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INDECOMPOSABLE POLYTOPES

BY

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ABSTRACT. The space of summands (with respect to vector addition) of a convex polytope in n dimensions is studied. This space is shown to be isomorphic to a convex pointed cone in Euclidean space. The extreme rays of this cone correspond to similarity classes of indecomposable polytopes. The decomposition of a polytope is described and a bound is given for the number of indecomposable summands needed. A means of determining indecomposability from the equations of the bounding hyperplanes is given.

1. Introduction. If A and B are convex compact subsets of E^n, their Minkowski sum, $A + B$, is defined

$$A + B = \{a + b: a \in A, b \in B\}.$$

Analogously $\lambda A = \{\lambda a: a \in A\}$ for $\lambda \geq 0$. C is said to be indecomposable if the decomposition $C = A + B$ is possible only when A and B are each of the form $\lambda A + \{x\}$ for some nonnegative λ's and vectors x.

Indecomposability was introduced by Gale [2] in an abstract which contains, without proof, numerous observations concerning decomposable and indecomposable sets. Grünbaum [3, p. 243, 244] mentions the addition of convex sets in connection with measures of symmetry and calls attention to the question of determining the necessary and sufficient conditions for a polytope to be indecomposable. This question was studied by Shephard [7] who found a sufficient condition which deals with the combinatorial type of the polytope. The purpose of this paper is to answer Grünbaum's question by giving the necessary and sufficient condition. We also show that any polytope can be written as a sum of indecomposable polytopes, and we give a sharp bound on the number required. This refines a result announced by Gale [2]. More complete details of the outline presented here can be found in [6]. McMullen [5] presents an alternative approach to the same problems.

2. Preliminaries. Rigorous study of polytopes can be fruitfully based on the support function and the tools of linear algebra. The support function of a compact convex set Q, denoted $h(Q, u)$ is defined on E^n as follows:

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$$h(Q,\ u) = \sup_{x \in Q} (x \cdot u).$$

In these terms, the supporting hyperplane to Q with outer normal u (u different from 0, the zero vector) has equation $x \cdot u = h(Q,\ u)$. We denote by $S(Q,\ u)$ the intersection of Q with this hyperplane. In symbols $S(Q,\ u) = \{x \in Q\colon x \cdot u = h(Q,\ u)\}$ when $u \neq 0$. If $u = 0$, $S(Q,\ u)$ is not defined.

We will write $R \leq Q$ provided $\dim S(R,\ u) \leq \dim S(Q,\ u)$ for all $u \neq 0$. The following theorem of Shephard [7] shows the value of this relation and is central to our methods:

Theorem 1. *There exists a polytope S so that $R + S = Q$ iff $R \leq Q$ and, for each $u \neq 0$ for which $S(Q,\ u)$ is an edge of Q, the corresponding face $S(R,\ u)$ is either a vertex or an edge of R no longer than $S(Q,\ u)$.*

The following result [6] (also discovered independently by Flowers [1]) will provide a useful alternative to the definition of the $\leq$ relation.

Theorem 2. *If R, Q are polytopes in E^n, $n \geq 2$, then the following are equivalent:*

(a) $R \leq Q$.

(b) $\dim S(Q,\ u) = 0$ *implies* $\dim S(R,\ u) = 0$.

(c) *If we denote by $N_P(F)$ the set of all outer normals to hyperplanes which intersect the polytope P along the face F, then for each $u \neq 0$, $N_Q(S(Q,\ u)) \subset N_R(S(R,\ u))$, and consequently $\{N_Q(Q')\colon Q'$ is a face of $Q\}$ is a refinement of $\{N_R(R')\colon R'$ is a face of $R\}$.*

If $Q \leq R$ and $R \leq Q$ we call R and Q locally similar and write $R \sim Q$. Local similarity is clearly an equivalence relation. The equivalence class determined by Q will be denoted $[Q]$. If $Q \leq R$ but it is not the case that $Q \sim R$, then we write $Q < R$.

3. **The summands of** P. For the remainder of this paper, P will denote a fixed n-dimensional polytope in E^n, where $n \geq 2$. $\{u_1, u_2, \cdots, u_f\}$ will denote a set of outer normals, one for each $n-1$ dimensional face of P. Henceforth, subscripted u's will always refer to this set of normals. The space of summands of P, denoted $\mathfrak{S}(P)$, is defined as follows.

Definition. $\mathfrak{S}(P) = \{R$: there exist a Q and $\lambda \geq 0$ such that $R + Q = \lambda P\}$. (Alternatively, by Theorem 1, $\mathfrak{S}(P) = \{Q\colon Q \leq P\}$.)

It is not hard to see that $\mathfrak{S}(P)$ consists entirely of polytopes and is a closed convex cone. Furthermore, all members of $\mathfrak{S}(P)$ are formed by intersecting half-spaces with outer normals among the u_i. That is

Lemma 1. *If* $R \in \mathfrak{S}(P)$, $R = \{x\colon x \cdot u_i \leq h(R, u_i)$ *for* $i = 1, 2, \cdots, f\}$.

Proof. Suppose R is n-dimensional and $R + Q = \lambda P$. R is the intersection of the half-spaces containing R which are determined by those hyperplanes generated by the $n-1$ dimensional faces (see Grünbaum [4, p. 31]). These half-spaces are among the half-spaces $\{x\colon x \cdot u_i \leq h(R, u_i)\}$; $i = 1, 2, \cdots, f$, for the following reason: for any u, $S(Q, u) + S(R, u) = S(\lambda P, u) = \lambda S(P, u)$ and so $\dim S(R, u) = n - 1$ implies $\dim S(P, u) = n - 1$ whence u is one of $u_1, u_2, \cdots, u_f$. Thus $\bigcap_1^f \{x\colon x \cdot u_i \leq h(R, u_i)\} \subset R$. But clearly $R \subset \bigcap_1^f \{x\colon x \cdot u_i \leq h(R, u_i)\}$.

Now if R is not of dimension n, it has dimension $< n$. Let $R' = R + P$, clearly an n-polytope since P is and since $R, P \subset E^n$. Since $h(R', u) = h(R, u) + h(P, u)$ we have

$$\begin{aligned} R + P = R' &= \{x\colon x \cdot u_i \leq h(R', u_i),\ i = 1, \cdots, f\} \\ &\supset \{x\colon x \cdot u_i \leq h(R, u_i),\ i = 1, \cdots, f\} + \{x\colon x \cdot u_i \leq h(P, u_i),\ i = 1, \cdots, f\} \\ &= \{x\colon x \cdot u_i \leq h(R, u_i),\ i = 1, \cdots, f\} + P. \end{aligned}$$

Therefore, $R \supset \{x\colon x \cdot u_i \leq h(R, u_i),\ i = 1, \cdots, f\}$. But by definition of the support function $R \subset \{x\colon x \cdot u_i \leq h(R, u_i),\ i = 1, \cdots, f\}$.

This lemma may be used to show:

Lemma 2. *Suppose* $F = S(R, u)$ *and is a face of* R, *where* $R \in \mathfrak{S}(P)$. *Let* U' *be the maximal subset of* $U = \{u_1, \cdots, u_f\}$ *with the property that* $F \subset \{x\colon x \cdot u_i = h(R, u_i)\}$ *for all* $u_i \in U'$ (*alternatively,* $F \subset S(R, u_i)$). *Then* $\dim F = n - \dim U'$.

Now we define a mapping $\Delta\colon \mathfrak{S}(P) \to E^f$ by

$$\Delta(R) = (h(R, u_1), h(R, u_2), \cdots, h(R, u_f)).$$

The mapping Δ is clearly linear and continuous (in terms of the Hausdorff metric on $\mathfrak{S}(P)$) because $R \to h(R, u)$ has these properties. Lemma 1 shows that Δ is 1-1. It is not hard to show that if $\Delta(Q_i) \to \Delta(Q)$ then $Q_i \to Q$ so Δ^{-1} is continuous as well. Consequently, we will sometimes work with the closed convex cone $\Delta\mathfrak{S}(P)$ instead of $\mathfrak{S}(P)$ so that we can use the tools of linear algebra. Results about $\mathfrak{S}(P)$ will translate immediately into results about $\Delta\mathfrak{S}(P)$ and vice versa.

Now for the main machinery. We will associate with each local similarity class $[R]$ a set of linear homogeneous equations and inequalities in f variables. It will turn out that the set of f-tuples of real numbers satisfying this system of equations and inequalities is precisely $\Delta[R]$. To define these equations and inequalities we associate with R certain distinquished subsets of the indices $1, 2, \cdots, f$ as follows. Suppose we have $n+1$ of these indices, say $1, 2, \cdots, n+1$, such that the hyperplanes $x \cdot u_i = h(R, u_i)$ for $i = 1, 2, \cdots, n$ have as their entire intersection a vertex of R, say r_0. Then we call $\{1, 2, \cdots, n+1\}$ a "vertex set" of indices for the polytope R. Such sets come in two types:

(1) $r_0 \cdot u_{n+1} = h(R, u_{n+1})$ (i.e., all hyperplanes are concurrent at r_0);

(2) $r_0 \cdot u_{n+1} < h(R, u_{n+1})$ (i.e., the hyperplanes are not all concurrent).

In case (1), we say $\{1, \cdots, n+1\}$ is a vertex set of type $E(R)$. In case (2), we say it is of type $I(R)$.

In either case, there is an $(n+1)$-tuple of scalars $\lambda_1, \cdots, \lambda_{n+1}$ such that $\sum_1^{n+1} \lambda_1 u_i = 0$. Since the hyperplanes $x \cdot u_i = h(R, u_i)$ for $i = 1, \cdots, n$ intersect in a point, the $u_1, \cdots, u_n$ are linearly independent and the $(n+1)$-tuple of λ_i's is essentially unique (to within a scalar multiple) and $\lambda_{n+1} \neq 0$. We can assume $\lambda_{n+1} > 0$. and henceforth we do so. In case (1):

$$0 = r_0 \cdot \left(\sum_1^{n+1} \lambda_i u_i\right) = \sum_1^{n+1} \lambda_i (r_0 \cdot u_i) = \sum_1^{n+1} \lambda_i h(R, u_i).$$

In case (2):

$$0 = r_0 \cdot \left(\sum_1^{n+1} \lambda_i u_i\right) = \sum_1^{n+1} \lambda_i (r_0 \cdot u_i) < \sum_1^{n+1} \lambda_i h(R, u_i).$$

Lemma 3. (1) *If $Q \leq R$ and J is a set of indices such that there exists a vertex r of R with $r \cdot u_i = h(R, u_i)$ when $i \in J$, then there exists a vertex q of Q with $q \cdot u_i = h(Q, u_i)$ when $i \in J$.*

(2) *If $Q < R$ then there exists a set of indices of type $I(R)$ which is also of type $E(Q)$.*

Proof. (1) Suppose $Q \leq R$ with $Q + T = \lambda R$ and $r = S(R, u)$, $q = S(Q, u)$, $t = S(T, u)$. Then $q + t = \lambda r$ and

$$(*) \qquad q \cdot u_i \leq h(Q, u_i), \qquad t \cdot u_i \leq h(T, u_i)$$

for each i by definition of the support function. Consequently

$$(**) \qquad \begin{aligned} \lambda r \cdot u_i &= q \cdot u_i + t \cdot u_i \leq h(Q, u_i) + h(T, u_i) \\ &= h(Q + T, u_i) = h(\lambda R, u_i) \end{aligned}$$

with equality iff equality holds in both of $(*)$. But for $i \in J$ we have equality in $(**)$ whence we also have it in $(*)$.

(2) If $Q < R$ then by Theorem 2 there exists u such that $S(Q, u) = q$, a vertex of Q, while $\dim S(R, u) > 0$. $S(R, u)$ therefore contains at least two distinct vertices of R, say r_1 and r_2. We can find indices, $1, 2, \cdots, n+1$, say, such that

(a) r_1 is the unique intersection of hyperplanes $x \cdot u_i = h(R, u_i)$, $i = 1, \cdots, n$;

(b) r_2 lies on $x \cdot u_{n+1} = h(R, u_{n+1})$ but r_1 does not. Now suppose $Q + T = \lambda R$ with $t_1, t_2 \in S(T, u)$ such that

$$q + t_1 = \lambda r_1, \qquad q + t_2 = \lambda r_2.$$

As before we have

$$\lambda r_j \cdot u_i = q \cdot u_i + t_j \cdot u_i \leq h(Q, u_i) + h(T, u_i) = h(\lambda R, u_i)$$

with equality iff $q \cdot u_i = h(Q, u_i)$ and $t_j \cdot u_i = h(T, u_i)$. But when $j = 1$ we have equality if $i = 1, \cdots, n$ and if $j = 2$ equality holds when $i = n + 1$. Consequently, $q \cdot u_i = h(Q, u_i)$ when $i = 1, \cdots, n + 1$, making that set of indices an $E(Q)$. But (a) and (b) guarantee that this set is an $I(R)$.

Lemma 4. *Let $\delta_1, \delta_2, \cdots, \delta_f$ be real numbers such that*

(a) *for every $E(R)$ with $\Sigma_{i \in E(R)} \lambda_i u_i = 0$, we have $\Sigma_{i \in E(R)} \lambda_i \delta_i = 0$, and*

(b) *for every $I(R)$ with $\Sigma_{i \in I(R)} \lambda_i u_i = 0$, we have $\Sigma_{i \in I(R)} \lambda_i \delta_i \geq 0$.*

Then if $Q = \{x\colon x \cdot u_i \leq \delta_i, \ i = 1, \cdots, f\}$, $Q \leq R$.

Proof. We will show that $\dim S(R, u) = 0$ implies $\dim S(Q, u) = 0$ and then rely upon Theorem 2. Suppose $S(R, u) = r_0$ and r_0 lies on $x \cdot u_i = h(R, u_i)$ for a maximal set of indices J. By Lemma 2 there exist at least n indices, say $1, 2, \cdots, n$, in J which correspond to a linearly independent set of u_i's. The corresponding $x \cdot u_i = \delta_i$, $i \in J$, are concurrent at a point q_0 (if J contains only n indices this is obvious; if J has more than n indices we rely on part (a) of the hypothesis).

We next show that $q_0 \in Q$. For if not, then

$$(*) \qquad q_0 \cdot u_i > \delta_i \quad \text{for at least one } i$$

where $i \notin J$, for convenience, say $i = n + 1$. Now $\{1, 2, \cdots, n + 1\}$ is either an $E(R)$ or an $I(R)$ (recall that $r_0 \cdot u_i = h(R, u_i)$ if $i = 1, 2, \cdots, n$). So if $\Sigma_1^{n+1} \lambda_i u_i = 0$, $\lambda_{n+1} > 0$, then $\Sigma_1^{n+1} \lambda_i \delta_i \geq 0$ by hypothesis. But $(*)$ yields

$$0 = q_0 \cdot \left(\sum_1^{n+1} \lambda_i u_i \right) = \sum_1^{n+1} \lambda_i (q_0 \cdot u_i) > \sum_1^{n+1} \lambda_i \delta_i,$$

a contradiction.

Now let q be any point of Q. We wish to show $q \cdot u \leq q_0 \cdot u$ with equality iff $q = q_0$ (here u is the earlier mentioned vector such that $r_0 = S(R, u)$). This will show that $S(Q, u) = q_0$ and $\dim S(Q, u) = 0$.

If $i \notin J$, $r_0 \cdot u_i < h(R, u_i)$ and so we can choose a nonnegative λ sufficiently small that

$$r_0 \cdot u_i + \lambda(q - q_0) \cdot u_i < h(R, u_i)$$

whenever $i \notin J$. If $i \in J$, we have

$$r_0 \cdot u_i + \lambda(q - q_0) \cdot u_i = h(R, u_i) + \lambda(q \cdot u_i) - \lambda\delta_i$$

for any $\lambda \geq 0$.

But since $q \in Q$, $q \cdot u_i \leq \delta_i$, so

$$r_0 \cdot u_i + \lambda(q - q_0) \cdot u_i \leq h(R, u_i) \quad \text{for } i \in J.$$

Therefore, if $\lambda \geq 0$ is sufficiently small,

$$r_0 \cdot u_i + \lambda(q - q_0) \cdot u_i \leq h(R, u_i),$$

for any i whatsoever. By Lemma 1 this means $r_0 + \lambda(q - q_0) \in R$ for the appropriate λ. But then, since $r_0 = S(R, u)$,

$$[r_0 + \lambda(q - q_0)] \cdot u \leq h(R, u) = r_0 \cdot u$$

with equality iff $r_0 + \lambda(q - q_0) = r_0$. Equivalently,

$$q \cdot u \leq q_0 \cdot u$$

with equality iff $q = q_0$. This finishes the proof.

Corollary 1. *If $Q, R \in \mathcal{S}(P)$ then the following are equivalent:*

(1) $Q \leq R$.

(2) (a) *Each $E(R)$ is an $E(Q)$, and* (b) *each $I(R)$ is an $E(Q)$ or an $I(Q)$.*

(3) (a) *For every $E(R)$ with $\Sigma_{i \in E(R)} \lambda_i u_i = 0$ we have $\Sigma_{i \in E(R)} \lambda_i h(Q, u_i) = 0$, and* (b) *for every $I(R)$ with $\Sigma_{i \in I(R)} \lambda_i u_i = 0$ we have $\Sigma_{i \in I(R)} \lambda_i h(Q, u_i) \geq 0$.*

Now to each R we can associate a set of linear homogeneous relations as follows. For an index set of the type $E(R)$ where $\Sigma_{i \in E(R)} \lambda_i u_i = 0$, assign the equation $\Sigma_{i \in E(R)} \lambda_i \xi_i = 0$ in the variables ξ_i. For an $I(R)$ corresponding to $\Sigma_{i \in I(R)} \lambda_i u_i = 0$ we assign $\Sigma_{i \in I(R)} \lambda_i \xi_i > 0$. Our last result shows that these relations depend only on $[R]$, that is if $Q \sim R$ then Q gives rise to these relations too. Define $e[R]$ and $i[R]$ respectively to be the essentially unique systems of equations and inequalities respectively corresponding to $E(R)$ and $I(R)$ respectively.

Lemma 5. (1) *$\Delta(R)$ is the solution set of $e[R] \cup i[R]$, a relatively open set in the flat determined by $e[R]$.*

(2) $Q < R$ *iff* $\Delta(Q) \in \text{rel bd}(\Delta[R])$.

Proof. Part 1 follows from the last result and Lemma 4.

As for (2), if $Q < R$ then part (2) of Lemma 3 gives $\Delta(Q) \in \text{rel bd}(\Delta[R])$. On the other hand, if $\Delta(Q) \in \text{rel bd}(\Delta[R])$, then $\Delta(Q) \notin \Delta[R]$ since $\Delta[R]$ is relatively open. Thus it is not the case that $Q \sim R$. But $\Delta(Q) \in \overline{\Delta[R]} \subset \mathcal{S}(R)$ so $Q \leq R$. Thus $Q < R$.

At this point it is possible to give a condition for indecomposability of a polytope based on the recognition that R is indecomposable iff $\Delta(R)$ lies on a face of $\Delta\mathcal{S}(P)$ of dimension n. We shall not prove this because, by restricting ourselves to polytopes with Steiner points at 0, we can get a more pleasing result. Similarly, we can now represent any ΔQ in $\Delta\mathcal{S}(P)$ as a positive combination of points lying on extreme rays of $\Delta\mathcal{S}(P)$. To achieve a better result we now investigate polytopes with Steiner points at 0.

4. **Polytopes with Steiner point at the origin.** The Steiner point of a polytope [8] may be defined

$$s(P) = \sum_{1}^{t} V(p_i)\, p_i \tag{4.1}$$

where $p_1, \cdots, p_t$ are the vertices of P and $V(p_i)$ is the ratio of the $n-1$ content of that portion of the unit $n-1$ sphere centered at 0 cut off by $N_p(p_i)$ to the $n-1$ content of the whole unit $n-1$ sphere.

The Steiner point mapping $P \to s(P)$ has some interesting properties [8] among which is: $s(P+Q) = s(P) + s(Q)$. Given a polytope P it is therefore clear that by choosing $x = -s(P)$, the translate $P + x$ is the unique translate of P whose Steiner point is at 0.

Henceforth, we restrict ourselves to sets with Steiner point at 0. In so doing, no decompositions of a set are really "lost" provided we are content not to distinguish between two decompositions, one of which consists of sets which are translates of the corresponding sets in the other decomposition. For if $A + B = C$ we have $s(C) = s(A) + s(B)$ whence $(A - s(A)) + (B - s(B)) = (C - s(C))$. The sets in the latter decomposition have Steiner points at 0. The appropriate definition of indecomposability becomes

Definition. A is indecomposable if $A = B + C$ is possible only if B and C are of the form λA for suitable nonnegative λ's.

We will use the following notations:

$$\mathcal{S}_0(P) = \{Q\colon Q \in \mathcal{S}(P) \text{ and } s(Q) = 0\}, \qquad [Q]_0 = \{R\colon R \in [Q] \text{ and } s(R) = 0\}.$$

We shall now show that restricting a polytope to have $s(R) = 0$ as well as $R \sim Q$ amounts to adding n more linear homogeneous equations to those defining $\Delta[Q]$.

Let $r_1, r_2, \cdots, r_t$ be the vertices of P. Each r_j can be gotten as the intersection of certain of the hyperplanes $x \cdot u_i = h(R, u_i)$ (see Lemma 1) and so has coordinates which are linear homogeneous functions of the quantities $h(R, u_i)$ (if we solve for a coordinate using Cramer's rule, one column of the determinant in the numerator has the $h(R, u_i)$ as entries). Furthermore, if $Q \sim R$ then to each vertex $r_j \in R$ there exists a vertex $q_j \in Q$ which is the intersection of the corresponding hyperplanes $x \cdot u_i = h(Q, u_i)$ (Corollary 1, part 2(a)). This q_j has coordinates which are the same linear homogeneous functions of the $h(Q, u_i)$.

By Theorem 2, $Q \sim P$ implies $V(p_j) = V(q_j)$. In view of the linear form of 4.1 and our remarks about the determination of the vertices by linear homogeneous functions, which functions are the same for all members of the local similarity class, it follows that, for polytopes Q in a fixed local similarity class $[P]$, the requirement $s(Q) = 0$ is equivalent to the requirement that the numbers $h(Q, u_i)$ satisfy n linear homogeneous equations in the variables $\xi_1, \xi_2, \cdots, \xi_f$. Call this set of equations $e_0[P]$.

Lemma 6. *The rank of the system* $e[P] \cup e_0[P]$ *is* $n + \operatorname{rank}(e[P])$.

Proof. If $Q \sim P$ and $x \in E^n$, we can find a suitable translate, Q', of Q with $s(Q') = x$. Simply take $Q' = Q + (x - s(Q))$. Thus the coordinates of the Steiner point are independently variable regardless of the local similarity class. Now each equation of $e_0[P]$ refers to one particular coordinate, so it follows that no equation of $e_0[P]$ can be deduced from any combination of equations in $e[P] \cup e_0[P]$ which does not include the equation in question. This yields the result.

Apparently we are faced with the inconvenience that, for different local similarity classes, we obtain different systems $e_0[P]$. Our next lemma shows that, while the equations may be different, the inconvenience this poses is only apparent.

Lemma 7. *Let* A *be the flat in* E^f *determined by equations* $e_0[P]$. *If* $Q \in \mathcal{S}(P)$ *then a necessary and sufficient condition for* $s(Q) = 0$ *is that* $\Delta(Q) \in A$.

Proof. We have already discussed the case where $Q \sim P$ so let us assume $Q < P$. Then by Lemma 5, $\Delta(Q) \in \operatorname{rel\,bd}(\Delta[P])$. Choose P' in $[P]$ with $s(P') = 0$ (i.e., $\Delta(P') \in \Delta[P] \cap A$). Since $[P]$ is relatively open (Lemma 5) $P' \in \operatorname{rel\,int}[P]$ and, if $0 < \lambda \leq 1$, $\lambda P' + (1 - \lambda) Q \in [P]$. Thus, for such λ,

$$\begin{aligned} \Delta(Q) \in A \quad & \text{iff } \Delta(\lambda P' + (1-\lambda)Q) \in A \\ & \text{iff } s(\lambda P' + (1-\lambda)Q) = 0 \\ & \text{iff } s(Q) = 0. \end{aligned}$$

As a result, we have

Corollary 2. (1) $\Delta\mathcal{S}_0(P) = \Delta\mathcal{S}(P) \cap A$, *a closed set*;

(2) $\Delta[Q]_0 = \Delta[Q] \cap A$, *a nonempty set*;

(3) $\Delta[Q]_0$ *is relatively open, i.e., open in the flat determined by the hyperplanes defined by the equations* $e[Q] \cap e_0[Q]$;

(4) *if* $R < Q$ *and* $R,\ Q \in \mathcal{S}_0(P)$ *then* $\Delta R \in \mathrm{rel\ bd}(\Delta[Q]_0)$.

Proof. Parts (1), (2), and (3) are quite straightforward. For part (4), since $\Delta[Q]_0$ is nonempty, it contains $\Delta(Q')$ say. All points of the form $\lambda\Delta(Q') + (1-\lambda)\Delta(R)$, $0 < \lambda \le 1$, lie in $\Delta[Q]_0$ so $\Delta(R)$ is a limit point of $\Delta[Q]_0$. But, by the convexity of $\Delta[R]_0$, no point of the form $\lambda\Delta(Q') + (1-\lambda)\Delta(R)$, $\lambda < 0$, lies in $\Delta[R]_0$. Thus $\Delta R \in \mathrm{rel\ bd}(\Delta[Q]_0)$.

Theorem 3. *The following are each necessary and sufficient for* P *to be indecomposable*:

(1) $\dim \mathcal{S}_0(P) = 1$;

(2) $f - n - \mathrm{rank}(e[P]) = 1$.

Proof. Suppose $\dim \mathcal{S}_0(P) = 1$. Then $\Delta\mathcal{S}_0(P)$ is a ray or a line. But it cannot be a line since $s(Q) = 0$ implies $h(Q, u_i) \ge 0$ for each i. Therefore, if $Q \in \mathcal{S}_0(P)$, $\Delta(Q) = \lambda\Delta(P)$ for $\lambda \ge 0$ whereupon $Q = \lambda P$. Conversely, if P is indecomposable, any Q in $\mathcal{S}_0(P)$ is of the form λP where $\lambda \ge 0$, and so $\Delta(Q) = \lambda\Delta(P)$, again making $\Delta\mathcal{S}_0(P)$ a ray whence $\dim \Delta\mathcal{S}_0(P) = 1$. Now $\dim \Delta\mathcal{S}_0(P) = \dim \Delta[P]_0$ (by part (4) of Corollary 2) and $\dim \Delta[P]_0 = f - n - \mathrm{rank}(e[P])$ by Lemma 6 and part (3) of Corollary 2, and a standard law of nullity argument.

A weaker version of the following theorem was announced by Gale [2].

Theorem 4. *Let* P *be an* n*-polytope with* f *faces. Then there exist indecomposable polytopes* $P_1, \cdots, P_k$ *so that* $P = P_1 + \cdots + P_k$. *Furthermore, they can be chosen so that* $k \le f - n - \mathrm{rank}(e[P])$.

Proof. Let $e_1, \cdots, e_n$ be a basis for E^n. $l(Q) = \Sigma_1^n[h(Q, e_i) + h(Q, -e_i)]$ is a continuous linear functional on $\mathcal{S}_0(P)$. Let β denote $\{Q \in \mathcal{S}_0(P)\colon l(Q) = 1\}$. Since $s(Q) = 0$ if $Q \in \beta$, we have $h(Q, e_i)$ and $h(Q, -e_i) \ge 0$ for each e_i. Consequently, each $h(Q, e_i)$ and $h(Q, -e_i) \le 1$. This places Q inside a parellelepiped of dimension 2 along each coordinate axis. Therefore $\Delta\beta$ is a bounded set in R^f. But since l is continuous, β is closed so $\Delta\beta$ is closed and bounded, hence compact. $\Delta\beta$ is also convex and clearly $\dim \Delta\beta = \dim \Delta\mathcal{S}_0(P) - 1$. Now, by Carathéodory's theorem, any point of $\Delta\beta$ can be written as a convex combination of at most $\dim \mathcal{S}_0(P)$ extreme points of $\Delta\beta$. Thus $P = \lambda_1 P_1 + \cdots + \lambda_k P_k$, where $0 \le \lambda_i \le 1$, $\Sigma_1^k \lambda_i = 1$, $k \le \dim \mathcal{S}_0(P) = f - n - \mathrm{rank}(e[P])$ and ΔP_i is extreme in $\Delta\beta$. But the indecomposable polytopes of $\mathcal{S}_0(P)$ are precisely those of the form λQ, $\lambda \ge 0$, where $\Delta(Q)$ is extreme in β.

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