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CONDITIONS FOR A TVS TO BE HOMEOMORPHIC WITH ITS COUNTABLE PRODUCT

BY

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ABSTRACT. C. Bessaga has given conditions for a Banach space to be homeomorphic with its countable product. In this paper, we extend and generalize these results to complete metric topological vector spaces by using infinite dimensional techniques.

1. Introduction. Let M be a topological vector space (TVS), and let M^ω denote its countable product. The hypothesis that M is homeomorphic to $(\cong) M^\omega$ is contained in several important theorems in infinite dimensional topology. For example, it is used in Henderson and Schori's paper [8]. In the following, we will establish equivalent conditions for a metric TVS M to be homeomorphic to M^ω.

If d is a metric on a TVS M we say that d is *translation invariant* if $d(x+y, y) = d(x, 0)$ for all $x, y \in M$. We say that d is *strictly monotone* if $0 \leq t < s$ implies $d(tx, 0) < d(s, 0)$ for all nonzero $x \in M$.

Let c_0 be the Banach space consisting of sequences of real numbers such that the absolute values tend toward zero. The norm is the supremum norm. Let l_p $(p > 0)$ be the collection of sequences of reals $\{x_i\}$ such that $\sum_{i=1}^\infty |x_i|^p < \infty$ with the quasi-norm being $\{\sum_{i=1}^\infty | \ |^p\}^{1/p}$. Finally, let $\{a_{\alpha n}\}_{\alpha, n = 1, 2, \cdots}$ be a matrix of positive real numbers satisfying:

(1) $a_{1n} \leq a_{2n} \leq \cdots$ $(n = 1, 2, \cdots)$, and

(2) $\sum_{n=1}^\infty a_{\alpha n}/a_{\alpha+1 n} < \infty$ $(\alpha = 1, 2, \cdots)$.

By $M(a_{\alpha n})$, we mean the collection of all real sequences, $x = \{x_i\}$, such that $|x|_\alpha = \sup_n a_{\alpha n}|x_n| < \infty$ for all α. The metric on $M(a_{\alpha n})$ is then just

$$d(x, y) = \sum_{\alpha=1}^\infty \frac{1}{2^\alpha} \frac{|x-y|_\alpha}{1 + |x-y|_\alpha}.$$

Given $\{(M^i, d^i) \mid i = 1, 2, \cdots\}$, a collection of complete metric TVS's such that each metric is translation invariant and strictly monotone, and given one of the coordinate spaces E listed above, we define $\sum_E M^i$ to be the set of sequences

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$\{x_i\}$, $x_i \in M^i$, such that $\{d^i(x_i, 0)\} \in E$. $\Sigma_E M^i$ has the topology induced from the metric $\rho_E(\{d^i(\ ,\ )\}, 0)$ where ρ_E is the metric on E. It is easy to show that $\Sigma_E M^i$ is a metric TVS. We will use the notation $\Sigma_E M$ when all the spaces $(M^i, d^i) = (M, d)$ for some fixed pair (M, d).

Bessaga has proven in [5] that:

Theorem. *Let X be a Banach space with norm $\|\ \|$. If $X \cong \Sigma_{l_1} X$ (using $\|-\|$ for d above), then $X \cong X^\omega$.*

As a consequence of this theorem, Bessaga has shown that, for $\aleph \geq \aleph_0$, $l_1(\aleph) \cong l_1(\aleph)^\omega$. Here $l_1(\aleph)$ is the set of $\aleph$-sequences of reals, $\{x_\lambda\}_{\lambda \in \mathrm{Card}\,(\aleph)}$ with at most countably many nonzero coordinates and with $\Sigma_{\lambda \in \mathrm{Card}\,(\aleph)} |x_\lambda| < \infty$. The norm is the obvious one.

Using entirely different techniques, we have proven the converse of this theorem, and, in fact, have shown

Theorem 0. *Let X be an infinite dimensional Banach space. Then $\Sigma_{l_1} X \cong X^\omega$ using any equivalent norm on X.*

We have also generalized this result to:

Theorem 1. *Let M be a complete metric TVS with translation invariant, strictly monotone metric d. Then $M \cong M^\omega$ if and only if both $M \cong l_1 \times M$ and $M \cong \Sigma_{l_1} M$.*

Note that we are *not* assuming that M is either locally convex or separable.

Remark 1. It was shown by R. D. Anderson [1] that all separable Fréchet (complete, locally convex, metric TVS) spaces are homeomorphic to l_1. Then, since every infinite dimensional Fréchet space F clearly contains a closed, separable, infinite dimensional subspace K, we know that every infinite dimensional Fréchet space contains a closed subspace homeomorphic to l_1. We may apply the Bartle-Graves-Michael theorem to the quotient space F/K to produce a homeomorphism between F and $K \times F/K$ (see [5, Theorem 8.1]). Since l_1 is obviously homeomorphic to $l_1 \times l_1$, we have:

$$F \cong K \times F/K \cong l_1 \times F/K \cong l_1 \times (l_1 \times F/K) \cong l_1 \times F.$$

Therefore, every infinite dimensional Fréchet space F satisfies $F \cong l_1 \times F$ and, in particular, so does every Banach space.

Remark 2. As a consequence of Theorem 6.7 [9] and of a result of Eidelheit and Mazur [7] all complete metric TVS's have a translation invariant, strictly monotone metric. Thus, every complete metric TVS has a metric satisfying the hypothesis of Theorem 1.

Now, the proof of Theorem 1 is similar in style to the Anderson-Bing paper [3]. We actually show that:

(*Step* 1) $l_1 \times \Sigma_{l_1} M \cong \Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M]$

(*Step* 2) $\cong \Sigma_{l_1}[(\Sigma_{l_1} l_1 \backslash \{0\}) \times (M \backslash \{0\})]$

(*Step* 3) $\cong \left[(\Sigma_{l_1} l_1)^{\omega} \backslash \left(\bigcup l^{ij}\right) \cup \left(\bigcup K^i\right)\right] \times [M \backslash \{0\}]^{\omega}$

(*Step* 4) $\cong (\Sigma_{l_1} l_1)^{\omega} \times M^{\omega}$

.(*Step* 5) $\cong M^{\omega}$,

where $l^{ij} = \{\{\eta^k\} \in [\Sigma_{l_1} l_1]^{\omega}: \pi_j(\eta^i) = 0\}$ using π_j to mean the projection onto the jth coordinate of $\Sigma_{l_1} l_1$, and where

$$K^i = \left\{\{r^i\} \in \left[\Sigma_{l_1} l_1\right]^{\omega}: \sum_{i=1}^{\infty} \frac{1}{1 + |r^i|} \leq i\right\}$$

using $|\ |$ to represent the norm on $\Sigma_{l_1} l_1$. The K^i's and l^{ij}'s occur as technical difficulties, in that we construct a map which, when restricted to the complement of these sets, is a homeomorphism.

As a consequence of the proof of Theorem 1, we also have the following:

Theorem 2. *Let* $\{(M^i, d^i) | i = 1, 2, \cdots\}$ *be a collection of complete metric TVS's such that each metric is translation invariant and strictly monotone. If* $M^j \cong l_1 \times M^j$ *for some* j, *then* $\Sigma_{l_1} M^i \cong \Pi_{i=1}^{\infty} M^i$.

We may generalize these results to obtain:

Theorem 3. *Let* E *and* E' *be any two of the spaces* c_0, l_p $(p > 0)$, $M(a_{\alpha n})$. *Let* M *be a complete metric TVS with translation invariant, strictly monotone metric* d. *Then* $M \cong M^{\omega}$ *if and only if both* $M \cong l_1 \times M$ *and* $M \cong \Sigma_E M$. *Also, if* $\{(M^i, d^i) | i = 1, 2, \cdots\}$ *is a collection of complete metric TVS's such that each* d^i *is translation invariant and strictly monotone, and if* $l_1 \times M^j \cong M^j$ *for some* j, *then* $\Sigma_E M^i \cong \Pi_{i=1}^{\infty} M^i \cong \Sigma_{E'} M^i$.

2. Lemmas. We say that a subset $K \subset M$ is *infinitely deficient* if there exists a homeomorphism h from M to $l_1 \times M$ such that $h(K) \subset \{0\} \times M$. We say that $K \subset M$ is *strongly negligible* if, for every open cover G of M there exists a homeomorphism h from M onto $M \backslash K$ such that for any $x \in M$ there is a $g \in G$ which contains both x and $h(x)$.

Lemma 1. *Let* M *be a complete metric space,* $K \subset M$ *a countable union of closed, infinitely deficient sets. Then* K *is strongly negligible in* M.

Proof. Lemma 1 in Cutler's paper [6] contains this result.

A set $K \subset M$ has *property* Z (is a Z-set) in M if for each nonempty, homotopically trivial, open set U in M it is true that $U \backslash K$ is nonempty and homotopically trivial.

We need the following two lemmas on Z-sets:

Lemma 2. *The following are equivalent in* l_1:

1. K *has property* Z *in* l_1,

2. K *has property* Z *locally in* l_1 (*i.e. each point of* K *lies in an open set* $\mathcal{O}$ *for which* $K \cap \mathcal{O}$ *has property* Z *in* $\mathcal{O}$).

Proof. See Lemma 1 [4].

Lemma 3. *Countable unions of Z-sets are strongly negligible from* l_1.

Proof. See Anderson's paper [2, Theorem 1].

Lemma 4 is a necessary technical lemma:

Lemma 4. *Let* M *be a metric TVS with translation invariant, strictly monotone metric* d. *Then, if* $\{x_n\}$ *is a sequence in* M *converging to* x *and* $\{t_n\}$ *is a sequence of nonnegative real numbers such that* $d(t_n x_n, 0)$ *converges to* $d(tx, 0)$, *then* $\{t_n x_n\}$ *converges to* tx.

Proof. Suppose $\{t_n\}$ does not converge to t. Then there exists a subsequence (call it $\{t_n\}$ again) such that

1. $t_n \geq t + \epsilon$ for some fixed $\epsilon > 0$, all n, or

2. $t_n \leq t - \epsilon$ for some fixed $\epsilon > 0$, all n.

But then

1. $d(t_n x_n, 0) \geq d((t+\epsilon)x_n, 0) \longrightarrow d((t+\epsilon)x, 0) > d(tx, 0)$, or

2. $d(t_n x_n, 0) \leq d((t-\epsilon)x_n, 0) \longrightarrow d((t-\epsilon)x, 0) < d(tx, 0)$.

Contradiction. Therefore, $\{t_n\}$ converges to t, and, hence $\{t_n x_n\}$ converges to tx.

Lemma 5. *Any complete metric TVS* M *is homeomorphic to* $\mathbf{R} \times M/\mathbf{R}$ *where* $\mathbf{R}$ *is the real line.*

Proof. In Michael's paper [10], Corollary 7.3 says: Let G be a metrizable group, and H be a closed subgroup which is isomorphic to the additive group of a complete, metrizable, locally convex TVS. Then there exists a cross section, i.e. a continuous function $\psi: G/H \longrightarrow G$ such that $\phi \circ \psi =$ identity where $\phi: G \longrightarrow G/H$ is the canonical map. Identifying $\mathbf{R}$ with a one-dimensional subspace of M, we then have the homeomorphism

$$h: M \longrightarrow M/\mathbf{R} \times \mathbf{R}$$

$$x \longrightarrow (\phi(x), x - \psi \circ \phi(x))$$

where ϕ is the canonical map, and where ψ is the cross section guaranteed in the above corollary.

Finally, we observe

Lemma 6. *Let M be a complete metric TVS with translation invariant, strictly monotone metric d. Then $\Sigma_{l_1}(M, d) = \Sigma_{l_1}(M, d/(1+d))$.*

Proof. The required isomorphism is the identity map. Just use the fact that if $\{x_i\}, \{y_i\} \in \Sigma_{l_1}(M, d)$ and $\Sigma_{i=1}^{\infty} d(x_i, y_i) < 1$, then

$$\frac{d(x_i, y_i)}{2} \le \frac{d(x_i, y_i)}{1 + d(x_i, y_i)} \le d(x_i, y_i), \quad \text{all } i.$$

The notion of summability is the same in both spaces. This can be seen by letting $\{y_i\} = 0$ above, and applying the inequality to all elements in the sequence whose absolute value is less than 1.

3. Proofs of Theorems. Note that it is enough to prove Theorem 1 to prove Theorem 0. This follows from the fact that any infinite dimensional Banach space B has an l_1 factor, i.e. $B \cong l_1 \times B$. (See Remark 1.) Thus $\Sigma_{l_1} B \cong B \times \Sigma_{l_1} B \cong l_1 \times B \times \Sigma_{l_1} B \cong l_1 \times \Sigma_{l_1} B$. Also, Theorem 2 can be obtained by replacing $\Sigma_{l_1} M$ by $\Sigma_{l_1} M^i$ in the proof of Theorem 1, and observing that the map is constructed coordinate-wise.

Proof of Theorem 1. Applying Lemma 6, we will assume that the metric d on M is bounded. We will first prove that $l_1 \times \Sigma_{l_1} M$ is homeomorphic to M^{ω}.

Step 1. $l_1 \times \Sigma_{l_1} M \cong \Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M]$.

Since $l_1 = \Sigma_{l_1}(\Sigma_{l_1} l_1)$, $l_1 \times \Sigma_{l_1} M = \Sigma_{l_1}(\Sigma_{l_1} l_1) \times \Sigma_{l_1} M$. But this is just $\Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M]$ when we use addition of the metrics, i.e. $d' = |\ | + d$ where $|\ |$ is the norm on $\Sigma_{l_1} l_1$ and d is the metric on M. This is the same space when we use $p = \max(|\ |, d)$ as well, since $p \le d' \le 2p$. We will use p in the following.

Step 2. $\Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M] \cong \Sigma_{l_1}[(\Sigma_{l_1} l_1 \backslash \{0\}) \times (M \backslash \{0\})]$.

Now

$$\Sigma_{l_1}[(\Sigma_{l_1} l_l \backslash \{0\}) \times (M \backslash \{0\})] = \Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M] \backslash [(\bigcup L^{ij}) \cup (\bigcup M^i)]$$

where

$$L^{ij} = \{\{\xi^k\} \in \Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M] \colon \pi_{\Sigma}(\xi^i) \text{ has } j\text{th coordinate } 0\},$$

$$M^i = \{\{\xi^k\} \in \Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M] \colon \pi_M(\xi^i) = 0\}.$$

Here π_Σ and π_M are the projections onto $\Sigma_{l_1} l_1$ and M. Note that the L^{ij} and M^i are all closed. They are infinitely deficient since $l_1 \cong l_1 \times l_1$ and $M \cong l_1 \times M$. Thus, by Lemma 1, $[(\bigcup L^{ij}) \cup (\bigcup M^i)]$ is strongly negligible in $\Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M]$. In particular, we have the required homeomorphism.

Step 3. $\Sigma_{l_1}[(\Sigma_{l_1} l_1 \backslash \{0\}) \times (M \backslash \{0\})] \cong [(\Sigma_{l_1} l_1)^\omega \backslash (\bigcup l^{ij}) \cup (\bigcup K^i)] \times [M \backslash \{0\}]^\omega$.

The l^{ij} and K^i are defined as follows:

$$l^{ij} = \{\{\eta^k\} \in [\Sigma_{l_1} l_1]^\omega : \pi_j(\eta^i) = 0\}$$

and

$$K^i = \left\{\{r^k\} \in [\Sigma_{l_1} l_1]^\omega : \sum_{k=1}^{\infty} \frac{1}{1 + |r^k|} \leq i\right\}.$$

Here, π_j is projection onto the jth coordinate.

Define

$$h : \Sigma_{l_1} [(\Sigma_{l_1} l_1 \backslash \{0\}) \times (M \backslash \{0\})] \to \left[(\Sigma_{l_1} l_1)^\omega \Big\backslash \left(\bigcup l^{ij}\right) \cup (K^i)\right] \times [M \backslash \{0\}]^\omega$$

$$\{x_i\} \mapsto (t_1 x_1, t_2 x_2, \cdots, t_n x_n, \cdots)$$

where $t_n \geq 0$ is chosen such that

$$p(t_n x_n, 0) = \frac{\sum_{i=n}^{\infty} p(x_i, 0)}{p(x_{n-1}, 0)} .$$

For $n = 1$, the denominator will be, by definition, 1. Note that x_n is actually a pair: $x_n = \{x_n^l, x_n^M\}$, with $x_n^l \in \Sigma_{l_1} l_1$ and $x_n^M \in M$. Thus $t_n x_n = \{t_n x_n^l, t_n x_n^M\}$.

Since p is strictly monotone and unbounded on rays, a unique such choice exists. The map h is continuous since it is coordinate-wise continuous by Lemma 4.

Define

$$g : [(\Sigma_{l_1} l_1) \times M]^\omega \to \Sigma_{l_1} [(\Sigma_{l_1} l_1) \times M]$$

by

$$\{z_i\} \mapsto (s_1 z_1, s_2 z_2, \cdots, s_n z_n, \cdots)$$

where $s_n \geq 0$ is chosen such that

$$p(s_n z_n, 0) = \frac{[\Pi_{i=1}^{n} p(z_i, 0)]}{[\Pi_{i=2}^{n+1} (1 + p(z_i, 0))]} .$$

Again, a unique such point may be chosen.

To show that g is into $\Sigma_{l_1}[(\Sigma_{l_1} l_1) \times M]$, observe that

$$\sum_{n=1}^{\infty}\left[\frac{\Pi_{i=1}^{n} p(z_i, 0)}{\Pi_{i=2}^{n+1}(1 + p(z_i, 0))}\right] = p(z_1, 0)\left\{\sum_{n=1}^{\infty}\left[\frac{\Pi_{i=2}^{n} p(z_i, 0)}{\Pi_{i=2}^{n+1}(1 + p(z_i, 0))}\right]\right\}.$$

Let $r_i = d(z_i, 0)$. Then, looking at the partial sums $\mathcal{S}^n$ of the bracket on the right ({ }), we have by induction

$$\mathcal{S}_n = 1 - r_2 \cdot r_3 \cdots r_n \cdot r_{n+1}/(1 + r_2)(1 + r_3) \cdots (1 + r_n)(1 + r_{n+1}).$$

Note that $0 < r_i/(1 + r_i) < 1$, all i. Therefore, letting $u_i = 1 - r_i/(1 + r_i) = 1/(1 + r_i)$, we have

$$\left(\frac{r_2}{1 + r_2}\right) \cdots \left(\frac{r_{n+1}}{1 + r_{n+1}}\right) = (1 - u_2) \cdots (1 - u_n)$$

$$\leq \exp(-u_2 - u_3 - \cdots - u_n) \quad \text{since } 0 < u_i < 1, \text{ all } i,$$

$$= \exp\left(-\frac{1}{1 + r_2} - \cdots - \frac{1}{1 + r_{n+1}}\right).$$

If $\Sigma_{i=1}^{\infty} 1/(1 + r_i) = \infty$, then $\exp(-1/(1 + r_2) - \cdots - 1/(1 + r_{n+1}))$ tends toward 0. Therefore, the sum in brackets is 1. If $0 < \Sigma_{i=1}^{\infty} 1/(1 + r_i) < \infty$, then $0 < \exp(-\Sigma 1/(1 + r_i)) < 1$. Hence, the sum is less than or equal to $1 - \exp(-\Sigma 1/(1 + r_i))$. Therefore,

$$\sum_{n=1}^{\infty}\left[\frac{\Pi_{i=1}^{n} p(z_i, 0)}{\Pi_{i=2}^{\infty}(1 + p(z_i, 0))}\right] \leq p(z_1, 0)\left[1 - \exp\left(-\sum_{i=2}^{\infty}\frac{1}{1 + p(z_i, 0)}\right)\right]$$

$$< \infty.$$

By Theorem 15.5 of [11], we also have that $\Pi_{i=2}^{\infty}(1 - u_i) > 0$ if $\Sigma_{i=2}^{\infty} u_i < \infty$.

Now, let $\tilde{M} = [(\Sigma_{l_1} l_1)^{\omega}\backslash(\bigcup l^{ij}) \cup (\bigcup K^i)] \times [M\backslash\{0\}]^{\omega}$. Then, we can show that

$$g|_{\tilde{M}} \circ h = \text{identity on } \Sigma_{l_1}[(\Sigma_{l_1} l_1\backslash\{0\}) \times (M\backslash\{0\})]$$

by writing out the composition coordinate-wise, and applying induction. Also, $g|_{\tilde{M}}$ is continuous since, given $\epsilon > 0$ and $\{z_j\}$, choose N such that

$$\sum_{i=N}^{\infty} p(g(\{z_j\})_i, 0) < \frac{\epsilon}{4},$$

where $g(\{z_j\})_i$ is the ith coordinate of $g(\{z_j\})$. Then pick $N_i(z_i)$, a neighborhood of the ith coordinate $(i = 1, \cdots, N)$, such that given $\{z_j'\}$ with $z_i' \in N_i$, $i = 1, 2, \cdots, N$, implies

$$p(g(\{z_j\})_i, g(\{z_j'\})_i) < \epsilon/4N, \qquad i = 1, \cdots, N - 1,$$

and, in addition, such that for N_1, we have $|p(z_1', 0) - p(z_1, 0)| < \epsilon/4N$. Then, given $\{z_j'\}$ such that $z_i' \in N_i$, $i = 1, \cdots, N$, implies

$$\sum_{i=1}^{\infty} p(g(\{z_j\})_i, g(\{z_j'\})_i)$$

$$\leq \sum_{i=1}^{N-1} p(g(\{z_j\})_i, g(\{z_j'\})_i) + \sum_{i=N}^{\infty} p(g(\{z_j\})_i, 0) + \sum_{i=N}^{\infty} p(0, g(\{z_j'\})_i)$$

$$\leq (N-1)(\epsilon/4N) + \epsilon/4 + \epsilon/4 + N(\epsilon/4N) < \epsilon.$$

This follows from:

$$\begin{array}{ccc} \sum_{i=1}^{\infty} p(g(\{z_j'\})_i, 0) & < & \sum_{i=1}^{\infty} p(g(\{z_j\})_i, 0) + \epsilon/4N \\ \| & & \| \\ p(z_1', 0) & & p(z_1, 0) + \epsilon/4N. \end{array}$$

Therefore, $\Sigma_{i=N}^{\infty} p(g(\{z_j'\})_i, 0) < \epsilon/4 + N(\epsilon/4N)$.

Now, to show that $h \circ g|_{\tilde{M}}$ = identity on $\tilde{M}$, we again write out the composition coordinate-wise and apply induction. Thus, we have that h is the required homeomorphism.

Step 4. $[(\Sigma_{l_1} l_1)^{\omega} \backslash (\bigcup l^{ij}) \cup (\bigcup K^i)] \times [M\backslash\{0\}]^{\omega} \cong (\Sigma_{l_1} l_1)^{\omega} \times M^{\omega}$.

We know that $\{0\}$ is strongly negligible from M. Therefore $(M\backslash\{0\})^{\omega} \cong M^{\omega}$. To show

$$\left[(\Sigma_{l_1} l_1)^{\omega} \backslash \left(\bigcup l^{ij}\right) \cup \left(\bigcup K^i\right)\right] \cong (\Sigma_{l_1} l_1)^{\omega}$$

we will show that each K^j and l^{ij} is a Z-set.

First, we claim each K^j is closed. To show this, pick $\{x^k\} \in (K^j)^c$ where $(K^j)^c$ is the complement of K^j. Then $\Sigma_{k=1}^{\infty} 1/(1 + |x^k|) > j$. Pick N such that $\Sigma_{k=1}^{N} 1/(1 + |x^k|) > j$. Then there exists a neighborhood U of $\{x^k\}_{k=1}^{N}$, $U \subset [\Sigma_{l_1} l_1]^N$ such that $\Sigma_{k=1}^{\infty} 1/(1 + |y^k|) > j$ for all $\{y^k\} \in U$. Then, let $\mathcal{O} = U \times \Pi_{N+1}^{\infty}[\Sigma_{l_1} l_1]$. $\mathcal{O}$ is open in $[\Sigma_{l_1} l_1]^{\omega}$, and $\{x^k\} \in \mathcal{O}$. Also $\mathcal{O} \cap K^j = \emptyset$. Thus K^j is closed. Moreover, K^j is nowhere dense since it contains no bounded sequences and is closed, and any open set must contain bounded sequences.

We claim K^j is a Z-set. Given a convex basic open set $\mathfrak{U}$ and a map $f: S^n \to \mathfrak{U}\backslash K^j$, look at $f(S^n)$. For all $f(s) \in \bigcup_{i>j} K^i$, pick a constant N such that

(1) $\Sigma_{i=1}^{N} 1/(1 + |f(s)_i|) > j$, and

(2) $\mathfrak{U}$ contains full factors from N on.

This can be done since $f(S^n)$ is compact and lies in $(K^j)^c$. Then to each $f(s) \in f(S^n)$ associate the point $\{x_k\}$ such that $(x_1, \cdots, x_N) = (f(s)_1, \cdots, f(s)_N)$ and $x_k = 0$ for all $k > N$. Then, the line segment from $f(s)$ to $\{x_k\}$ is not in K^j. But then, since $([\Sigma_{l_1} l_1]^\omega)^f = \{\{\eta^k\} \in [\Sigma_{l_1} l_1]^\omega : \eta^i = 0$ for all but a finite number of i's$\}$ is contractible in itself, we may define a map $\tilde{f}: E^{n+1} \to \mathfrak{U} \backslash K^j$ with $\tilde{f}|S^n = f$. Thus, by Lemma 2, each K^j is a Z-set.

To show that each l^{ij} is a Z-set, we can use an argument similar to the above. Just keep away from the appropriate 0 coordinate.

Thus, by Lemma 3 $[(\bigcup l^{ij}) \cup (\bigcup K^i)]$ is strongly negligible, and we have the homeomorphism.

Step 5. $(\Sigma_{l_1} l_1)^\omega \times M^\omega \cong M^\omega$.

$\Sigma_{l_1} l_1 = l_1$. Therefore $(\Sigma_{l_1} l_1)^\omega \times M^\omega = l_1^\omega \times M^\omega$. By Lemma 5, we have $M \cong \mathrm{R} \times M/\mathrm{R}$. Thus $M^\omega \cong (\mathrm{R} \times M/\mathrm{R})^\omega \cong \mathrm{R}^\omega \times (M/\mathrm{R})^\omega$. By Anderson's paper [1] we have $\mathrm{R}^\omega \cong l_1 \cong l_1^\omega$. Thus $M^\omega \cong \mathrm{R}^\omega \times (M/\mathrm{R})^\omega \cong \mathrm{R}^\omega \times \mathrm{R}^\omega \times (M/\mathrm{R})^\omega \cong l_1 \times (\mathrm{R} \times M/\mathrm{R})^\omega = l_1^\omega \times M^\omega$. Thus $M^\omega \cong l_1^\omega \times M^\omega \cong (\Sigma_{l_1} l_1)^\omega \times M^\omega$.

From Step 1 through Step 5, we have $l_1 \times \Sigma_{l_1} M \cong M^\omega$. Also, Step 5 shows that $M^\omega \cong l_1 \times M^\omega$.

If we assume $M \cong M^\omega$, then $M \cong M^\omega \cong l_1 \times M^\omega \cong l_1 \times M$. Hence $M \cong M^\omega \cong l_1 \times \Sigma_{l_1} M \cong l_1 \times M \times \Sigma_{l_1} M \cong M \times \Sigma_{l_1} M \cong \Sigma_{l_1} M$.

If we assume $M \cong l_1 \times M$ and $M \cong \Sigma_{l_1} M$, then we have $M \cong \Sigma_{l_1} M \cong M \times \Sigma_{l_1} M \cong l_1 \times M \times \Sigma_{\omega_1} M \cong l_1 \times \Sigma_{l_1} M \cong M^\omega$.

But, then, we have proven Theorem 1.

Proof of Theorem 3. Let E and E' be any two of the spaces listed. If F is a coordinate space, we define $F^+ = \{\{x^i\} \in F : x^i \geq 0$ for all $i\}$. Then, Bessaga has shown, in [5], that there exists a homeomorphism $h: E^+ \to (E')^+$ such that, for $\{x^i\} \in E^+$, $x^j = 0$ if and only if $\pi_j(h(\{x^i\})) = 0$. π_j is projection onto the jth coordinate.

Let d' be the max metric on $l_1 \times M$, and define

$$H: \Sigma_E(l_1 \times M) \to \Sigma_{E'}(l_1 \times M)$$

$$\{x_i\} \mapsto \{t_i x_i\}$$

where each $t_i \geq 0$ is chosen such that $\{d'(t_i x_i, 0)\} = h(\{d'(x_i, 0)\})$.

Then H is 1-1 since, if $\{t_i x_i\} = H(\{x_i\}) = H(\{y_i\}) = \{s_i y_i\}$, then $h(\{d'(x_i, 0)\}) = h(\{d'(y_i, 0)\})$. Hence, $d'(x_i, 0) = d'(y_i, 0)$ for all i. But then $\{x_i\} = \{y_i\}$ since d' is strictly monotone.

H is onto since, given $\{y_i\} \in \Sigma_{E'}(l_1 \times M)$, choose $\{s_i y_i\}$ such that $\{d'(s_i y_i, 0)\} = h^{-1}(\{d'(y_i, 0)\})$. Then $H(\{s_i y_i\}) = \{y_i\}$ since d' is strictly monotone. This also shows how to define H^{-1}.

Now H and H^{-1} are continuous by Lemma 4. Note that the same proof works if we allow M to change.

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