

COUNTABLE UNIONS OF TOTALLY PROJECTIVE GROUPS⁽¹⁾

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ABSTRACT. Let the p -primary abelian group G be the set-theoretic union of a countable collection of isotype subgroups H_n of countable length. We prove that if H_n is totally projective for each n , then G must be totally projective. In particular, an ascending sequence of isotype and totally projective subgroups of countable length leads to a totally projective group. The result generalizes and complements a number of theorems appearing in various articles in the recent literature. Several applications of the main result are presented.

In [6], the author proved that if a p -primary abelian group G is the set-theoretical union of a countable number of pure subgroups H_n , then G is a direct sum of cyclic groups if H_n is for each n . In the present paper, we generalize this result by proving that if the subgroups H_n are isotype then G must be totally projective provided that H_n is totally projective of countable length for each n . Although we do not resolve the case where the subgroups H_n are allowed to have arbitrary length, the restriction to countable length is enough to unify a number of recent theorems from [1], [6], [7], [9], [10] and [11]. This theorem also provides the basis for some additional new results. We remark that even if the theorem is valid for groups of length Ω , few, if any, applications should be expected for this case because such groups tend not to be *countable* set-theoretic unions of significantly simpler isotype and totally projective subgroups.

For terminology, as well as for the fundamental techniques, the reader is referred to [2, Chapter V] or to original sources such as [5] and [7]. In particular, we shall assume the equivalence of being totally projective and satisfying the author's third axiom of countability. It will be tacitly assumed throughout that all groups are p -primary abelian groups.

Although it is not truly essential, we employ the following lemma in the proof of our main theorem.

Lemma 1. *Let G be a p -primary group of countable length and let*

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$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots, \quad \alpha < \mu,$$

be a chain (indexed by an arbitrary ordinal μ) of nice subgroups of G satisfying the following conditions:

- (i) $A_{\alpha+1}/A_\alpha$ is countable.
- (ii) $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ when β is a limit.
- (iii) $G = \bigcup_{\alpha < \mu} A_\alpha$.

Then G is totally projective.

Proof. If $\mu \leq \Omega$, then it follows at once that G satisfies the third axiom of countability. However, for an arbitrary μ , it is not immediately evident that conditions (i)–(iii) imply the third axiom of countability.

Nevertheless, under the hypothesis, we can embed (by extending inductively from A_α to $A_{\alpha+1}$ a height-preserving monomorphism into a sufficiently large totally projective group) the group G as an isotype subgroup of a totally projective group. Since the length of G is countable, G itself is totally projective by our theorem in [3].

Theorem 1. Suppose that the p -primary abelian group G is the set-theoretical union of a countable number of isotype subgroups H_n . If H_n is totally projective of countable length for each n , then G is totally projective.

Proof. First, observe that G must have countable length. Let the length of G be λ and let the subgroups H_n be indexed by the nonnegative integers. For each $n < \omega$, let $H_n = \sum_{i \in I(n)} C_i$ where C_i is countable for each i . Suppose that we have developed a chain

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_\alpha \subseteq \cdots, \quad \alpha < \tau,$$

of subgroups of G satisfying the following conditions:

- (1) $A_{\alpha+1}/A_\alpha$ is countable.
- (2) $A_\beta = \bigcup_{\alpha < \beta} A_\alpha$ when β is a limit.
- (3) $A_\alpha \cap H_n = \sum_{i \in I(n, \alpha)} C_i$ for each n and α , where $I(n, \alpha)$ is a subset of $I(n)$.
- (4) $\langle p^\gamma G, A_\alpha \rangle \cap \langle H_n, A_\alpha \rangle = \langle p^\gamma H_n, A_\alpha \rangle$ for each n and α and for each $\gamma \leq \lambda$.

If τ is a limit ordinal, we define $A_\tau = \bigcup_{\alpha < \tau} A_\alpha$ and observe that conditions (1)–(4) continue to hold for the extended chain of subgroups A_α . Now suppose that $\tau - 1$ exists. For an arbitrary countable extension C of $A_{\tau-1}$ in G , there exists a countable extension K of C such that $K \cap H_n = \sum_{i \in J(n)} C_i$, for each n , where $J(n)$ is a countable extension of $I(n, \tau - 1)$; we assume the reader is familiar with the back-and-forth procedure involved here (if this is not the case, see, for example, [2, Chapter V]).

We shall now give the essence of the argument that there exists a countable extension B of $K \supseteq C \supseteq A_{\tau-1}$ in G such that

$$\langle p^\gamma G, B \rangle \cap \langle H_n, B \rangle = \langle p^\gamma H_n, B \rangle$$

for all $\gamma \leq \lambda$ and all $n < \omega$. Let $\{k_i\}_{i < \omega}$ be a set of representatives for the cosets of $K/A_{\tau-1}$. For each triple (i, n, γ) with $i, n < \omega$ and $\gamma \leq \lambda$ such that $z + b_n \in k_i + A_{\tau-1}$ with $z \in p^\gamma G$ and $b_n \in H_n$, we choose a single representative $z = z_{i, n, \gamma}$ for the triple (i, n, γ) . Thus there are only a countable number of such representatives. Letting $B_1 = \langle K, z_{i, n, \gamma} \rangle$, we observe that

$$\langle p^\gamma G, K \rangle \cap \langle H_n, K \rangle \subseteq \langle p^\gamma H_n, B_1 \rangle.$$

If we let B_1 replace K and then in turn let B_{j+1} replace B_j where $B_{j+1} = \langle B_j, z_{i, n, \gamma} \rangle$, we see that $B = \bigcup_{j < \omega} B_j$ has the desired properties.

By alternately considering conditions (3) and (4), we obtain in G a countable extension A_τ of $A_{\tau-1}$ containing C that satisfies both conditions (3) and (4). Hence a chain satisfying conditions (1)–(4) can be developed until G is exhausted. We shall show that A_α is necessarily nice in G for each α . It suffices to show

$$(I) \quad p^\gamma(G/A_\alpha) = \langle p^\gamma G, A_\alpha \rangle / A_\alpha$$

for all $\gamma \leq \lambda$, which will be proved by induction on γ in conjunction with the condition

$$(II) \quad p^\gamma(G/A_\alpha) \cap \langle H_n, A_\alpha \rangle / A_\alpha = \langle p^\gamma H_n, A_\alpha \rangle / A_\alpha = p^\gamma(\langle H_n, A_\alpha \rangle / A_\alpha).$$

Obviously, for a given γ the second equality in condition (II) is a consequence of the first. However, the second equality is valid, independent of the first, because of condition (3). Therefore, we observe that condition (II) complements condition (I) extremely well inasmuch as (II) automatically survives limit ordinals whereas (I) automatically survives isolated ordinals. Thus if (I) should fail, it must fail for the first time at a limit ordinal δ . We claim that (II) must hold for $\gamma = \delta$. Since δ is a limit, it suffices to prove that (II) holds for all $\gamma < \delta$. By the choice of δ , condition (I) holds for all $\gamma < \delta$. Hence, if $\gamma < \delta$, we have that

$$p^\gamma(G/A_\alpha) \cap \langle H_n, A_\alpha \rangle / A_\alpha = (\langle p^\gamma G, A_\alpha \rangle \cap \langle H_n, A_\alpha \rangle) / A_\alpha,$$

but by condition (4) the latter is equal to $\langle p^\gamma H_n, A_\alpha \rangle / A_\alpha$. Thus condition (II) holds for $\gamma = \delta$. This implies that

$$p^\delta(G/A_\alpha) \subseteq \bigcup_{n < \omega} \langle p^\delta H_n, A_\alpha \rangle / A_\alpha \subseteq \langle p^\delta G, A_\alpha \rangle / A_\alpha$$

which proves that A_α is nice in G for each α . The proof of the theorem is now a consequence of Lemma 1.

Remark. In the proof of Theorem 1, Lemma 1 could be omitted by directly

exhibiting the validity of the third axiom of countability. However, since the proof of Lemma 1 is so short, we prefer to use it.

It was first proved in [7, Theorem 3.10] that if G is a summable p -primary group of countable length λ , then G is totally projective provided that $G/p^\alpha G$ is totally projective for each $\alpha < \lambda$. Later, C. Megibben [10, Theorem A] gave a different and more detailed proof of this theorem and referred to the result as "the generalized Kulikov criterion". We shall exhibit some of the strength of Theorem 1 by showing that it quickly yields the proof of the generalization of Kulikov's criterion just mentioned. At the same time, a very small (and purely artificial) improvement is effected.

Theorem 2. *Suppose that G is a summable p -primary group of countable length λ . Then G is totally projective provided that $G/p^\alpha G$ is totally projective for each limit ordinal $\alpha < \lambda$.*

Proof. The proof is by induction on λ . Since G is totally projective if both $p^\alpha G$ and $G/p^\alpha G$ are, we may assume that λ is a limit ordinal. Let $\lambda(1) < \lambda(2) < \dots < \lambda(n) < \dots$ be an increasing sequence of ordinals whose limit is λ . Choose $H_1 \subseteq H_2 \subseteq \dots \subseteq H_n \subseteq \dots$ to be an increasing sequence so that H_n is $p^{\lambda(n)}$ -high in G and so that $H[p] = G[p]$, where $H = \bigcup_{n < \omega} H_n$. Since H is pure, $H = G$. Thus, in view of Theorem 1, it suffices to prove that H_n is totally projective for each n . However, we know that H_n is isomorphic to an isotype subgroup of $G/p^{\lambda(n)}G$ under the natural map. Since the latter is totally projective, H_n is too [3].

We shall now generalize Theorem 2 in [6], which states that if the p -primary group G has cardinality $\aleph_\alpha$ where α is cofinal with ω , then G is a direct sum of cyclic groups provided each subgroup of G having cardinality less than $\aleph_\alpha$ is a direct sum of cyclic groups.

Theorem 3. *Let G be a p -primary abelian group of cardinality $\aleph_\alpha$ where α is cofinal with ω . If each subgroup of G having cardinality less than $\aleph_\alpha$ is contained in a totally projective subgroup of G having countable length, then G is necessarily totally projective.*

Proof. Suppose that H_n is a subgroup of G having cardinality $\aleph_{\alpha(n)}$, where $\alpha(1) < \alpha(2) < \dots < \alpha(n) < \dots$ is an increasing sequence of ordinals whose limit is α . According to [4], H_n is contained in an isotype subgroup K_n of G having the same cardinality as that of H_n , which is $\aleph_{\alpha(n)}$. By hypothesis, K_n is contained in a totally projective subgroup G_n of G having countable length. Since K_n is isotype in G_n , K_n is also totally projective. It follows that G is the union of a countable ascending sequence $K_1 \subseteq K_2 \subseteq \dots \subseteq K_n \subseteq \dots$ of isotype and totally projective subgroups K_n . By Theorem 1 G is totally projective.

One can also apply Theorem 1 to show that certain subgroups of totally

projective groups are again totally projective. A paper by F. F. Kamalov [9] establishes the following. If G is totally projective of length less than ω^2 and if H is a subgroup of G such that $p^\alpha G/p^\alpha H$ is countable for all α , then H is also totally projective. A far-reaching generalization of Kamalov's result has been provided by B. L. Edington [1]. As we have indicated, Theorem 1 can be used to establish results of this kind. Details in this connection will appear elsewhere.

As was pointed out in [10, Lemma 6.1], if A and B are p^α -high subgroups of G and if $\alpha = \beta + n$ where β is a limit and $n < \omega$, then there exists a height-preserving isomorphism between $(A/p^\beta A)[p]$ and $(B/p^\beta B)[p]$. More generally, it is not difficult to show that there exists a height-preserving isomorphism between $(A/p^\beta A)[p]$ and $(B/p^\beta B)[p]$ for every limit ordinal β .

R. Nunke [12] and C. Megibben [10] have given independent proofs that all p^α -high subgroups of G are direct sums of countable groups if one p^α -high subgroup of G is a direct sum of countable groups. Since the theorem is trivially true (in view of the results of [7]) if $\alpha = \Omega$, there is no loss of generality in assuming that α is countable. Hence, in the light of the preceding remarks, this result is encompassed in the next theorem.

Theorem 4. *Let H and K be p -groups of countable type τ and suppose that H is totally projective. If, for each ordinal $\alpha \leq \tau$, there exists a height-preserving isomorphism between $(H/p^{\omega_\alpha} H)[p]$ and $(K/p^{\omega_\alpha} K)[p]$, then K is totally projective and $H \cong K$.*

Proof. By hypothesis, there exists a height-preserving isomorphism between $H[p]$ and $K[p]$. Thus the groups H and K have the same Ulm invariants, and the two are isomorphic if K is totally projective.

We shall prove that K is totally projective by induction on τ . Since H is summable, K must be summable. If $\tau = 1$, then K is a direct sum of cyclic groups. Thus we may assume that $\tau > 1$. Next, assume that $\tau - 1$ exists. For simplicity of notation, let G^α denote $p^{\omega_\alpha} G$. We observe that

$$H/H^{\tau-1}/(H/H^{\tau-1})^\alpha \cong H/H^\alpha$$

for all $\alpha \leq \tau - 1$, and likewise, of course, for K . By the induction hypothesis, $K/K^{\tau-1}$ is totally projective. Moreover, $K^{\tau-1}$ is a direct sum of cyclic groups since $K^{\tau-1}[p]$ and $H^{\tau-1}[p]$ have a height-preserving isomorphism between them. Therefore K is totally projective. Now we consider the case that τ is a limit ordinal. Let $H[p] = \sum S_\alpha$ and $K[p] = \sum T_\alpha$ be decompositions of $H[p]$ and $K[p]$, respectively, such that the nonzero elements of S_α and T_α , respectively, have height exactly α . Letting $1 \leq \tau(1) < \tau(2) < \dots < \tau(n) < \dots$ be an increasing sequence of ordinals whose limit is τ , we choose ascending sequences

$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ and $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$ where A_n and B_n , respectively, are chosen maximal with respect to $A_n[p] = \sum_{\alpha < \omega \tau(n)} S_\alpha$ and $B_n[p] = \sum_{\alpha < \omega \tau(n)} T_\alpha$. Note that $H = \bigcup_{n < \omega} A_n$ and $K = \bigcup_{n < \omega} B_n$ since $\bigcup_{n < \omega} A_n$ and $\bigcup_{n < \omega} B_n$ are pure subgroups of H and K , respectively, containing $H[p]$ and $K[p]$.

We wish to show that B_n is totally projective and our plan is to use the induction hypothesis of the present theorem. Thus we need to show the existence of a height-preserving isomorphism between $(A_n/A_n^\alpha)[p]$ and $(B_n/B_n^\alpha)[p]$ for each $\alpha \leq \tau(n)$. In order to do so we need to consider two cases. If $\alpha = \tau(n)$, then there is a height-preserving isomorphism between $(A_n/A_n^\alpha)[p] = A_n[p]$ and $(B_n/B_n^\alpha)[p] = B_n[p]$ by the choice of A_n and B_n . Thus it suffices to consider the case that $\alpha < \tau(n)$. Recall that $H = \langle A_n, H^\alpha \rangle$ since $\alpha < \tau(n)$ and since A_n is $p^{\omega \tau(n)}$ -high in H . Similarly, $K = \langle B_n, K^\alpha \rangle$. Therefore,

$$H/H^\alpha \cong A_n/A_n^\alpha \quad \text{and} \quad K/K^\alpha \cong B_n/B_n^\alpha.$$

We conclude that there exists a height-preserving isomorphism between $(A_n/A_n^\alpha)[p]$ and $(B_n/B_n^\alpha)[p]$ if $\alpha < \tau(n)$. By the induction hypothesis, B_n is totally projective. Since B_n is isotype in K and since $K = \bigcup_{n < \omega} B_n$, Theorem 1 implies that K is totally projective.

We should remark that an alternate proof of Theorem 4 could be obtained from [10, Theorem 6.7].

Although it is not our purpose to do so here, the so-called C_λ -theory initiated by Hill and Megibben [8] and later developed more fully by Megibben [11] could be established now using Theorem 1 to great advantage. In fact a restricted technical version of Theorem 1 (concerning λ -high towers) was an essential requirement in the original development of the C_λ -theory [11].

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