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ISOMETRIES OF *-INVARIANT SUBSPACES(1)

BY

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ABSTRACT. We consider families of increasing *-invariant subspaces of $H^2(D)$, and from these we construct canonical isometries from certain L^2 spaces to H^2. We give necessary and sufficient conditions for these maps to be unitary, and discuss the relevance to a problem concerning a concrete model theory for a certain class of operators.

1. Introduction. Let H^2 denote the usual Hardy class of functions holomorphic in the unit disc D. Beurling showed [2] that any closed subspace invariant under multiplication by z is of the form $s(z)H^2$, where s is inner. Here we consider the *-invariant space $M = (sH^2)^\perp$, where s is a singular inner function. It is well known (see [5] for details) that

$$s(z) = \exp\left[-\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta)\right],$$

where σ is a finite positive singular measure.

In §2, we decompose M into a "continuous chain" of increasing *-invariant subspaces, and from this chain we construct a canonical isometry from a certain L^2 space onto M. This generalizes a map used by Ahern and Clark [1], and Kriete [6]. In §3, we give necessary and sufficient conditions for this map to be unitary, and in §4, we examine some measure theoretic implications of these conditions. In §5 we generalize our methods, and finally, in §6 we show relations of these isometries to concrete canonical models of a class of operators defined by Kriete [7], and point out relations of our work to his.

We consider only singular *-invariant subspaces since in the general case, $\phi = s \cdot B$ where s is singular, and

$$B(z) = \prod_n \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z},$$

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and by writing $(\phi H^2)^\perp = (BH^2)^\perp \oplus B(sH^2)^\perp$, we can consider separately the singular and Blaschke product cases [1]. If $\{(B_n H^2)^\perp\}$ is any family of increasing $*$-invariant subspaces of $(BH^2)^\perp$, it follows that each B_n is a subproduct of B. Relabeling if necessary, we assume B_n has zeroes $a_1, \cdots, a_{n-1}$, and then $\{b_j\}_{j=1}^n$, $b_j(z) = (1 - |a_j|^2)^{1/2} B_j(z)(1 - \bar{a}_j z)^{-1}$, forms an orthonormal basis for $(B_n H^2)^\perp$ [11]. Then $(V\{c_n\})(z) = \sum_{n=0}^\infty c_n b_n(z)$ maps l^2 unitarily onto $(BH^2)^\perp$ in a canonical manner. We thus restrict ourselves to the singular case where families of subspaces are uncountable, and hence such natural orthonormal bases do not exist.

2. **Constructing isometries.** For σ a positive singular Borel measure on the unit circle T (which we identify with $[0, 2\pi]$), we say $\{\sigma_\lambda\}_{\lambda \in T}$ is a (*right*) *continuous chain* if

(i) $\sigma_{2\pi} = \sigma$, σ_0 is the zero measure,

(ii) if $\lambda < \mu$, $(\sigma_\mu - \sigma_\lambda)$ is a positive Borel measure,

(iii) $d(\lambda) = \sigma_\lambda(T)$ is a (right) continuous function of λ.

We note that (i) implies that $\sigma_\lambda \ll \sigma_\mu$, $\lambda < \mu$, and hence the only possible atoms of σ_λ are atoms of σ. In what follows, the subscript λ will implicitly range over $[0, 2\pi]$.

Given a singular inner function

$$s(z) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma(\theta)\right]$$

and a right continuous chain $\{\sigma_\lambda\}$, let

$$s_\lambda(z) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\sigma_\lambda(\theta)\right],$$

$$M_\lambda = (s_\lambda H^2)^\perp, \text{ and}$$

P_λ be orthogonal projection on M_λ.

We denote $M_{2\pi}$ by M and $P_{2\pi}$ by P, and note that since s_λ is a singular inner function dividing s, $\{M_\lambda\}$ is an increasing family of $*$-invariant subspaces of M. From Beurling's theorem [2] and the continuity condition on $d(\lambda) = \sigma_\lambda(T)$, we have the following proposition.

Proposition 2.1. (i) $M_\lambda = \bigcap_{\mu > \lambda} M_\mu$.

(ii) *If* $\{\sigma_\lambda\}$ *is a continuous chain, then* $(\bigcup_{\mu < \lambda} M_\mu)$ *is dense in* M_λ.

Details of the proof can be found in [7]. Thus, 2.1 shows that the increasing family of projections $\{P_\lambda\}$ is right continuous in the strong operator topology. For $z \in D$ fixed, and 1 the constant function,

$$(P_\lambda 1)(z) = 1 - s_\lambda(z)s_\lambda(0) = \mu_z([0, \lambda])$$

is a complex Borel measure on $[0, 2\pi]$.

$$\nu_z([0, \lambda]) = \int_0^{2\pi} \frac{d\sigma_\lambda(\theta)}{1 - ze^{-i\theta}}$$

is also a Borel measure, and a simple computation shows that $\mu_z(E) = \int_E 2s_\lambda(z)s_\lambda(0)\, d\nu_z(\lambda)$, i.e., $d\mu_z(\lambda) = 2s_\lambda(z)s_\lambda(0)\, d\nu_z(\lambda)$. Thus, $\nu = \nu_0$ and $\mu = \mu_0$ are equivalent, i.e., mutually absolutely continuous, positive measures, and $\nu((a, b]) = \sigma_b(T) - \sigma_a(T)$.

Proposition 2.2. *There exists* $F(z, \lambda)$ *such that for each* $z \in D$, $F(z, \lambda) \in L^\infty(\nu)$ *and* $d\nu_z(\lambda) = F(z, \lambda)\, d\nu(\lambda)$.

Proof. Fix $z \in D$ and let $C_z = \sup_\theta |1/(1 - ze^{-i\theta})|$. Then for $(a, b] \subset T$,

$$|\nu_z((a, b])| = \left| \int_0^{2\pi} \frac{d(\sigma_b - \sigma_a)(\theta)}{1 - ze^{-i\theta}} \right| \leq C_z \nu((a, b]),$$

so $|\nu_z(E)| \leq C_z \nu(E)$ for all $E \subset T$. $F(z, \lambda)$ is just the Radon-Nikodým derivative of $\nu_z \ll \nu$.

Thus, $\mu_z \ll \mu$, so for $c \in L^2(\mu)$ we define

$$(Vc)(z) = \int_0^{2\pi} c(\lambda)\, d\mu_z(\lambda) = 2 \int_0^{2\pi} c(\lambda) s_\lambda(z) s_\lambda(0) F(z, \lambda)\, d\nu(\lambda).$$

Proposition 2.3. V: $L^2 \to M$ *is an isometry.*

Proof. Let $\chi_{(a,b]}$ be the characteristic function of $(a, b]$, and S the closed linear span of all such χ. Since $V\chi_{(a,b]} = P_b 1 - P_a 1$ is the projection of 1 onto $M_b \ominus M_a$, V maps S isometrically into M. For $c \in L^2(\mu)$, $c_n \to c$, $\{c_n\} \subset S$, we have $\{Vc_n\}$ Cauchy in M, and since for $z \in D$ fixed, $(Vc_n)(z) \to (Vc)(z)$, V is an isometry on all of $L^2(\mu)$.

Proposition 2.4. *Let* $c \in L^2(\mu)$.

(i) *If* $c(\lambda) = 0$ *a.e. for* $\lambda > a$, *then* $(Vc) \in M_a$.

(ii) *If* $c(\lambda) = 0$ *a.e. for* $\lambda \leq a$, *then* $(Vc) \in M_a^\perp = s_a H^2$.

Proof. For continuous c, (Vc) is the limit of Riemann sums and the proposition is clear. The general case follows by continuity.

Corollary 2.5. *Let* Q_λ: $L^2(\mu) \to L^2(\mu)$ *by*

$$(Q_\lambda c)(x) = c(x), \qquad x \le \lambda,$$
$$= 0, \qquad x > \lambda.$$

Then $P_\lambda V = VQ_\lambda$.

Proof. $(Vc) = \int_{[0,\lambda]} c(x)\, d(P_\lambda 1) + \int_{(\lambda,2\pi]} c(x)\, d(P_\lambda 1)$. Since the first summand is in M_λ and the second is in $M_\lambda^\perp$, we have $(P_\lambda V_c) = \int_{[0,\lambda]} c(x)\, d(P_\lambda 1) = V(Q_\lambda c)$. Since c is arbitrary, the proposition follows.

3. **Conditions for unitary maps.** We know that $V(L^2(\mu))$ is a closed subspace of M, and we now consider when V is actually onto. It is clear that a necessary condition for this is that $\{\sigma_\lambda\}$ be a continuous chain, since if $\sigma_\lambda(T)$ has a jump at λ_0, define

$$\tilde{\sigma}(E) = \lim_{\epsilon \to 0^+} \sigma_{\lambda - \epsilon}(E)$$

and let

$$\tilde{s}(t) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\tilde{\sigma}(\theta)\right].$$

Then $\tilde{s}$ is a singular inner function and $V(L^2(\mu|_{[0,\lambda_0)})) \subset (\tilde{s}H^2)^\perp$ and $V(L^2(\mu|_{(\lambda_0,2\pi]})) \subset s_{\lambda_0}H^2$, so $N = (s_{\lambda_0}H^2)^\perp \ominus (\tilde{s}H^2)^\perp$ is an infinite dimensional subspace of M, since $\sigma_{\lambda_0} \ne \tilde{\sigma}$, and N cannot be contained in $V(L^2)$. Thus, we now assume that all chains $\{\sigma_\lambda\}$ are continuous, which is equivalent to assuming that μ and ν are nonatomic measures.

For $\zeta \in D$, let $K_\zeta(z) = (1 - \overline{s(\zeta)}s(z))/(1 - \bar{\zeta}z)$ be the projection of the reproducing kernel $(1 - \bar{\zeta}z)^{-1}$ onto M. One can see that K_ζ is in the range of V iff

$$(*) \qquad F(z, \lambda) + \bar{F}(\zeta, \lambda) - 1 = F(z, \lambda)\, \bar{F}(\zeta, \lambda)(1 - \bar{\zeta}z) \quad \text{a.e. } [\nu]$$

holds for all $z \in D$. Since the linear span of the K_ζ is dense in M, V is onto iff $(*)$ holds for all $\zeta \in D$, which is equivalent to $F(z, \lambda) = (1 - ze^{-if(\lambda)})^{-1}$ for some real f. Details for this are found in [10].

We get this result more simply by following the methods used by Kriete [7]. Kriete constructs a unitary map

$$\mathcal{F}\colon M \to \mathfrak{D} = \int_0^{2\pi} \oplus L^2(\nu_\lambda)\, d\nu(\lambda),$$

where $\mathfrak{D}$ is a direct integral space. The measures ν_λ are defined by the relation

$$\int_T h(\theta)\, d\sigma_\lambda(\theta) = \int_0^\lambda \left(\int_T h(\theta)\, d\nu_s(\theta)\right) d\nu(s)$$

for all $h \in C(T)$ [7, p. 133]. It is easy to see that our isometry V is the unitary map $\mathfrak{F}^{-1}$ restricted to $\mathfrak{D}_\lambda$, the set of functions in $\mathfrak{D}$ depending only on λ, i.e., if $h(\lambda, \theta) \in \mathfrak{D}_\lambda \subset \mathfrak{D}$, then for each λ_0, $h(\lambda_0, \theta) \in L^2(\nu_{\lambda_0})$ is constant a.e. $[\nu_{\lambda_0}]$. Thus, V is onto iff $\mathfrak{D}_\lambda = \mathfrak{D}$. As Kriete remarks [7, p. 137], this holds iff $\nu_\lambda = \delta_{f(\lambda)}$, a unit point mass at $f(\lambda)$. If we consider $h_z(\theta) = (1 - ze^{-i\theta})^{-1} \in C(T)$, we see that

$$\int_0^{2\pi} (1 - ze^{-i\theta})^{-1} d\sigma_\lambda(\theta) = \int_0^\lambda \left(\int_0^{2\pi} (1 - ze^{-i\theta})^{-1} d\nu_s(\theta)\right) d\nu(s),$$

so we have $F(z, \lambda) = \int_0^{2\pi}(1 - ze^{-i\theta})^{-1} d\nu_\lambda(\theta)$. Hence, we have

Proposition 3.1. *V is onto iff there is a real f such that $F(z, \lambda) = (1 - ze^{-if(\lambda)})^{-1}$ a.e. $[\nu]$, i.e.*

$$d\left[\int_0^{2\pi} (1 - ze^{-i\theta})^{-1} d\sigma_\lambda(\theta)\right] = (1 - ze^{-if(\lambda)})^{-1} d\left[\int_0^{2\pi} d\sigma_\lambda(\theta)\right].$$

We note that $K_0 = 1 - s(z)s(0) = P1 = V(1)$ is always in the range of V. From this, it follows that $V(L^2)$ cannot be *-invariant unless V is onto.

4. Descriptions of the chains of measures. We now examine more closely what

$$(**) \qquad d\left[\int_0^{2\pi} \frac{d\sigma_\lambda(\theta)}{1 - ze^{-i\theta}}\right] = \frac{1}{1 - ze^{-if(\lambda)}} d\nu(\lambda)$$

implies about the chain $\{\sigma_\lambda\}$.

Proposition 4.1. *(**) holds iff for all λ, $\sigma_\lambda(E) = \nu(f^{-1}(E) \cap [0, \lambda])$, for all $E \subset T$.*

Proof. Suppose (**) holds, so that $\nu_\lambda = \delta_{f(\lambda)}$. Then for $E \subset T$, consider $h(\theta) = \chi_E(\theta)$ in the equation defining ν_λ. Thus,

$$\begin{aligned}\sigma_\lambda(E) &= \int_T \chi_E(\theta)\, d\sigma_\lambda(\theta) = \int_0^\lambda \left(\int_T \chi_E(\theta)\, d\nu_s(\theta)\right) d\nu(s) \\ &= \int_0^\lambda \left(\int_E d\delta_{f(s)}\right) d\nu(s) \\ &= \int_0^\lambda \chi_{f^{-1}(E)}(s)\, d\nu(s) = \nu(f^{-1}(E) \cap [0, \lambda]).\end{aligned}$$

Conversely, suppose $\sigma_\lambda(E) = \nu(f^{-1}(E) \cap [0, \lambda])$. Then we get $\int_0^\lambda \nu_s(E)\, d\nu(s) = \int_0^\lambda \chi_{f^{-1}(E)}(s)\, d\nu(s)$. Since λ is arbitrary, we have that $\nu_s(E) = \chi_{f^{-1}(E)}(s)$ a.e. $[\nu]$, so $\nu = \delta_{f(s)}$ and (**) follows.

A general finite positive singular σ can be written as $\sigma = \sum_j a_j\delta_j + \tilde{\sigma}$, where $a_j > 0$, $\sum a_j < \infty$, δ_j is a unit point mass at θ_j, and $\tilde{\sigma}$ is a continuous singular measure. Then if $\{\sigma_\lambda\}$ is a continuous chain, $\sigma_\lambda = \sum \alpha_j(\lambda)\delta_j + \tilde{\sigma}_\lambda$, where for each j, α_j is a continuous increasing function of λ, $\alpha_j(0) = 0$, $\alpha_j(2\pi) = a_j$, and $\{\tilde{\sigma}_\lambda\}$ is a continuous chain for $\tilde{\sigma}$. If we let $\nu_j([0, \lambda]) = \alpha_j(\lambda)$, and $\tilde{\nu}([0, \lambda]) = \tilde{\sigma}_\lambda(T)$, then $d\nu(\lambda) = d[\sigma_\lambda(T)] = \sum_j d\nu_j(\lambda) + d\tilde{\nu}(\lambda)$.

Proposition 4.2. *Using the above notation,* (**) *holds if and only if the measures* ν_j, $j = 1, 2, \cdots$, *and* $\tilde{\nu}$ *are mutually singular,* $f(\theta) = \theta_j$ *a.e.* $[\nu_j]$, *and* $\tilde{\sigma}_\lambda(E) = \tilde{\nu}(f^{-1}(E) \cap [0, \lambda])$.

Proof. (**) implies that

$$\int_0^{2\pi} \frac{d\tilde{\sigma}_\lambda(\theta)}{1 - ze^{-i\theta}} + \sum_j \frac{\alpha_j(\lambda)}{1 - ze^{-i\theta_j}} = \sum_j \int_0^\lambda \frac{d\nu_j(\theta)}{1 - ze^{-i\theta}} + \int_0^\lambda \frac{d\tilde{\nu}(\theta)}{1 - ze^{-i\theta}}.$$

Since each side is a holomorphic function of z, we equate the nth derivatives evaluated at 0, and we get

$$\int_0^{2\pi} e^{-in\theta} d\tilde{\sigma}_\lambda(\theta) + \sum_j e^{-in\theta_j}\alpha_j(\lambda) = \sum_j \int_0^\lambda e^{-in\theta} d\nu_j(\theta) + \int_0^\lambda e^{-in\theta} d\tilde{\nu}(\theta).$$

Taking complex conjugates, and then considering linear combinations and monotone limits yields, for any bounded Borel function b,

$$\int_0^{2\pi} b(\theta)\, d\tilde{\sigma}_\lambda(\theta) + \sum_j b(\theta_j)\alpha_j(\lambda) = \sum_j \int_0^\lambda b(f(\theta_j))\, d\nu_j + \int_0^\lambda b(f(\theta))\, d\tilde{\nu}(\theta).$$

Let $b_K(\theta) = \chi_{\{\theta_K\}}(\theta)$ and $B_K = f^{-1}(\{\theta_K\})$. Then

$$\alpha_K(\lambda) = \int_0^\lambda d\nu_K(\theta) = \int_0^\lambda \chi_{B_K}(\theta)\left(\sum_j d\nu_j(\theta) + d\tilde{\nu}(\theta)\right).$$

Since λ is arbitrary, we have

$$\int_E d\nu_K(\theta) = \sum_j \int_{E\cap B_K} d\nu_j(\theta) + \int_{E\cap B_K} d\tilde{\nu}(\theta).$$

Hence, ν_K is carried in B_K, and

$$\tilde{\nu}(B_K) = \nu_j(B_K) = 0 \quad \text{if } \alpha \neq K.$$

Clearly $f(\theta) = \theta_K$ a.e. $[\nu_K]$, and considering only $(\bigcup_K B_K)^c$, Proposition 4.1 completes the proof.

We note that given any Borel measurable $f: [0, 2\pi] \xrightarrow{\text{onto}} [0, 2\pi]$, and any continuous singular σ, there is a Borel measure ν such that $\nu(f^{-1}(E)) = \sigma(E)$ [9]. If we choose such a ν and define $\sigma_\lambda(E) = \nu(f^{-1}(E) \cap [0, \lambda])$, then $\{\sigma_\lambda\}$ is a continuous chain with corresponding $F(z, \lambda) = (1 - ze^{-if(\lambda)})^{-1}$. Thus, all chains arise in this manner, and any onto f may occur.

It is difficult to determine whether a given chain $\{\sigma_\lambda\}$, where σ is continuous, satisfies the condition of Proposition 4.2. We now consider those chains obtained by letting $\sigma_\lambda = \sigma|_{A_\lambda}$, i.e., σ restricted to A_λ, where $\{A_\lambda\}$ is an increasing family of Borel sets. We see that if $(A_\lambda - B_\lambda) \cup (B_\lambda - A_\lambda)$ is countable for all λ, then $\sigma|_{A_\lambda} = \sigma|_{B_\lambda}$ for all continuous σ, so it suffices to consider collections $\{A_\lambda\}$ modulo this equivalence relation. We now suppose that $\{\sigma|_{A_\lambda}\}$ is a chain which satisfies (**) for all continuous singular σ, and we characterize the collection $\{A_\lambda\}$.

We first note that continuity of the chain $\{\sigma_\lambda\}$ implies that $S_\lambda = (A_\lambda - \bigcup_{\mu<\lambda} A_\mu)$ and $D_\lambda = (\bigcap_{\mu>\lambda} A_\mu - A_\lambda)$ are at most countable for all λ. (If not, one could find a continuous σ carried in S_{λ_0} or D_{λ_0}, and $a(\lambda) = \sigma_\lambda(T)$ would jump at λ_0.) Hence $\{A_\lambda\}$ is equivalent to $\{\bigcap_{\mu>\lambda} A_\mu\}$, and we may assume that $A_\lambda = \bigcap_{\mu>\lambda} A_\mu$. We may also assume that $A_0 = \emptyset$ and $A_{2\pi} = [0, 2\pi]$.

For $x \in [0, 2\pi]$, let $\lambda_x = \inf\{\lambda | x \in A_\lambda\}$, and define $\gamma(x) = \lambda_x$. Then $\gamma^{-1}(\lambda_x) = \{y | y \in A_{\lambda_x}, \lambda < \lambda_x \Rightarrow y \notin A_\lambda\} = S_{\lambda_x}$, and $\gamma^{-1}([0, \lambda]) = \bigcup_{\mu<\lambda} S_\mu = \bigcup_{\mu<\lambda}(A_\mu - \bigcup_{\nu<\mu} A_\nu) = A_\lambda$, so γ is Borel measurable and $\gamma^{-1}(p)$ is countable for all p. Hence, there exists a countable (perhaps finite) collection of disjoint Borel sets $\{B_j\}$ with $\bigcup_j B_j = [0, 2\pi]$, $\gamma_j = \gamma|_{B_j}$ 1-1, $\gamma(B_j) \supset \gamma(B_{j+1})$ [4]. For any continuous σ, define σ_j by

$$\sigma_j(S) = \sigma(\gamma_j^{-1}(S)) = \sigma^{\gamma_j}(S), \qquad S \subset T.$$

Let $\beta_j: [0, 2\pi] \to [0, 2\pi]$ be defined by

$$\begin{aligned} \beta_j(x) &= \gamma_j^{-1}(x) \quad \text{if } x \in \gamma_j(B_j), \\ &= 0 \qquad\qquad \text{if } x \notin \gamma_j(B_j). \end{aligned}$$

Then $\sigma_j(S) = \sigma(\beta_j(S))$ since $\sigma(\{0\}) = 0$.

Now, suppose $S_\lambda = (A_\lambda - \bigcup_{\mu<\lambda} A_\mu)$ has at least two elements for all $\lambda \in E$, where E is uncountable. Then $\{B_j\}$ has (at least) two uncountable sets, B_1 and B_2. Clearly we can choose a continuous singular σ carried on $\tilde{S} = \gamma_1^{-1}(\gamma_2(B_2)) \cup B_2$ such that $\sigma(\gamma_1^{-1}(S)) = \sigma_1(S) = \sigma(\gamma_2^{-1}(S)) = \sigma_2(S)$ for all $S \subset T$. Then $\sigma_j(S) = 0$

if $j > 2$, and $\nu([0, \lambda]) = \sigma_\lambda(T) = \sigma(A_\lambda) = \sigma(\gamma^{-1}[0, \lambda]) = \Sigma_j \sigma_j([0, \lambda]) = (\sigma_1 + \sigma_2)([0, \lambda]) = 2\sigma_1([0, \lambda])$.

Since $\{\sigma_\lambda\}$ satisfies (**) for some f, we have

$$\int_0^\lambda \frac{d\nu(\theta)}{1 - ze^{-if(\theta)}} = \int_0^{2\pi} \frac{d\sigma_\lambda(\theta)}{1 - ze^{-i\theta}} = \int_{A_\lambda} \frac{d\sigma(\theta)}{1 - ze^{-i\theta}}$$

$$= \sum_j \int_{\gamma_j^{-1}([0,\lambda])} \frac{d\sigma(\theta)}{1 - ze^{-i\theta}}$$

$$= \int_0^\lambda \frac{d\sigma_1(\theta)}{1 - ze^{-i\beta_1(\theta)}} + \int_0^\lambda \frac{d\sigma_2(\theta)}{1 - ze^{-i\beta_2(\theta)}} + 0$$

$$= \frac{1}{2} \int_0^\lambda [(1 + ze^{-i\beta_1(\theta)})^{-1} + (1 + ze^{-i\beta_2(\theta)})^{-1}]\, d\nu(\theta).$$

Comparing nth derivatives at $z = 0$, we have, since λ is arbitrary,

$$2e^{-inf(\theta)} = e^{-in\beta_1(\theta)} + e^{-in\beta_2(\theta)} \quad \text{a.e. } [\nu].$$

Thus, $\beta_1(\theta) = \beta_2(\theta)$ a.e. $[\nu]$, which is impossible since $B_1 \cap B_2 = \emptyset$ and $\beta_i(\theta) \in B_i$. Thus, the set E is countable, and by taking an equivalent collection $\{A_\lambda\}$, we may assume $E = \emptyset$.

Proposition 4.3. *Suppose $\{A_\lambda\}$ induces a unitary map $V: L^2(\mu) \to M$ for all continuous singular σ. Then*

$$\left(A_\lambda - \bigcup_{\mu>\lambda} A_\mu\right) = \{p_\lambda\}, \qquad \lambda \in E,$$

$$= \emptyset, \qquad \lambda \in E^c,$$

for some set E. Further, (**) *holds with $f(\lambda) = p_\lambda$, $A_\lambda = f(E \cap [0, \lambda])$, $\nu(S) = \sigma(f(E \cap S))$, and $\nu(E^c) = 0$.*

Proof. We have proved all but the final statement:

$$E = \left\{\lambda \,\middle|\, S_\lambda = \left(A_\lambda - \bigcup_{\mu>\lambda} A_\mu\right) = \{f(\lambda)\}\right\}$$

$$= \{\lambda | \lambda = \lambda_x = \gamma(x) \text{ for some } x \in [0, 2\pi]\} = \gamma([0, 2\pi]).$$

Let $x \in A_\lambda$. Then $x \in A_{\lambda_x}$ for $\lambda_x \le \lambda$, so $x \in S_{\lambda_x}$. Thus, $S_{\lambda_x} = \{x\}$ and $x = f(\lambda_x) \in f(E \cap [0, \lambda])$. Let $x \in f(E \cap [0, \lambda])$. Then $x = f(\lambda_x)$, $\lambda_x \le \lambda$, $\lambda_x \in E$. Hence, $x \in A_{\lambda_x} \subseteq A_\lambda$, so $A_\lambda = f(E \cap [0, \lambda])$. $\sigma_\lambda(S) = \nu[f^{-1}(S) \cap [0, \lambda]]$, so

$\nu(S) = \sigma(f(S \cap E))$ for all $S \subset T$ since $f|_E$ is 1-1.

Thus, $\nu(E^c) = 0$.

Conversely, if $\{A_\lambda\}$ is an increasing family of Borel sets with

(i) $A_0 = \emptyset$, $A_{2\pi} = [0, 2\pi]$,

(ii) $\bigcap_{\mu>\lambda} A = A_\lambda$, and

(iii)
$$A_\lambda - \bigcup_{\mu>\lambda} A_\mu = \{p_\lambda\}, \quad \lambda \in E,$$
$$= \emptyset, \quad \lambda \in E^c,$$

for some Borel set E, we can define $f(\lambda) = p_\lambda$ if $\lambda \in E$, $f(\lambda) = 0$ if $\lambda \in E^c$. Then f is Borel measurable by Kuratowski's isomorphism theorem [8]. If we let $\sigma_\lambda = \sigma|_{A_\lambda}$, then $\{\sigma_\lambda\}$ is a continuous chain satisfying (**), and hence induces an onto map V.

5. **More general isometries.** The isometries we have defined are closely related to the map $U = \int e^{i\lambda}\, dP_\lambda \colon M \to M$. (This is defined as the limit in the strong operator topology of appropriate simple functions. See [3] for details.) This leads us to examine the special role played by the function 1.

Proposition 5.1. *$V\colon L^2(\mu) \to M$ is onto if and only if 1 is cyclic for $\int e^{i\lambda}\, dP_\lambda$. More precisely,*

$$\text{Range}(V) = \text{span}\{U^n 1\}_{n=-\infty}^{\infty}.$$

Proof. Since $d(P_x \int_0^{2\pi} c(\lambda)\, dP_\lambda f) = c(x)\, dP_x f$, we have $U^n = \int e^{in\lambda}\, dP_\lambda$, $n = 0, \pm 1, \cdots$. Thus, $U^n 1 = V(e^{in\lambda})$. The proposition now follows since $\{e^{in\lambda}\}$ is dense in $L^2(\mu)$.

For $f \in M$, define $\mu_f([0, \lambda]) = (P_\lambda f, f)$. Then μ_f is a positive Borel measure and we have $V_f\colon L^2(\mu_f) \to M$ defined by $(V_f c)(z) = \int c(\lambda)\, d(P_\lambda f)(z)$. Analogous to $V = V_1$, we have

Proposition 5.2. *$V_f\colon L^2(\mu_f) \to M$ is an isometry. $VQ_\lambda = P_\lambda V$, where $Q_\lambda\colon L^2(\mu_f) \to L^2(\mu_f|_{[0,\lambda]})$ by restriction, and* Range $(V_f) = \text{span}\{U^n f\}_{n=-\infty}^{\infty}$.

Now, given $\{\sigma_\lambda\}$, $\{s_\lambda\}$, and $\{M_\lambda\}$, we can ask whether some $V_f\colon L^2(\mu_f) \to M$ is unitary. We show below that $V = V_1\colon L^2(\mu) \to M$ is the best candidate for a unitary map.

Lemma 5.2. *Let $g_n(z) = z^n$. Then for $z \in D$, $n \geq 0$, $\mu_z^{(n)}([0, \lambda]) = (P_\lambda g_n)(z)$ is a complex Borel measure. For any $z \in D$, $n \geq 0$, $\mu_z^{(n)} \ll \mu$.*

Proof. The proof follows by a simple induction.

Proposition 5.3. *Suppose V_f: $L^2(\mu_f) \to M$ is onto. Then V: $L^2(\mu) \to M$ is onto.*

Proof. If V_f is onto, there is a $c_1 \in L^2(\mu_f)$ with

$$(V_f c_1) = \int_0^{2\pi} c_1(\lambda)\, dP_\lambda f = 1 - s(z)s(0) = \int_0^{2\pi} d(P_\lambda 1)(z).$$

Then, $d(P_\lambda 1)(z) = c_1(\lambda)\, d(P_\lambda f)(z)$ for $z \in D$. In particular, $c_1(\lambda)\, d(P_\lambda f)(0) = d\mu(\lambda)$.

Suppose $\mu_f(E) = 0$, $F \subset E$. Then $0 = \|\chi_F\|^2_{L^2(\mu_f)} = \|V_f\chi_F\|^2_{H^2}$, so $V\chi_F \equiv 0$. Thus, $0 = (V\chi_F) = \int_F d(P_\lambda f)(0)$, so $\mu(F) = 0$ and $\mu \ll \mu_f$.

Suppose $\mu(E) = 0$. Then by Lemma 5.2, $\int \chi_E(\lambda)\, dP_\lambda$: $M \to M$ annihilates z^n for all $n \geq 0$, and is hence the zero operator. Hence, $\int_E dP_\lambda f \equiv 0 = (V_f \chi_E)$, and $\mu_f \ll \mu$. Thus, given $c \in L^2(\mu_f)$, c/c_1 is well defined with respect to the measure algebra of μ, and

$$\|c\|_{L^2(\mu_f)} = \int |c(\lambda)|^2\, d\mu_f(\lambda) = \int |c(\lambda)|^2\, d(P_\lambda f, f)$$

$$= \int |c(\lambda)|^2 \frac{1}{|c_1(\lambda)|^2}\, d(P_\lambda 1, 1) = \left\|\frac{c}{c_1}\right\|^2_{L^2(\mu)}.$$

Thus, $V_1(c/c_1) = V_f(c)$ so V_1 is onto.

6. **Conclusion.** Let $\{\sigma_\lambda\}$ be a chain yielding a unitary V: $L^2(d\mu) \xrightarrow{\text{onto}} M$, with $F(z, \lambda) = (1 - ze^{-if(\lambda)})^{-1}$. Then, by techniques similar to those of Ahern and Clark [1], who used $\sigma_\lambda = \sigma|_{(0,\lambda)}$ which corresponds to $f(\lambda) = (1 - ze^{-i\lambda})^{-1}$, we have that

$$(V^* g)(\lambda) = \lim_{r \to 1} (2\pi s_\lambda(0))^{-1} \int_0^{2\pi} g(e^{i\theta})\, \overline{s_\lambda}(re^{i\theta})(1 - re^{if(\lambda)})^{-1}\, d\theta.$$

If T: $M \to M$ is the restricted shift defined by $Tg = P_M zg$, $g \in M$, and M, K: $L^2(\mu) \to L^2(\mu)$ are defined by

$$(Mc)(\lambda) = e^{if(\lambda)} c(\lambda), \qquad (Kc)(\lambda) = 2\int_0^\lambda e^{\sigma_t(T) - \sigma_\lambda(T)} c(t)\, d\mu(t),$$

then $T = V(I - K)MV^*$. Thus, we say that V implements the concrete model theory for $((I-K)M)^*$. This is a special case of a class of operators considered by Kriete [7]. In the nononto case, $V(L^2(\mu))$ is the image of a certain one-dimensional section of $\mathfrak{D}$. It is not clear whether this space has any special significance with respect to restricted shifts or $*$-invariant subspaces.

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