SQUARE INTEGRABLE REPRESENTATIONS OF SEMISIMPLE LIE GROUPS[1]

BY

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ABSTRACT. Let D be a bounded symmetric domain. Let G be the universal covering group of the identity component $A_0(D)$ of the group of all holomorphic diffeomorphisms of D onto itself. In this case, any G-homogeneous vector bundle $E \to D$ admits a natural structure of G-homogeneous holomorphic vector bundles. The vector bundle $E \to D$ must be holomorphically trivial, since D is a Stein manifold. We exhibit explicitly a holomorphic trivialization of $E \to D$ by defining a map $\Phi\colon G \to GL(V)$ (V being the fiber of the vector bundle) which extends the classical "universal factor of automorphy" for the action of $A_0(D)$ on D. Then, we study the space H of all square integrable holomorphic sections of $E \to D$. The natural action of G on H defines a unitary irreducible representation of G. The representations obtained in this way are square integrable over G/Z (Z denotes the center of G) in the sense that the absolute values of their matrix coefficients are in $L_2(G/Z)$.

1. **Introduction.** Now, let G be any connected covering group of $A_0(D)$. Choose a base point $o \in D$ and represent $D = G/K$.

When G admits a finite dimensional faithful representation (in which case G has infinite fundamental group), M. S. Narasimhan and K. Okamoto [9] have shown that most of the discrete classes of G (obtained by Harish-Chandra in [6]) can be realized on spaces of square integrable harmonic forms of type $(0, q)$ with coefficients in homogeneous holomorphic vector bundles on G/K arising from finite dimensional irreducible unitary representations of K.

If we assume that G has a finite dimensional faithful representation we can consider a complex form G_c of G and realize G/K as an open G-orbit in a complex manifold G_c/L where $L = K_c P_+$ (see §2) is a closed complex Lie subgroup of G_c. Homogeneous vector bundles are obtained as associated bundles $E_\tau \to D$ to the principal K-bundle $G \to D$ where τ is a continuous representation of K on a finite dimensional complex vector space V. If τ denotes an irreducible unitary

representation of K, then τ is uniquely extended to a holomorphic representation $\tilde{\tau}$ of L which is trivial on P_+. Then the homogeneous vector bundle $E_\tau \to G/K$ is the restriction of the homogeneous holomorphic vector bundle $E_{\tilde{\tau}} \to G_c/L$ to the open submanifold G/K of G_c/L. In this way $E_\tau \to G/K$ picks up a structure of a homogeneous holomorphic vector bundle.

When we drop the requirement that G have a complex form, we are led to consider the following more general problem.

The existence and uniqueness of homogeneous holomorphic vector bundle structures for homogeneous vector bundles over complex manifolds homogeneous under a Lie group G. This is solved in [10].

In §2 we apply these results to the case when D is a bounded symmetric domain and G is the universal covering group of $A_0(D)$. In this case, the homogeneous vector bundle $E_\tau \to D$ admits a natural structure of G-homogeneous holomorphic vector bundle which is described in Theorem 2.1. In the second part of §2 we are concerned with the problem of constructing explicitly a holomorphic trivialization of $E_\tau \to D$. This is achieved by defining a map $\Phi: G \to Gl(V)$ with the properties stated in Theorem 2.3. Because of this, one is in good position to study the space H of all square integrable holomorphic sections $E_\tau \to D$ (whenever τ is a unitary irreducible representation of K). The natural action of G on H defines a unitary irreducible representation of G (cf. Theorem 3.1). The representations obtained in this way are unitarily equivalent to those studied in [3] and [4]. These representations are square integrable over G/Z, where Z denotes the center of G.

An application of our constructions to the case where G is a connected covering group of $A_0(D)$ implies that the Narasimhan-Okamoto result now extends without change to the case where G has finite center. The extension to an arbitrary G depends only on the generalization of the Harish-Chandra character theory of the discrete series representations, [5], [6] to that of square integrable ones.

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2. Homogeneous holomorphic vector bundles over bounded symmetric domains.
A *bounded domain* is a bounded, open connected subset of the complex vector space C^N, N being a positive integer. A bounded domain D is called *symmetric* if each $z \in D$ is an isolated fixed point of an involutive holomorphic diffeomorphism s_z of D onto itself. If such a diffeomorphism s_z exists it is unique. Let G be a connected *covering group* of the identity component $A_0(D)$ of the group of all holomorphic diffeomorphisms of D onto itself. Then G is transitive on D.

Choose a base point $o \in D$ and represent $D = G/K$. Let σ denote the automorphism $g \to s_0 g s_0$ of $A_0(D)$ and let $\mathfrak{g}_0$ denote the Lie algebra of $A_0(D)$. If we put $\dot\sigma = d\sigma_e$ we have the direct decomposition

$$(2.1) \qquad\qquad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

where $\mathfrak{k}_0$ coincides with the Lie algebra of K and the $+1$ eigenspace of $\dot\sigma$ and $\mathfrak{p}_0$ is the -1 eigenspace of $\dot\sigma$. Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{p}_0$ respectively. We denote by π the canonical projection mapping G onto D. We may identify $\mathfrak{p}_0$ with the real tangent space D_0 of D at the point o by the mapping $Y \to d\pi_e Y_e$ $(Y \in \mathfrak{p}_0)$. We denote by T_z (resp. $\bar{T}_z$) the holomorphic (resp. the antiholomorphic) tangent space of D at z. Put

$$(2.2) \qquad \mathfrak{p}_- = \{Y \in \mathfrak{p}: d\pi_e Y_e \in T_0\}, \qquad \mathfrak{p}_+ = \{Y \in \mathfrak{p}: d\pi_e Y_e \in \bar{T}_0\}.$$

As a corollary of Theorem 3.6 of [10] we have the following result:

2.1. Theorem. *Let τ be a continuous representation of K on a finite dimensional vector space, $E_\tau \to D$ the associated G-homogeneous complex vector bundle. Then $E_\tau \to D$ has a natural structure of G-homogeneous holomorphic vector bundle for which the holomorphic sections s over an open set $U \subset D$ are characterized by*

section: $X|_s + \tau(X)|_s = 0$ *for all* $X \in \mathfrak{k}_0$,
holomorphic: $X|_s = 0$ *for all* $X \in \mathfrak{p}_+$.

Proof. In [10] we have defined a complex subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ by

$$\mathfrak{l} = \{X \in \mathfrak{g}: X_0 = 0\}.$$

From the definitions it follows easily that $\mathfrak{l} = \mathfrak{k} + \mathfrak{p}_+$. Since $\mathfrak{l}$ is invariant under the complex linear extension of $\dot\sigma$ it follows that $\mathfrak{p}_+ = \mathfrak{l} \cap \mathfrak{p}$ and therefore $[\mathfrak{k}, \mathfrak{p}_+] \subset [\mathfrak{k}, \mathfrak{p}] \cap \mathfrak{l} \subset \mathfrak{p} \cap \mathfrak{l} = \mathfrak{p}_+$. Now we want to prove that $\mathfrak{p}_+$ is an abelian ideal of $\mathfrak{l}$. Let J_0 denote the complex structure of D_0 and let $\mathfrak{k}_0^*$ denote the Lie algebra of the linear isotropy group K^* corresponding to K. Then J_0 belongs to the center c_0 of $\mathfrak{k}_0^*$ (see [7]). Identifying the Lie algebras $\mathfrak{k}_0$ and $\mathfrak{k}_0^*$, for $X, Y \in \mathfrak{p}_+$ we have

$$0 = [J_0, [X, Y]] = [[J_0, X], Y] + [X, [J_0, Y]] = -2i[X, Y]$$

since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. This shows that $[\mathfrak{p}_+, \mathfrak{p}_+] = 0$.

Since K is connected there is a unique extension λ of τ from K to $\mathfrak{l}$ such that $\lambda|\mathfrak{p}_+ = 0$. Now Theorem 3.6 of [10] provides the required G-homogeneous holomorphic structure for $E_\tau \to D$. This completes the proof of the theorem.

Let M be a connected complex manifold with a Hermitian structure; M is said to be a *Hermitian symmetric space* if each point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry s_p of M.

For any bounded domain D the Bergman kernel function of D defines a Kählerian structure on D, i.e. a Hermitian structure such that the almost complex structure J of D is invariant under parallelism. With respect to this structure any holomorphic diffeomorphism of D onto itself is an isometry, therefore any bounded symmetric domain is a Hermitian symmetric space. Moreover, D is a Hermitian symmetric space of noncompact type, which means that the Lie algebra g_0 of $A_0(D)$ is semisimple without compact factors and that (2.1) is a Cartan decomposition of g_0.

Conversely, if M is a Hermitian symmetric space of noncompact type, then there exist a bounded symmetric domain D and a holomorphic diffeomorphism of M onto D. We need to recall the Harish-Chandra realization of M as a bounded symmetric domain D in order to exhibit explicitly a holomorphic trivialization of $E_\tau \to D$. See [7] for details.

As before, let $A_0(M)$ be the identity component of the group of all isometries of M. Let g_0 denote its Lie algebra and let

$$g_0 = \mathfrak{k}_0 + \mathfrak{p}_0$$

be the Cartan decomposition of g_0 which arises from the symmetry with respect to some point o in M. Let c_0 be the center of $\mathfrak{k}_0$ and let $\mathfrak{h}_0$ be some maximal abelian subalgebra of $\mathfrak{k}_0$. Then $c_0 \subset \mathfrak{h}_0$ and $\mathfrak{h}_0$ is a maximal abelian subalgebra of g_0.

Let g be the complexification of g_0 and let c, $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{p}$ be the complex subspaces of g spanned by c_0, $\mathfrak{h}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$. Now $\mathfrak{h}$ is a Cartan subalgebra of g. Let Δ denote the set of nonzero roots of g with respect to $\mathfrak{h}$. Since $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}$ and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$, it is clear that either a root subspace $g^\alpha \subset \mathfrak{k}$ or $g^\alpha \subset \mathfrak{p}$ ($\alpha \in \Delta$). A root $\alpha \in \Delta$ is called *compact* or *noncompact* according to $g^\alpha \subset \mathfrak{k}$ or $g^\alpha \subset \mathfrak{p}$. Moreover, $[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+$ (see (2.2)), therefore there exists a subset $P_n \subset \Delta$ such that

$$\mathfrak{p}_+ = \sum_{\alpha \in P_n} g^\alpha.$$

Since $\mathfrak{p}_+ = \nu_0(\mathfrak{p}_-)$ and $\nu_0(g^\alpha) = g^{-\alpha}$ ($\alpha \in \Delta$) where ν_0 denotes the conjugation of g with respect to g_0, we see that $\mathfrak{p}_- = \Sigma_{\alpha \in P_n} g^{-\alpha}$.

We can introduce a linear ordering in the dual of the real vector space $i\mathfrak{h}_0$ such that the set P of all positive roots contains P_n. We choose an ordering with this property and whenever necessary we will be referring to it.

Let G_c denote the simply connected Lie group with Lie algebra $\mathfrak{g}$. Let P_-, P_+, K_c denote the analytic subgroups of G_c corresponding to the subalgebras $\mathfrak{p}_-$, $\mathfrak{p}_+$, and $\mathfrak{k}$, respectively. ($\mathfrak{p}_-$ and $\mathfrak{p}_+$ are abelian subspaces of $\mathfrak{p}$, cf. the proof of Theorem 2.1.) Let G_0, K_0 denote the analytic subgroups of G_c corresponding to $\mathfrak{g}_0$ and $\mathfrak{k}_0$, respectively. The mapping $(q, k, p) \to qkp$ is a holomorphic diffeomorphism of $P_- \times K_c \times P_+$ onto an open submanifold of G_c, containing G_0. For $x \in G_0$, let $\zeta(x)$ denote the unique element in P_- such that $x \in \zeta(x)K_cP_+$. P_- and P_+ are simply connected abelian Lie groups; let $\log: P_- \to \mathfrak{p}_-$ be the inverse of the exponential mapping. The mapping

$$xK_0 \to \log \zeta(x), \qquad x \in G_0,$$

is a holomorphic diffeomorphism of G_0/K_0 onto a bounded domain D in the complex vector space $\mathfrak{p}_-$.

From now on, G will denote the *universal covering group of* $A_0(M)$, and K will denote the isotropy subgroup of G at $o \in M$. Then we can make the identification $M = G/K$. Let $g \to \bar{g}$, $g \in G$, be the *covering homomorphism* $G \to G_0$. The mapping

$$\psi: gK \to \log \zeta(\bar{g}), \qquad g \in G,$$

is, therefore, a holomorphic diffeomorphism of G/K onto D.

Let τ be a representation of K on the finite dimensional complex vector space V. We want to define a holomorphic equivalence between the holomorphic vector bundles $E_\tau \to G/K$ and the trivial vector bundle $D \times V \to D$. This amounts to defining a holomorphic diffeomorphism Ψ such that the diagram

(2.3)
$$
\begin{array}{ccc}
G \times_\tau V & \xrightarrow{\Psi} & D \times V \\
\downarrow & & \downarrow \\
G/K & \xrightarrow{\psi} & D
\end{array}
$$

commutes and induces linear isomorphisms between fibers. To define such a map Ψ is equivalent to defining a map $\Phi: G \to \mathrm{Gl}(V)$ such that

$$\Phi(gk) = \Phi(g)\tau(k) \quad \text{for all } g \in G; \text{ all } k \in K;$$

Ψ and Φ being related by

(2.4)
$$\Psi[g, v] = (\psi(gK), \Phi(g)v), \qquad g \in G, v \in V.$$

To motivate our definition of the map Φ in the general case let us consider first the following example: The holomorphic tangent space of G/K at eK has been identified with $\mathfrak{p}_-$, and for every $k \in K$ the differential of the map $gK \to kgK$ at eK corresponds to the automorphism of $\mathfrak{p}_-$ induced by $\mathrm{Ad}_G(k)$. This

implies that the holomorphic tangent bundle of G/K is the homogeneous vector bundle E_τ, where τ denotes the representation of K on $\mathfrak{p}_-$ induced by Ad_G. Let $\rho(g)$ denote the action of G on D given by

$$\rho(g)\psi(xK) = \psi(gxK), \qquad g, x \in G.$$

Since the tangent bundle of D is already presented as a trivial vector bundle we just have to compute the differential of $\rho(g)$ at the origin 0.

2.2. **Lemma** (cf. Lemma 1.9 of [1]). *Let $\kappa\colon G \to K_c$ be defined by $\bar{g} \in P_-\kappa(g)P_+$. Then*

$$d\rho(g)_0 = \mathrm{Ad}_{G_c}(\kappa(g))|\mathfrak{p}_- \quad \textit{for all } g \in G.$$

Proof. Given $X \in \mathfrak{p}_-$, by definition we have

$$\rho(g)(tX) = \log\left(\zeta(\bar{g}\,\exp tX)\right),$$

for a sufficiently small t. Let $\bar{g} = q\kappa(g)p$ with $q \in P_-$ and $p \in P_+$, then

$$\bar{g}\,\exp tX = q\kappa(g)p\,\exp tX = q\kappa(g)(\exp t\,\mathrm{Ad}(p)X)p.$$

Since $[\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm$ and $[\mathfrak{p}_\pm, \mathfrak{p}_\pm] = \{0\}$, for $Y \in \mathfrak{p}_+$ we have

$$\mathrm{Ad}(\exp Y)X = X + [Y, X] + \tfrac{1}{2}[Y, [Y, X]].$$

Therefore we can write

$$\mathrm{Ad}(p)X = X + Z \quad \text{with } Z \in \mathfrak{k} + \mathfrak{p}_+.$$

Putting these things together we obtain

$$
\begin{aligned}
d\rho(g)_0 X &= \left[d/dt\,\log\left(\zeta(\bar{g}\,\exp tX)\right)\right]_{t=0} \\
&= \left[d/dt\,\log\left(\zeta(q\kappa(g)(\exp t(X + Z))p)\right)\right]_{t=0} \\
&= \left[d/dt\,\log\left(\zeta(q(\exp t\,\mathrm{Ad}(\kappa(g))X\,\exp t\,\mathrm{Ad}(\kappa(g))Z)\kappa(g)p)\right)\right]_{t=0} \\
&= \left[d/dt\,\log\left(q\,\exp t\,\mathrm{Ad}(\kappa(g))X\right)\right]_{t=0} = \mathrm{Ad}(\kappa(g))X,
\end{aligned}
$$

where we have used the fact that $K_c P_+$ is a subgroup of G_c.

Lemma 2.2 tells us how we can define a holomorphic equivalence between the holomorphic tangent bundle E_τ of G/K and the trivial bundle $D \times \mathfrak{p}_- \to D$. We can just take as the map $\Phi\colon G \to \mathrm{Gl}(\mathfrak{p}_-)$ of (2.4) the map defined by

$$\Phi(g) = \mathrm{Ad}_{G_c}(\kappa(g))|\mathfrak{p}_-, \qquad g \in G.$$

The situation in this example is particularly simple because the representation τ of K factors thru the restriction to K_0 of a representation of K_c. In the general case one has to be more careful.

The Lie algebra $\mathfrak{k}_0$ being the Lie algebra of a compact Lie group, e.g. K_0, is the direct sum $\mathfrak{k}_0 = c_0 + [\mathfrak{k}_0, \mathfrak{k}_0]$ of its center and its derived algebra $\mathfrak{k}_0' = [\mathfrak{k}_0, \mathfrak{k}_0]$. If K' denotes the analytic subgroup of K which corresponds to $\mathfrak{k}_0'$, then K' is a compact, simply connected semisimple Lie group. Let C be the analytic subgroup of K with Lie algebra c_0. Then C is a simply connected abelian Lie group and $K = C \times K'$ (direct product). Let $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$ be the derived algebra of $\mathfrak{k}$ and let $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{k}'$. Then $\mathfrak{h} = c + \mathfrak{h}'$. Since the compact roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ can be identified with the roots of $\mathfrak{k}'$ with respect to $\mathfrak{h}'$, it follows that a linear functional Λ on $\mathfrak{h}$ coincides on $\mathfrak{h}'$ with the highest weight of some irreducible representation of K' if and only if $2\Lambda(H_\alpha)/\alpha(H_\alpha)$ is a nonnegative integer for every compact positive root α (H_α is the unique element in $\mathfrak{h}$ such that $B(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$; B denotes the Killing form on $\mathfrak{g} \times \mathfrak{g}$). Therefore the irreducible representations of K are in one-to-one correspondence $\Lambda \leftrightarrow \tau_\Lambda$ with the linear functionals Λ on $\mathfrak{h}$ such that $2\Lambda(H_\alpha)/\alpha(H_\alpha)$ is a nonnegative integer for every compact positive root α.

Let $\alpha_1, \cdots, \alpha_r$ be the set of all simple roots of $\mathfrak{g}$; here we are referring to the set P of all positive roots introduced before. Assume that $\alpha_1, \cdots, \alpha_t$ ($t \leq r$) are all the noncompact positive roots among these. Define a linear functional Λ_0 on $\mathfrak{h}$ by

$$\Lambda_0(H_{\alpha_i}) = 0, \qquad 1 \leq i \leq t,$$

$$\Lambda_0(H_{\alpha_i}) = \Lambda(H_{\alpha_i}), \qquad t < i \leq r.$$

Now, there exists a finite dimensional irreducible representation τ of G_c with highest weight Λ_0. Let v be a nonzero highest weight vector. Let V denote the linear span of $\{\tau(k)v: k \in K_c\}$; V is a K_c-invariant subspace. Let $\tau_{\Lambda_0}(k) = \tau(k)|V$, for all $k \in K_c$, then (V, τ_{Λ_0}) is an irreducible K_c-module with highest weight Λ_0. Let τ_Λ be the representation of K on V defined by

$$(2.5) \qquad \tau_\Lambda(k) = e^{\lambda(H)}\tau_{\Lambda_0}(\bar{k}), \qquad k \in (\exp H)K', \; H \in c_0,$$

where $\lambda = \Lambda - \Lambda_0$. We claim that τ_Λ is an irreducible representation of K on V with highest weight Λ. In fact, it is an irreducible representation, because τ_{Λ_0} is irreducible, and, if $H \in c_0$,

$$\dot{\tau}_\Lambda(H) = [d/dt \; e^{\lambda(tH)}\tau_{\Lambda_0}(\exp_{K_0} tH)]_{t=0} = \lambda(H) + \dot{\tau}_{\Lambda_0}(H),$$

if $X \in \mathfrak{k}_0'$,

$$\dot{\tau}_\Lambda(X) = [d/dt \; \tau_{\Lambda_0}(\exp_{K_0} tX)]_{t=0} = \dot{\tau}_{\Lambda_0}(X).$$

Therefore

$$\dot{\tau}_\Lambda(H)v = (\lambda + \Lambda_0)(H)v = \Lambda(H)v \quad \text{if } H \in c_0,$$

$$\dot{\tau}_\Lambda(H)v = \Lambda_0(H)v = \Lambda(H)v \qquad \text{if } H \in \mathfrak{h}_0' = \mathfrak{h}_0 \cap \mathfrak{k}_0',$$

hence, the highest weight of τ_Λ is Λ.

We recall that we have defined the map $\kappa\colon G \to K_c$ by

$$\bar{g} = q\kappa(g)p, \qquad q \in P_-, \ \kappa(g) \in K_c, \ p \in P_+.$$

Let $\tilde{K}_c$ denote the universal covering group of K_c. Since G is simply connected the map κ can be lifted, in a unique way, to a map $\tilde{\kappa}\colon G \to \tilde{K}_c$ such that $\tilde{\kappa}(e) = e$ and making the following diagram commutative:

$$
\begin{array}{c}
\tilde{K}_c \\
{\scriptstyle \tilde{\kappa}}\nearrow \quad \downarrow \\
G \underset{\kappa}{\to} K_c.
\end{array}
$$

We have the direct decomposition $\mathfrak{k} = c + \mathfrak{k}'$, hence, if $\tilde{K}_c'$ and $\tilde{C}$ denote the analytic subgroups of $\tilde{K}_c$ which correspond to the subalgebras $\mathfrak{k}'$ and c respectively we have that $\tilde{K}_c = \tilde{C} \times \tilde{K}_c'$ (direct product). Let us define a map $\gamma\colon G \to c$ by

$$\tilde{\kappa}(g) \in \exp(\gamma(g))\tilde{K}_c', \quad \text{for all } g \in G.$$

Now we are in a position to define our desired map $\Phi\colon G \to \mathrm{Gl}(V)$. Let

$$(2.6) \qquad\qquad \Phi(g) = e^{\lambda(\gamma(g))}\tau_{\Lambda_0}(\kappa(g)), \ g \in G.$$

We want to point out the following properties of the map Φ.

2.3. Theorem. (i) $\Phi(e) = I$ *identity transformation of* V.

(ii) $\Phi(k_1 g k_2) = \tau_\Lambda(k_1)\Phi(g)\tau_\Lambda(k_2)$ *for all* $g \in G$; $k_1, k_2 \in K$.

(iii) $X\Phi = 0$ *for all* $X \in \mathfrak{p}_+$.

Moreover any continuously differentiable function $\Phi_1\colon G \to L(V)$ *which satisfies properties* (i), (ii) *and* (iii) *coincides with the function* Φ.

Before giving the proof of Theorem 2.3 we will state and prove the following lemma.

2.4. Lemma. (i) $\kappa(k_1 g k_2) = \bar{k}_1 \kappa(g)\bar{k}_2$ *for all* $g \in G$; $k_1, k_2 \in K$.

(ii) $d\kappa_g(X) = 0$ *for all* $g \in G$; $X \in \mathfrak{p}_+$.

Proof. For $g \in G$, let $\bar{g} = q\kappa(g)p$ where $q \in P_-$, $p \in P_+$. Given $k_1, k_2 \in K$, write

$$\overline{k_1 g k_2} = \overline{k}_1 \overline{g} \overline{k}_2 = \overline{k}_1 q \overline{k}_1^{-1} \overline{k}_1 \kappa(g) \overline{k}_2 \overline{k}_2^{-1} p \overline{k}_2.$$

Since $kP_{\pm}k^{-1} = P_{\pm}$ for all $k \in K_c$, we conclude that $\kappa(k_1 g k_2) = \overline{k}_1 \kappa(g) \overline{k}_2$. To prove (ii) it is convenient to introduce the following notation: let $\alpha: G \to G_c$ be the map defined by $\alpha(g) = \overline{g}$ and let $\beta: P_- K_c P_+ \to K_c$ be the map defined by $\beta(qkp) = k$ whenever $q \in P_-$, $k \in K_c$ and $p \in P_+$. Suppose $X_1, X_2 \in \mathfrak{g}_0$, then

$$da_g(X_1 + iX_2) = [d/dt\ \alpha(g \exp tX_1)]_{t=0} + i[d/dt\ \alpha(g \exp tX_2)]_{t=0}$$

$$= [d/dt\ (\alpha(g) \exp_{G_c} tX_1)]_{t=0} + i[d/dt\ (\alpha(g) \exp_{G_c} tX_2)]_{t=0}$$

$$= [d/dt\ (\alpha(g) \exp_{G_c} t(X_1 + iX_2))]_{t=0}.$$

Now

$$d\kappa_g(X_1 + iX_2) = d\beta da_g(X_1 + iX_2)$$

$$= d\beta[d/dt\ (\alpha(g) \exp_{G_c} t(X_1 + iX_2))]_{t=0}$$

$$= [d/dt\ (\beta(\alpha(g) \exp_{G_c} t(X_1 + iX_2)))]_{t=0}.$$

If $X_1 + iX_2 \in \mathfrak{p}_+$ then

$$\beta(\alpha(g) \exp_{G_c} t(X_1 + iX_2)) = \beta(q\kappa(g)p \exp_{G_c}(X_1 + iX_2)) = \kappa(g),$$

since P_+ is a group, and therefore $d\kappa_g(X_1 + iX_2) = 0$.

Proof of Theorem 2.3. From part (i) of Lemma 2.4 and the uniqueness of path lifting it follows that $\tilde{\kappa}(k_1 g k_2) = \tilde{\kappa}(k_1)\tilde{\kappa}(g)\tilde{\kappa}(k_2)$ for all $g \in G$; $k_1, k_2 \in K$; which in turns implies that

$$\gamma(k_1 g k_2) = \gamma(k_1) + \gamma(g) + \gamma(k_2) \quad \text{for all } g \in G;\ k_1, k_2 \in K.$$

Also, we need to observe that if $k = (\exp H)k'$, $H \in \mathfrak{c}_0$ and $k' \in K'$, then the uniqueness of path lifting implies that $\tilde{\kappa}(k) = \tilde{\kappa}(\exp H)\tilde{\kappa}(k')$, and moreover that $\tilde{\kappa}(k') \in K_c'$ and $\tilde{\kappa}(\exp H) = \exp_{\tilde{K}_c}(H)$. Hence $\gamma(k) = H$ if $k \in (\exp H)K'$, $H \in \mathfrak{c}_0$.

Therefore, going back to the definition (2.6) and using part (i) of Lemma 2.4 again, we have

$$\Phi(k_1 g k_2) = e^{\lambda(\gamma(k_1 g k_2))} \tau_{\Lambda_0}(\kappa(k_1 g k_2))$$

$$= e^{\lambda(\gamma(k_1))} \tau_{\Lambda_0}(\overline{k}_1)\Phi(g)e^{\lambda(\gamma(k_2))}\tau_{\Lambda_0}(\overline{k}_2)$$

$$= e^{\lambda(H_1)} \tau_{\Lambda_0}(\overline{k}_1)\Phi(g)e^{\lambda(H_2)}\tau_{\Lambda_0}(\overline{k}_2)$$

if $k_j \in (\exp H_j)K'$, $H_j \in \mathfrak{c}_0$, $j = 1, 2$. By taking a look at (2.5) we realize that part (ii) of the theorem is proved. We will prove now that $X\Phi = 0$ for all $X \in \mathfrak{p}_+$. Let $\rho: \tilde{K}_c \to K_c$ denote the covering projection. Then

$$d\rho(d\tilde{\kappa}_g(X)) = d\kappa_g(X) = 0 \quad \text{for all } g \in G, \ X \in \mathfrak{p}_+.$$

Since $d\rho$ is an isomorphism, it follows that $d\tilde{\kappa}_g(X) = 0$ for all $X \in \mathfrak{p}_+$. Let δ: $\tilde{K}_c \to c$ denote the map defined by $\beta((\exp H)k) = H$ $(H \in c, k \in \tilde{K}'_c)$. Then

$$d\gamma_g(X) = d\beta(d\tilde{\kappa}_g(X)) = 0, \quad \text{for all } g \in G, \ X \in \mathfrak{p}_+.$$

Therefore,

$$[X\Phi](g) = \lambda(d\gamma_g(X))\Phi(g) + e^{\lambda(\gamma(g))}(d\tau_{\Lambda_0})_{\kappa(g)}d\kappa_g(X) = 0$$

for all $g \in G$, $X \in \mathfrak{p}_+$.

Finally, we have to prove the uniqueness of the map Φ. Let $\Phi_1 \colon G \to L(V)$ ($L(V)$ denotes the space of all linear maps of V into V) be any continuously differentiable map with the properties (i), (ii) and (iii) of the Theorem 2.3. Then the formula

$$(2.7) \qquad F(\psi(gK)) = \phi_1(g)\Phi(g)^{-1}$$

defines a continuously differentiable map $F \colon D \to L(V)$ such that

$$(2.8) \qquad F(k \cdot z) = \tau_\Lambda(k)F(z)\tau_\Lambda(k)^{-1} \quad \text{for all } k \in K; \ z \in D.$$

In (2.2) we have identified $\mathfrak{p}_-$ (resp. $\mathfrak{p}_+$) with the holomorphic (resp. antiholomorphic) tangent space of D at 0, via the canonical projection $\pi \colon G \to D$. Then, since the holomorphic structure of D is G-invariant, we have $d\pi_g X_g \in T_{\pi(g)}$ (resp. $d\pi_g X_g \in \overline{T}_{\pi(g)}$) if $X \in \mathfrak{p}_-$ (resp. $X \in \mathfrak{p}_+$). Thus a continuously differentiable vector valued function on D is holomorphic if and only if $(d\pi_g X_g)f = 0$ for all $g \in G$, $X \in \mathfrak{p}_+$.

From (2.7) it follows that

$$(d\pi_g X_g)F = [X\Phi_1](g)\Phi(g)^{-1} - \Phi_1(g)\Phi(g)^{-1}[X\Phi](g)\Phi^{-1}(g)$$

which shows that F is holomorphic since $X\Phi = X\Phi_1 = 0$ for all $X \in \mathfrak{p}_+$.

Let us point out that the action of K on D is given by

$$k \cdot z = \text{Ad}(k)z \quad \text{for all } k \in K, \ z \in D.$$

Let C_0 be the analytic subgroup of K_0 with Lie algebra c_0. Then C_0 is a torus. If we decompose $\mathfrak{p}_-$ into irreducible submodules under the action of C_0, the characters which thus appear are nontrivial, since they correspond to the noncompact simple roots which have a nonzero restriction to c_0. Therefore, if f is a holomorphic vector valued function on D then the Cauchy integral formula tells us that

$$(2.9) \qquad f(0) = \int_{C_0} f(\text{Ad}(b)z)\,db \quad \text{for all } z \in D$$

where db denotes the normalized Haar measure of C_0.

We apply (2.9) to the holomorphic function F defined in (2.7):

$$F(0) = \int_{C_0} F(\mathrm{Ad}\,(b)z)\,db = \int_{C_0} F(z)\,db = F(z) \quad \text{for all } z \in D;$$

here we have used (2.8) and the fact that $\tau_\Lambda(k)$ is a scalar multiple of the identity for every k in the center of K. Thus we have proved that $F(z) = F(0) = \Phi_1(e)\Phi(e)^{-1}$ $= I$ (identity of $\mathrm{Gl}(V)$) for all $z \in D$ and therefore $\Phi_1(g) = \Phi(g)$ for all $g \in G$. This completes the proof of the Theorem 2.3.

The map $\Phi\colon G \to \mathrm{Gl}(V)$ defined in (2.6), and since property (ii) of Theorem 2.3 holds, certainly defines a C^∞-equivalence Ψ (see 2.4) between the homogeneous holomorphic vector bundle $E_\Lambda \to G/K$ induced by the irreducible representation τ_Λ of K on V, and the trivial vector bundle $D \times V \to D$. Therefore, the pair (ψ, Ψ) induced a bijection $s \leftrightarrow s'$ between C^∞-sections of $E_\Lambda \to G/K$ and those of $D \times V \to D$. If we define the functions $f_s\colon G/K \to V$ and $f_{s'}\colon D \to V$ by

$$s(gK) = [g, f_s(g)] \quad \text{and} \quad s'(z) = (z, f_{s'}(z)) \quad \text{for all } g \in G,\ z \in D$$

then f_s and $f_{s'}$ are related by

(2.10) $$f_{s'}(\psi(gK)) = \Phi(g)f_s(g) \quad \text{for } s \leftrightarrow s' \text{ and all } g \in G.$$

According to Theorem 2.1 a C^∞-section s of $E_\Lambda \to G/K$ is holomorphic if and only if $Xf_s = 0$ for all $X \in \mathfrak{p}_+$. From (2.10) it follows that

$$(d\pi_g X_g)f_{s'} = [X\Phi](g)f_s(g) + \Phi(g)[Xf_s](g).$$

Part (iii) of Theorem 2.3 implies now that $[Xf_s](g) = 0$ if and only if $(d\pi_g X_g)f_{s'}$ $= 0$. In other words, the bijection $s \leftrightarrow s'$ restricts to a bijection between the holomorphic sections of $E_\Lambda \to G/K$ and the holomorphic sections of $D \times V \to D$. Hence Ψ is holomorphic.

As an application of the uniqueness part of Theorem 2.3, we will point out here a functional equation satisfied by the map $\Phi\colon G \to \mathrm{Gl}(V)$ defined in (2.6).

Let τ_Λ be an irreducible *unitary* representation of K on V and let $\chi_\Lambda\colon K \to C$ denote its character. Let $\Phi\colon G \to \mathrm{Gl}(V)$ denote the unique map satisfying properties (i), (ii) and (iii) of Theorem 3.3. Then $\overline{\chi_\Lambda(k)}\Phi(xkg)$ depends only on $\bar{k}$. Let $d\bar{k}$ denote the Haar measure on K_0 normalized by $\int_{K_0} d\bar{k} = 1$.

2.5. Theorem.

(2.11) $$(\dim V) \int_{K_0} \overline{\chi_\Lambda(k)}\,\Phi(xkg)\,d\bar{k} = \Phi(x)\Phi(g) \quad \textit{for all } x, g \in G.$$

Proof. Fix $x \in G$. Let

$$F(g) = (\dim V)\left(\int_{K_0} \overline{\chi_\Lambda(k)} \Phi(xkg)\, d\bar{k}\right) \Phi(g)^{-1}.$$

Then it is easy to check that

$$F(bgk) = F(g) \quad \text{for all } g \in G,\ k \in K,\ b \in \text{center of } K,$$
$$XF = 0 \qquad \text{for all } X \in \mathfrak{p}_+.$$

These were the two properties we used in Theorem 2.3 to conclude that $F(g) = F(e)$ for all $g \in G$. Now we compute

$$F(e) = (\dim V)\Phi(x) \int_{K_0} \overline{\chi_\Lambda(k)} r_\Lambda(k)\, d\bar{k}$$

$$= (\dim V)\Phi(x) \int_{K_0} \overline{\chi_{\Lambda_0}(k)} r_{\Lambda_0}(k)\, dk = \Phi(x)$$

(see 2.5), as a consequence of Schur orthogonality relations for the irreducible representation r_{Λ_0} of the compact group K_0 on V. This completes the proof of the theorem.

Let G be a connected Lie group, K a compact subgroup. A *spherical function* of the pair (G, K) is a complex valued continuous function f on G, not identically 0 such that

$$(2.12) \qquad \int_K f(xkg)\, dk = f(x)f(g).$$

Spherical functions have the properties (a) $f(e) = 1$ and (b) $f(k_1 g k_2) = f(g)$ for all $g \in G$; $k_1, k_2 \in K$. If we recall the properties (i) and (ii) of Theorem 2.3 the connection between (2.11) and (2.12) is transparent.

3. **Square integrable holomorphic sections.** In this chapter we will be referring to the situation and notation introduced in §2. In particular, $M = G/K$ denotes a Hermitian symmetric space of noncompact type, and G denotes the universal covering group of $A_0(M)$. Λ denotes the highest weight of an irreducible representation r_Λ of K on V, and $E_\Lambda \to G/K$ denotes the homogeneous holomorphic vector bundle over G/K induced by r_Λ (cf. Theorem 2.1).

To define a G-invariant Hermitian structure on $E_\Lambda \to G/K$ it is necessary and sufficient to have a K-invariant inner product on V. The one-to-one correspondence between these structures is given by

$$(3.1) \qquad ([g, v], [g, v']) = (v, v') \quad \text{for all } [g, v], [g, v'] \in E_\Lambda.$$

For an irreducible representation r_Λ of K on V we can define a K-invariant inner product on V if and only if Λ is a *real* linear functional on $\mathfrak{h}$, i.e. $\Lambda(H_\alpha)$ is a real number for all $\alpha \in \Delta$. Thus, from now on Λ will denote a real linear

functional on $\mathfrak{h}$ such that $2\Lambda(H_\alpha)/\alpha(H_\alpha)$ is a nonnegative integer for every compact positive root α. We choose a K-invariant inner product $(\ ,\ )$ on V, which is unique up to a multiplicative constant, and we introduce the corresponding G-invariant Hermitian structure on $E_\Lambda \rightarrow G/K$ (see 3.1).

Let H_Λ denote the space of all holomorphic sections s of $E_\Lambda \rightarrow G/K$ such that

$$(3.2) \qquad \int_{G/K} (s(x), s(x))\, dx < \infty,$$

dx denotes the volume element of the Hermitian symmetric space G/K. It is not difficult to show that H_Λ is a Hilbert space with the norm (3.2). Moreover the natural action of G on the sections of $E_\Lambda \rightarrow G/K$, given by

$$(gs)(x) = gs(g^{-1}x), \qquad g \in G,\ x \in G/K,$$

defines a unitary representation π_Λ of G on H_Λ.

By making use of the holomorphic trivialization (2.3) defined by (2.4) and (2.6) the space H_Λ can be identified with the space of all holomorphic functions $f: D \rightarrow V$ such that

$$\int_D (f(z), f(z))_z\, dz < \infty,$$

where $(v, w)_z = (\Phi(g)^{-1}v, \Phi(g)^{-1}w)$, $z = g \cdot 0$, $v, w \in V$, and dz is the volume element of D corresponding to dx under the map ψ. The action of G is given by

$$(3.3) \qquad (uf)(g \cdot 0) = \Phi(g)\Phi(u^{-1}g)^{-1} f(u^{-1}g \cdot 0), \qquad u, g \in G,$$

and in particular for $k \in K$ we have

$$(3.4) \qquad (kf)(z) = \tau_\Lambda(k)f(\mathrm{Ad}\,(k)^{-1}z), \qquad z \in D,$$

since the action of K on D is given by the adjoint representation of K on $\mathfrak{p}_-$ restricted to D; see also Theorem 2.3.

3.1. Theorem (cf. Theorem 2 of [2]). *The representation π_Λ of G on H_Λ is irreducible. Moreover, $H_\Lambda \neq \{0\}$ if and only if H_Λ contains the constant functions.*

Proof. Let H be a nonzero closed invariant subspace of H_Λ. Take a function $f \in H$ such that $f(0) \neq 0$. For a fixed $b \in C_0$ the formula (3.4) shows that the function $z \rightarrow f(\mathrm{Ad}\,(b)z)$ is also in H, and therefore the constant function $z \rightarrow f(0)$ belongs to H (see 2.9). Thus any two nonzero closed invariant subspaces of H_Λ have a nonzero intersection. Since π_Λ is unitary, the theorem is completely proved.

Theorem 3.1 leads us to consider the integral

(3.5) $\int_D (v, v)_z \, dz = \int_{G_0} (\Phi(g)^{-1}v, \Phi(g)^{-1}v) \, d\bar{g}, \qquad v \in V;$

note that $(\Phi(g)^{-1}v, \Phi(g)^{-1}v)$ depends only on $\bar{g}$. To compute this, one transforms the integration over G_0 as follows. (Cf. [7].) Let $a_{\mathfrak{p}_0}$ be a maximal abelian subspace of $\mathfrak{p}_0$ and extend $a_{\mathfrak{p}_0}$ to a maximal abelian subalgebra a_0 of $\mathfrak{g}_0$. Let a be the subalgebra of $\mathfrak{g}$ generated by a_0. Then a is a Cartan subalgebra of $\mathfrak{g}$ and let $a^* = \Sigma \, RH_\alpha$ where α runs over the set of all nonzero roots of $\mathfrak{g}$ with respect to a. Then $a_{\mathfrak{p}_0} \subset a^*$ and compatible orderings can be introduced in the dual spaces of $a_{\mathfrak{p}_0}$ and a^*. Let Q denote the set of positive roots which do not vanish identically on $a_{\mathfrak{p}_0}$. The Killing form of $\mathfrak{g}_0$ gives rise to a Euclidean metric on $a_{\mathfrak{p}_0}$. Let dH denote the Riemannian measure on $a_{\mathfrak{p}_0}$ induced by this metric. Then if f is a continuous function with compact support on G_0

(3.6) $\int_{G_0} f(y) \, dy = \int_{a_{\mathfrak{p}_0}} \left| \prod_{\alpha \in Q} \sinh \alpha(H) \right| dH \int_{K_0 \times K_0} f(k_1 \exp H k_2) \, dk_1 \, dk_2$

(dy and dk suitable normalized Haar measures).

In the special situation here, it is possible to select $a_{\mathfrak{p}_0}$ with particular reference to Δ. In fact one can find noncompact positive roots $\gamma_1, \cdots, \gamma_s$ such that $\gamma_i \pm \gamma_j$ is never a root for $i \neq j$ and the real subspace $a_{\mathfrak{p}_0}$ generated by $X_{\gamma_i} + X_{-\gamma_i}$ is a maximal abelian subspace of $\mathfrak{p}_0$ (see [7]).

In order to reduce the problem of computing (3.5) to an integral computed in [4], we will consider the homogeneous holomorphic vector bundle $E_{\Lambda^*} \to G/K$ associated to the *contragredient* representation of τ_Λ instead of $E_\Lambda \to G/K$.

Let V^* denote the dual space of V. The map $v \to 1_v$ defined by $1_v(w) = (w, v)$ defines a real isomorphism of V onto V^*. We can define an inner product on V^* by

$$(1_v, 1_w) = (w, v), \qquad v, w \in V.$$

Given a linear transformation $T: V \to V$ then $T^*: V \to V$ will denote the adjoint of T and $^tT: V^* \to V^*$ will denote the transpose transformation. One can easily check that $^tT 1_v = 1_{T^* v}$. The map $\Phi^*: G \to \mathrm{Gl}(V^*)$ corresponding to the contragredient representation of τ_Λ (see 2.6) is given by

$$\Phi^*(g) = e^{-\lambda(\gamma(g))} \, {}^t\tau_{\Lambda_0}(\kappa(g))^{-1}, \qquad g \in G,$$

as one can be convinced by using the uniqueness part of Theorem 2.3.

The function we now want to integrate over G_0 is

$$(\Phi^*(g)^{-1}1_v, \Phi^*(g)^{-1}1_v) = |e^{\lambda(\gamma(g))}|^2({}^t\tau_{\Lambda_0}(\kappa(g))1_v, {}^t\tau_{\Lambda_0}(\kappa(g))1_v)$$

$$= |e^{\lambda(\gamma(g))}|^2({}^1\tau_{\Lambda_0}(\kappa(g))^*v, {}^1\tau_{\Lambda_0}(\kappa(g))^*v)$$

$$= |e^{\lambda(\gamma(g))}|^2(\tau_{\Lambda_0}(\kappa(g))^*v, \tau_{\Lambda_0}(\kappa(g))^*v), \quad g \in G.$$

For $k_1, k_2 \in K$ and $H \in \mathfrak{a}_{\mathfrak{p}_0}$ we have

$$(\Phi^*(k_1 \exp H k_2)^{-1}1_v, \Phi^*(k_1 \exp H k_2)^{-1}1_v)$$

$$= |e^{\lambda(\gamma(\exp H))}|^2(\tau_{\Lambda_0}(\kappa(\exp H))^*\tau_{\Lambda_0}(\bar{k}_1^{-1})v, \tau_{\Lambda_0}(\kappa(\exp H))^*\tau_{\Lambda_0}(\bar{k}_1^{-1})v)$$

$$= e^{2\lambda(\gamma(\exp H))}\|\tau_{\Lambda_0}(\kappa(\exp H))\tau_{\Lambda_0}(\bar{k}_1^{-1})v\|^2$$

where we have taken into account the fact that $\lambda(\gamma(\exp H))$ is real and that $\tau_{\Lambda_0}(\kappa(\exp H))$ is selfadjoint for all $H \in \mathfrak{a}_{\mathfrak{p}_0}$.

According to (3.6) we have to compute

$$(3.7) \qquad \int_{K_0}\|\tau_{\Lambda_0}(\kappa(\exp H))\tau_{\Lambda_0}(k)v\|^2 \, dk = (\dim V)^{-1}\chi_{\Lambda_0}(\kappa(\exp H)^2)$$

if $\|v\| = 1$, where χ_{Λ_0} denotes the character of τ_{Λ_0}. This is a straightforward consequence of the Schur orthogonality relations for the irreducible representation τ_{Λ_0} of the compact group K_0 on V.

Thus the problem of computing

$$\int_D (1_v, 1_v)_z \, dz$$

has been reduced to the more concrete problem of evaluating

$$\int_{\mathfrak{a}_{\mathfrak{p}_0}} e^{2\lambda(\gamma(\exp H))}\chi_{\Lambda_0}(\kappa(\exp H)^2) \prod_{\alpha \in Q} |\sinh \alpha(H)| \, dH.$$

Let $f_{v'}$ denote the constant function $f(z) = v'$, $z \in D$, $v' \in V^*$, $\|v'\| = 1$ and let ρ denote the half sum of all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. H_{Λ^*} will denote the space of all square integrable holomorphic sections of $E_{\Lambda^*} \to G/K$ (see 3.2).

3.2. **Theorem** (cf. Theorem 4, [4]). *In order to have $H_{\Lambda^*} \neq \{0\}$ it is necessary and sufficient that $\Lambda(H_\beta) + \rho(H_\beta) < 0$ for all noncompact positive roots β. Moreover, it is possible to normalize the Haar measure on G_0 in such a way that (when $H_{\Lambda^*} \neq \{0\}$)*

$$(3.8) \qquad \|f_{v'}\|^2 = \int_{G_0}\|\Phi^*(g)^{-1}v'\|^2 \, d\bar{g}$$

$$= (\dim V)|\prod_{\alpha \in P} (\Lambda(H_\alpha) + \rho(H_\alpha))/\rho(H_\alpha)|^{-1}.$$

(We recall that P denotes the set of all positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.)

3.3. Remark. Let $v_0 \neq 0$ be a highest weight vector of the irreducible representation τ_Λ. From (3.3) and (3.4) one easily deduces that

$$\pi_\Lambda(X_\alpha)f_{v_0} = 0 \quad \text{and} \quad \pi_\Lambda(H_\alpha)f_{v_0} = \Lambda(H_\alpha)f_{v_0}$$

for all positive roots α of $\mathfrak{g}$ with respect to $\mathfrak{h}$. That is to say that π_Λ has an *extreme vector* f_{v_0} of weight Λ. Conversely, the representations π_Λ constructed here are precisely all the irreducible unitary representations of G which admit an extreme vector.

3.4. Remark. If we realized G/K as a bounded domain in $\mathfrak{p}_+$ instead of in $\mathfrak{p}_-$ then we do not need to pass to the contragredient representation of τ_Λ, and the statement of Theorem 3.2 is still correct by replacing the space $H_{\Lambda*}$ by H_Λ.

Square integrable representations. Let G be a connected semisimple Lie group and let Z denote its center. Then if π is an irreducible unitary representation of G on a Hilbert space H we can find a unitary character η_π of Z such that $\pi(z) = \eta_\pi(z)\pi(e)$ for $z \in Z$. Let Z_0 be a subgroup of Z of finite index. Let $g \rightarrow \bar{g}$ denote the natural projection of G onto $G_0 = G/Z_0$. Given two nonzero elements f_1 and f_2 in H, it is clear that $|(\pi(g)f_1, f_2)|$ $(g \in G)$ depends only on $\bar{g}$ and thus defines a continuous function of G_0. We say that π is *square integrable*, if there exist two elements $f_1 \neq 0$, $f_2 \neq 0$ in H such that

$$\int_{G_0} |(\pi(g)f_1, f_2)|^2 \, d\bar{g} < \infty.$$

It is obvious that this definition does not depend on the choice of the subgroup Z_0 as long as Z/Z_0 is finite.

Let π and π' be two square integrable representations of G on the Hilbert spaces H and H' respectively. Then if their central characters coincide on Z_0 it is clear that $(\pi(g)f_1, f_2)\overline{(\pi'(g)f_1', f_2')}$ depends only on $\bar{g}$ and therefore may be regarded as a function on G_0 $(f_1, f_2 \in H; f_1', f_2' \in H'; g \in G)$. Then the following Schur orthogonality relations hold (cf. [4]). If π and π' are not equivalent

$$\int_{G_0} (\pi(g)f_1, f_2)\overline{(\pi'(g)f_1', f_2')} \, d\bar{g} = 0$$

for all $f_1, f_2 \in H$ and $f_1', f_2' \in H'$, and

$$(3.9) \qquad \int_{G_0} (\pi(g)f_1, f_2)\overline{(\pi(g)f_1', f_2')} \, d\bar{g} = d_\pi^{-1}(f_1, f_1')\overline{(f_2, f_2')}$$

$(f_1, f_2, f_1', f_2' \in H)$, where d_π is a positive constant called the *formal degree* of π.

Clearly d_π depends on the choice of Z_0 and the normalization of the Haar measure of G_0. However once these have been fixed, it is obvious that two equivalent square integrable representations have the same formal degree.

Now we are going to prove that the representation $\pi_{\Lambda*}$ of G on $H_{\Lambda*}$ is square integrable and compute its formal degree.

Let us assume that $H_\Lambda \neq \{0\}$. Then according to the Theorem 3.1, for any $v \in V$ the constant function $f_v \in H_\Lambda$, hence

$$\int_{G_0} (\Phi(xg)^{-1}v, \Phi(g)^{-1}w)\,d\bar{g} = (\pi_\Lambda(x^{-1})f_v, f_w) < \infty \quad \text{all } x \in G; \ v, w \in V$$

(see 3.3). Therefore, $(\Phi(g)^{-1})^*\Phi(xg)^{-1}$ as a function of $\bar{g}$ is integrable over G_0 for all $x \in G$. Let

$$\tilde{F}(x) = \Phi(x) \int_{G_0} (\Phi(g)^{-1})^*\Phi(xg)^{-1}\,d\bar{g}.$$

One can easily check that

(i) $\tilde{F}(k_1 x k_2) = \tau_\Lambda(k_1)\tilde{F}(x)\tau_\Lambda(k_1)^{-1}$ for all $x \in G$; $k_1, k_2 \in K$.

(ii) $X\tilde{F} = 0$ for all $X \in \mathfrak{p}_+$

(cf. Theorem 3.3). Thus $\tilde{F}$ defines a holomorphic function F on D, namely

$$F(\psi(gK)) = \tilde{F}(g) \quad \text{all } g \in G.$$

The Cauchy's integral formula (2.9) and property (i) give

$$F(0) = \int_{C_0} F(\mathrm{Ad}(b)z)\,db = F(z) \quad \text{for all } z \in D;$$

we have taken into account the fact that $\tau_\Lambda(k)$ is a scalar whenever k is in the center of K. Therefore, $\tilde{F}(x)$ is equal to a constant. To evaluate this constant $\tilde{F}(e)$ we observe that

$$\tilde{F}(e) = \tau_\Lambda(k)\tilde{F}(e)\tau_\Lambda(k)^{-1} \quad \text{for all } k \in K$$

(cf. (i)). Hence $\tilde{F}(e)$ is a scalar. To evaluate the scalar we just take a vector $v \in V$ of unit length and compute

$$(\tilde{F}(e)v, v) = \int_{G_0} (\Phi(g)^{-1}v, \Phi(g)^{-1}v)\,d\bar{g} = \|f_v\|^2.$$

We have proved

3.5. Lemma. *If the space $H_\Lambda \neq \{0\}$ and $v \in V$ is such that $\|v\| = 1$ then*

$$\int_{G_0} (\Phi(g)^{-1})^*\Phi(xg)^{-1}\,d\bar{g} = \|f_v\|^2\Phi(x)^{-1} \quad \text{for all } x \in G.$$

We now consider the space $H_{\Lambda*}$ associated with the contragredient representation of τ_Λ and we assume that $H_{\Lambda*} \neq \{0\}$. To prove that the representation $\pi_{\Lambda*}$ of G is square integrable we will consider

$$(3.10) \qquad \int_{G_0} |(\pi_{\Lambda*}(x)f_{v'}, f_{v'})|^2\,d\bar{x}, \quad v' \in V^*, \ \|v'\| = 1.$$

According to Lemma 3.5 we have

$$(\pi_{\Lambda^*}(x)f_{v'}, f_{v'}) = \int_{G_0} (\Phi^*(x^{-1}g)^{-1}v', \Phi^*(g)^{-1}v')d\bar{g}$$

$$= \|f_{v'}\|^2 (\Phi^*(x^{-1})^{-1}v', v')$$

hence, the integral (3.10) is equal to

(3.11) $$\|f_{v'}\|^4 \int_{G_0} |(\Phi^*(x)^{-1}v', v')|^2 d\bar{x}.$$

By a computation similar to the one we did before one obtains that

(3.12)
$$|(\Phi^*(k_1 \exp H k_2)^{-1}v', v')|^2$$
$$= e^{2\lambda(\gamma(\exp H))}|(\tau_{\Lambda_0}(\bar{k}_1)\tau_{\Lambda_0}(\kappa(\exp H))\tau_{\Lambda_0}(\bar{k}_2)v, v)|^2$$

for all $k_1 k_2 \in K$; $H \in \alpha_{\mathfrak{p}_0}$ and $v' = 1_v$. Using the Schur orthogonality relations of τ_{Λ_0} on K_0 it follows that

(3.13)
$$\int_{K_0 \times K_0} |(\tau_{\Lambda_0}(k_1)\tau_{\Lambda_0}(\kappa(\exp H))\tau_{\Lambda_0}(k_2)v, v)|^2 dk_1 dk_2$$
$$= (\dim V)^{-1} \int_{K_0} \|\tau_{\Lambda_0}(\kappa(\exp H))\tau_{\Lambda_0}(k_2)v\|^2 dk_2$$
$$= (\dim V)^{-1} \chi_{\Lambda_0}(\kappa(\exp H)^2)$$

(see (3.7)). Making use of (3.6), (3.11), (3.12) and (3.13) we obtain

$$\int_{G_0} |(\pi_{\Lambda^*}(x)f_{v'}, f_{v'})|^2 dx = (\dim V)^{-1}\|f_{v'}\|^6 < \infty.$$

Therefore, π_{Λ^*} is a square integrable representation of G and according to (3.8) and (3.9) its formal degree is given by

$$d_{\pi_{\Lambda^*}} = (\dim V)\|f_{v'}\|^{-2} = \left|\prod_{\alpha \epsilon P} (\Lambda(H_\alpha) + \rho(H_\alpha))/\rho(H_\alpha)\right|.$$

Note the similarity of this formula with the formula of Weyl which gives the dimension of an irreducible representation of a compact semisimple Lie group in terms of its highest weight.

3.6. **Remark.** The Hodge star operator induces a conjugate linear isomorphism between the Hilbert space $H_\Lambda = H_2^{0,0}(E_\Lambda)$ of all square integrable holomorphic sections of $E_\Lambda \to G/K$ and the space $H_2^{0,n}(L \otimes E_\Lambda^*)$ of all square integrable harmonic $(0, n)$ forms on G/K with values in $L \otimes E_\Lambda^*$. L denotes the canonical line bundle of G/K and E_Λ^* denotes the complex dual bundle of E_Λ. Thus the action of G on $H_2^{0,n}(L \otimes E_\Lambda^*)$ defines also a unitary irreducible representation of G which is square integrable over G/Z.

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