring then Q is a classical quotient ring.*

Proof. The result is clear.

Our next corollary is due to Johnson [8, Theorem 4.4] and Sandomierski [16, Theorem 1.6].

Corollary 4.4. *If R is semiprime and Q is completely reducible then Q is a classical quotient ring.*

Proof. The proof is clear.

A solid Goldie ring is one with the maximum condition on rationally closed right ideals [6]. The solid Goldie rings are Goldie rings but not conversely [18]. The injective ones are quasi-Frobenius [1]. The next corollary was proven independently by Shock [18] (announced in [17]) and by Tachikawa [20].

Corollary 4.5. *A ring R has a classical quasi-Frobenius quotient ring if and only if R is a solid Goldie with the dense extension property and the prime radical of R is the right singular ideal.*

Proof. Recall R is solid Goldie if and only if Q is. If Q is quasi-Frobenius then $Z(Q) \cap R = Z(R)$ and $Z(R)$ is nilpotent. The proof is clear.

The dual notion of the solid Goldie ring is the ring with the minimum condition on rationally closed right ideals. Assume that R has this latter property. If R is singular $(Z(R) = (0))$, then the rationally closed right ideals are closed and the property coincides with the finite dimensional property. The other extreme is that R contains no proper dense right ideal and all right ideals are rationally closed; in this case R is Artinian. These rings contain the Artinian rings. The injective ones are quasi-Frobenius. In an Artinian ring a nonzero divisor is invertible. In R a nonzero divisor is invertible in Q as we see below.

Theorem 4.6. *In a ring with the minimum condition on rationally closed right ideals, a right nonzero divisor is invertible in Q.*

Proof. Suppose that $b \in R$ and $r(b) = (0)$. It suffices to show that bR is a dense right ideal; see Remark 3.1. Assume that $bR \supset b^2R \supset b^3R \supset \cdots$ is a strictly decreasing sequence, otherwise b is invertible in R. Clearly, $(b^n)^{-1}b^{n+1}R = bR$. If bR were not dense then the rational closure of b^nR would properly contain the rational closure of $b^{n+1}R$ (cf. [18, Lemma 3.1(2)]). We would then have a strictly decreasing sequence of rationally closed right ideals, a contradiction. Therefore, bR is dense.

Corollary 4.7. *A Cl-ring with the minimum condition on rationally closed right ideals has a classical Cl-Artinian quotient ring. The converse holds.*

Proof. The proof is clear.

We state the dual to Corollary 4.5.

Corollary 4.8. *A ring R has a classical quasi-Frobenius quotient ring if and only if R satisfies the dense extension property and R satisfies the minimum condition on rationally closed right ideals and the prime radical is the right singular ideal.*

Proof. The proof is clear.

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