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PROPERTY SUV^∞ AND PROPER SHAPE THEORY

BY

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ABSTRACT. A class of spaces called the SUV^∞ spaces has arisen in the study of a possibly noncompact variant of cellularity. These spaces play a role in this new theory analogous to that of the UV^∞ spaces in cellularity theory. Herein it is shown that the locally compact metric space X is an SUV^∞ space if and only if there exists a tree T such that X and T have the same proper shape. This result is then used to classify the proper shapes of the SUV^∞ spaces, two such being shown to have the same proper shape if and only if their end-sets are homeomorphic. Also, a possibly noncompact analog of property UV^n, called SUV^n, is defined and it is shown that if X is a closed connected subset of a piecewise linear n-manifold, then X is an SUV^n space if and only if X is an SUV^∞ space. Finally, it is shown that a locally finite connected simplicial complex is an SUV^∞ space if and only if all of its homotopy and proper homotopy groups vanish.

1. Introduction. An important notion in the study of embeddings of compact metric spaces has been that of property UV^∞. (Definitions of this and other terms used in this introduction, as well as references, are included in the text.) This notion first arose in the study of cellularity, and is connected with shape theory through the result that the compact metric space X has property UV^∞ if and only if X has the shape of a singleton.

In studying a generalization of cellularity, called quasi-cellularity, to noncompact sets, Hartley defined a possibly noncompact variant of property UV^∞, called property SUV^∞, and showed that property SUV^∞ plays a role in quasi-cellularity theory analogous to that played by property UV^∞ in cellularity theory.

Our main result connects property SUV^∞ with *proper* shape theory. In particular, we show, in §3, that if X is a locally compact metric space, then X has property SUV^∞ if and only if X has the proper shape of a locally finite connected and simply connected 1-complex. We use this result, and some of the results of §2, to give a complete classification of proper shapes among the SUV^∞ spaces

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via the well-known theory of ends. To be precise, we show that the SUV^{∞} spaces X and Y have the same proper shape if and only if their end-sets are homeomorphic. In §4 we define a property, called SUV^n, which is analogous to property UV^n for compact metric spaces, and show how this leads to a characterization of the finite-dimensional SUV^{∞} spaces. Finally, we show that a locally finite connected simplicial complex has property SUV^{∞} if and only if its homotopy and proper homotopy groups of all dimensions vanish.

We remark that *all spaces considered in this paper are metrizable*. In particular, ANR and AR mean, respectively, absolute neighborhood retract and absolute retract for *metric spaces*.

I would like to express my thanks to Professor B. J. Ball for his valuable remarks on a draft version of this paper. In particular, he pointed out the existence of Kuperberg's paper [9] and how the notion of unstable set could be used to simplify the statement of Lemma (2.1).

2. **Trees and proper maps.** A locally finite, connected and simply connected simplicial 1-complex is called a *tree*. In this section we shall classify (in Theorem (2.3)) the proper homotopy classes of maps of a (sufficiently nice) space into a tree and use this classification to study the relation of proper homotopy equivalence on the class of all trees. While our techniques are applicable to some other special types of spaces, we shall restrict ourselves to trees for convenience.

Perhaps we should recall that a map $f: X \to Y$ is *proper* if $f^{-1}(C)$ is compact for every compact set $C \subset Y$, and that maps $f: X \to Y$ and $g: X \to Y$ are *properly homotopic* if there exists a proper map $\Phi: X \times I \to Y$ such that $\Phi(x, 0) = f(x)$ and $\Phi(x, 1) = g(x)$ for all $x \in X$. (I denotes the interval $[0, 1]$.) The latter determines an equivalence relation on the proper maps from X to Y, and we write $f \cong_p g$ provided f and g are properly homotopic. We also recall that spaces X and Y are said to be of the same *proper homotopy type* if there exist proper maps $f: X \to Y$ and $g: Y \to X$ such that $gf \cong_p i_X$ and $fg \cong_p i_Y$. (i_Z denotes the identity map on the space Z.) In this case we write $X \cong_p Y$. If only $gf \cong_p i_X$ is known to hold, we say that Y *properly homotopically dominates* X and we write $X \leq_p Y$ or $Y \geq_p X$.

Suppose X is a space and $X' \subset X$. Then X' is an *unstable* subset of X if there exists a homotopy $H: X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$ and $H(x, t) \in X - X'$ for all $x \in X$ and $0 < t \leq 1$. The following lemma will be crucial to the proofs of Theorems (2.3) and (2.4) (cf. Theorem 2 of [9]).

(2.1) **Lemma.** *Suppose X is a compact metric absolute retract and X' is an unstable subset of X. Then if A is a compact metric space, B is a closed subset of A, and $f: B \to X$ is a map, there exists a map $f^*: A \to X$ such that $f^*|B = f$ and $f^*(A - B) \subset X - X'$.*

Proof. Let $H: X \times I \to X$ be a homotopy such that $H(x, 0) = x$ for all $x \in X$ and $H(x, t) \in X - X'$ for all $x \in X$ and $0 < t \leq 1$. Let $\phi: A \to I$ be a map such that $\phi(a) = 0$ if and only if $a \in B$, and let $\hat{f}: A \to X$ be an extension of f. Now, define $f^*: A \to X$ by $f^*(a) = H(\hat{f}(a), \phi(a))$. Then f^* is easily seen to have the desired properties.

We now briefly discuss what will prove to be the key geometric idea behind our classification scheme—the notion of "ends". Suppose X is a semicompact (rim compact, locally peripherally compact) space. Then the *Freudenthal compactification* of X, here denoted FX, is the least upper bound of all compactifications Y of X such that $\operatorname{ind}(Y - X) = 0$. In order that FX be metrizable, it is necessary and sufficient that X be separable and metrizable, and that QX, the space of quasi-components of X, be compact. For a more complete discussion of FX, see [6] or [8]. We shall refer to $FX - X$ as the space of *ends* of X, here denoted by EX. We note that EX is homeomorphic with a closed subset of the Cantor set.

Suppose X and Y are semicompact separable metric spaces, QX and QY are compact, and $f: X \to Y$ is a proper map. Then f has a unique extension to a map of pairs $Ff: (FX, EX) \to (FY, EY)$. If $g: X \to Y$ is a proper map and $f \simeq_p g$, then $Ff \mid EX = Fg \mid EX$. Also, the assignment $f \to Ff$ is functorial; that is, $Fi_X = i_{FX}$ and, if $f: X \to Y$ and $g: Y \to Z$ are proper maps, then $F(gf) = (Fg)(Ff)$. These facts are discussed in some detail in §4 of [3] (see, particularly, Lemmas 4.2 and 4.3).

Now suppose T is a tree. Then the following facts are easily established:

(i) *FT is a compact metric absolute retract;*

(ii) *FT is contractible to a point via a homotopy $K: FT \times I \to FT$ such that $K(x, t) \notin ET$ for all $x \in FT$ and $0 < t \leq 1$.*

Thus

(2.2) **Lemma.** *If T is a tree, then FT is a compact metric absolute retract and ET is an unstable subset of FT.*

We now give the main result of this section, which states that, under suitable restrictions on X, the proper homotopy classes of maps of X into the tree T are in 1-1 correspondence with the space ET^{EX}.

(2.3) **Theorem.** *Suppose X is a locally compact separable metric space such that QX is compact. Suppose further that T is a tree and that $g: X \to T$ and $h: X \to T$ are proper maps. Then $g \simeq_p h$ if and only if $Fg \mid EX = Fh \mid EX$. Furthermore, if $k_0: EX \to ET$ is a map, then there exists a proper map $k: X \to T$ such that for all $e \in EX$, $k_0(e) = Fk(e)$.*

Proof. Suppose first of all that $g \simeq_p h$. Then (as noted in the remarks following Lemma (2.1)) by Lemma 4.3 of [3], $Fg \mid EX = Fh \mid EX$.

Suppose then that $Fg\,|\,EX = Fh\,|\,EX$. Let $A = FX \times I$ and let $B = (FX \times \{0\}) \cup (EX \times I) \cup (FX \times \{1\})$. Define $f\colon B \to FT$ by

$$f(b) = \begin{cases} Fg(z) & \text{if } b = (z, 0) \in FX \times \{0\} \\ Fg(z) = Fh(z) & \text{if } b = (z, t) \in EX \times I \\ Fh(z) & \text{if } b = (z, 1) \in FX \times \{1\}. \end{cases}$$

Then f is continuous and, by Lemmas (2.2) and (2.1), there exists a map $f^*\colon A \to FT$ such that $f^*|B = f$ and $f^*(A - B) \subset FT - ET$. Now, let $G\colon X \times I \to T$ be defined by $G(x, t) = f^*(x, t)$ for all $(x, t) \in X \times I$. Then G is a proper homotopy joining g and h.

Finally, suppose $k_0\colon EX \to ET$ is a map. Since EX is a compact subspace of FX, by Lemmas (2.2) and (2.1), there exists a map $\hat{k}\colon FX \to FT$ such that $\hat{k}(e) = k_0(e)$ for all $e \in EX$ and $\hat{k}(X) \subset FT - ET = T$. Define $k\colon X \to T$ by $k(x) = \hat{k}(x)$ for all $x \in X$. Then k is proper and, if $e \in EX$, $Fk(e) = \hat{k}(e) = k_0(e)$.

Given a compact 0-dimensional metric space D, it is easy to construct a tree T such that $ET \approx D$. (We use $\approx$ for the relation of homeomorphism.) This, along with the next result, implies that the proper homotopy types of trees are in 1-1 correspondence with the homeomorphism classes of closed subsets of the Cantor set.

(2.4) **Theorem.** *Suppose T_1 and T_2 are trees. Then $T_1 \simeq_p T_2$ if and only if $ET_1 \approx ET_2$.*

Proof. Suppose first of all that $T_1 \simeq_p T_2$. Then, by Corollary 4.9 of [3], $ET_1 \approx ET_2$. (The proof is an easy consequence of the facts remarked on after Lemma (2.1).)

Now suppose $ET_1 \approx ET_2$. Let $h\colon ET_1 \to ET_2$ be a homeomorphism and denote h^{-1} by g. By Lemma (2.1), there exists a proper map $h^*\colon T_1 \to T_2$ such that if $e \in ET_1$, $Fh^*(e) = h(e)$, and a proper map $g^*\colon T_2 \to T_1$ such that if $e \in ET_2$, $Fg^*(e) = g(e)$. If $e \in ET_1$, then

$$Fi_{T_1}(e) = e = (gh)(e) = g(h(e)) = g(Fh^*(e))$$

$$= Fg^*(Fh^*(e)) = (Fg^*)(Fh^*)(e) = F(g^*h^*)(e).$$

Thus, $Fi_{T_1}|ET_1 = F(g^*h^*)|ET_1$ and, by Theorem (2.3), $i_{T_1} \simeq_p g^*h^*$. Similarly $i_{T_2} \simeq_p h^*g^*$, thereby showing that $T_1 \simeq_p T_2$.

By the proof of Theorem (2.4), we also have the following.

(2.5) **Corollary.** *Suppose T_1 and T_2 are trees. Then $T_1 \geq_p T_2$ if and only if ET_2 embeds in ET_1.*

(2.6) **Corollary.** *Suppose T_1 and T_2 are trees. Then either $T_1 \geq_p T_2$ or $T_2 \geq_p T_1$.*

Proof. By Corollary (2.5), it is only necessary to show that if P and Q are closed subsets of the Cantor set, then either P embeds in Q or Q embeds in P. Suppose first that one of P or Q, say P, is uncountable. Then P contains a Cantor set, so Q embeds in P. On the other hand, if P and Q are both countable, then each of P and Q is homeomorphic to a countable ordinal space, and it follows that one embeds in the other.

Finally, we remark that it is easy to construct trees T_1 and T_2 such that $T_1 \geq_p T_2$ and $T_2 \geq_p T_1$, but $T_1 \not\approx_p T_2$. We simply construct T_1 so that ET_1 is homeomorphic to the Cantor set, and T_2 so that ET_2 is homeomorphic to the sum of the Cantor set and an isolated point.

3. **Proper shape and property SUV^∞.** Suppose M is a topological space and $X \subset M$. Then X is said to have *property UV^∞ in M* if for each open set U containing X, there is an open set V containing X such that $V \subset U$ and V is contractible in U. This property has been particularly useful for compact X. See, for example, [1] or [12].

Property UV^∞ is an embedding property, rather than an intrinsic one. For example, if X denotes the "topologist's $\sin(1/x)$ curve" in R^2, then X has property UV^∞ in R^2 but X does not have property UV^∞ in X. However, suppose X is a closed subset of the ANR M and X has property UV^∞ in M. It follows that if X' is a closed subset of the ANR M' and $X \approx X'$, then X' has property UV^∞ in M' [12]. Thus it is customary to say X *has property UV^∞*, for which we shall write $X \in UV^\infty$, if X embeds as a closed UV^∞ subset of some ANR.

Now suppose M is a space and $X \subset M$. Then X is said to have *property SUV^∞ in M* if for each closed neighborhood U of X, there exist a closed neighborhood V of X lying interior to U, a tree T, maps $f: V \twoheadrightarrow T$ and $g: T \to U$, and a proper homotopy $H: V \times I \to U$ such that H_0 is the inclusion and $H_1 = gf$. Since f is onto and H_1 is proper, it easily follows that f and g are proper. As in the case of property UV^∞, property SUV^∞ is not intrinsic. However, having property SUV^∞ is invariant under (closed) embeddings into locally compact ANR's [7], and we say that X *has property SUV^∞*, and write $X \in SUV^\infty$, provided that X embeds as a closed SUV^∞ subset of some locally compact ANR. It follows easily that if $X \in SUV^\infty$, then $X \in UV^\infty$ and, for X compact, $X \in UV^\infty$ if and only if $X \in SUV^\infty$. However, there exist noncompact UV^∞ spaces (e.g. the plane) which do not have property SUV^∞.

Fox X compact and finite dimensional, $X \in UV^\infty$ if and only if X embeds as a cellular subset of a finite-dimensional manifold [10] (cf. also [11]). In [7] a (possibly noncompact) version of cellularity, called quasi-cellularity, is defined and studied. A compact subset of an n-manifold is quasi-cellular if and only if it is cellular, and it is shown in [7] that for X locally compact and finite dimensional, $X \in SUV^\infty$ if and only if X embeds as a quasi-cellular subset of some finite-dimensional manifold. Thus there is a strong parallel between properties UV^∞ and SUV^∞ in the compact and noncompact cases, respectively. In this section we show a further analog as described below.

For X compact, $X \in UV^\infty$ if and only if X has the shape of a point [4], [13]. We are now ready to state and prove our main result, namely that for X locally compact, $X \in SUV^\infty$ if and only if X has the proper shape of a tree. (This will be slightly strengthened in Corollary (3.5).) Following this we will apply the results of §2 to give the proper shape classification of the SUV^∞ spaces. For notations and terminology associated with the theory of proper shape, see [2] or [3].

(3.1) **Theorem.** *Suppose X is a locally compact metrizable space. Then $X \in SUV^\infty$ if and only if there exists a tree T_0 such that $Sh_p X = Sh_p T_0$.*

Proof. First suppose that there exists a tree T_0 such that $Sh_p X = Sh_p T_0$. Let Q denote the Hilbert cube and $K = Q \times [0, 1)$. Since $Sh_p X = Sh_p T_0$, X is connected, and it follows that X embeds as a closed subset of K. Hence, we may suppose that X is a closed subspace of K and that there exist proper fundamental nets $\underline{f} = \{f_\lambda \mid \lambda \in \Lambda\}$ and $\underline{g} = \{g_\delta \mid \delta \in \Delta\}$ from X to T_0 in (K, T_0) and T_0 to X in (T_0, K), respectively, such that $\underline{g}\underline{f} \simeq_p \underline{i}_X$ and $\underline{f}\underline{g} \simeq_p \underline{i}_{T_0}$ (see §5 of [3]).

To show that $X \in SUV^\infty$ it is sufficient to show that X has property SUV^∞ in K. To this end, let U be a closed neighborhood of X. Since $\underline{g}\underline{f} \simeq_p \underline{i}_X$, there exist a closed neighborhood V of X and indices $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$, such that if $\lambda \geq \lambda_0$ and $\delta \geq \delta_0$, then $g_\delta f_\lambda \mid V \simeq_p i_V$ in U. Without loss of generality, we may assume that V is connected and lies in the interior of U. Let $H: V \times I \to U$ be a proper homotopy such that H_0 is the inclusion and $H_1 = g_{\delta_0} f_{\lambda_0} \mid V$. Let $T = f_{\lambda_0}(V)$. Since V is connected, T is homeomorphic to a tree and thus it follows from the commutative diagram

$$\begin{tikzcd} V \arrow[rr, two heads, "f_{\lambda_0}"] \arrow[dr, "H_1"'] & & T \arrow[dl, "g_{\delta_0}"] \\ & U & \end{tikzcd}$$

that X has property SUV^∞ in K. (It is worth noting here that the hypothesis $\underline{f}\underline{g} \simeq_p \underline{i}_{T_0}$ remained unused. This will be remarked upon in Corollary (3.5) below.)

Now suppose $X \in SUV^\infty$. Then X is connected. Hence, X embeds in K and we may, in fact, suppose that X is embedded in K in such a way that the closure of X in $Q \times I$ is FX. Let T_0 be a tree such that $ET_0 = EX$. We shall construct proper fundamental nets $\underline{f}: X \to T_0$ in (K, T_0) and $\underline{g}: T_0 \to X$ in (T_0, K) such that $\underline{g}\,\underline{f} \simeq_p \underline{i}_X$ in (K, K) and $\underline{f}\,\underline{g} \simeq_p \underline{i}_{T_0}$ in (T_0, T_0), thus showing that $Sh_p X = Sh_p T_0$ (§5 of [3]).

By Theorem (2.3), there exists a proper map $f: X \to T_0$ such that Ff agrees with the identity on EX. Let $\{f^*\} = \underline{f}: X \to T_0$ denote a proper fundamental net from X to T_0 in (K, T_0) generated by f and having a single element.

Let $\Delta = \{U \subset K \mid U$ is a closed neighborhood of X in K and $EU = EX\}$. Then Δ is a cofinal system of closed neighborhoods of X in K and is a directed set under inclusion. Now fix $\delta = U \in \Delta$. Since $X \in SUV^\infty$, there exist a closed neighborhood V_δ of X lying in the interior of U, a tree T_δ, proper maps $\hat{f}_\delta: V_\delta \twoheadrightarrow T_\delta$ and $\hat{g}_\delta: T_\delta \to U$, and a proper homotopy $H^\delta: V_\delta \times I \to U$ such that H_0^δ is the inclusion and $H_1^\delta = \hat{g}_\delta \hat{f}_\delta$. We may, without loss of generality, assume that $V_\delta \in \Delta$. Then, restricting domains and ranges, we have the commutative diagram

$$\begin{tikzcd} EV_\delta \arrow[rr, two heads, "F\hat{f}_\delta"] \arrow[dr, "FH_1^\delta"'] & & ET_\delta \arrow[dl, "F\hat{g}_\delta"] \\ & EU & \end{tikzcd}$$

But $EU = EV_\delta$ and, since $H_1^\delta \simeq_p i_{V_\delta}$ in U, $FH_1^\delta = i_{EU}: EV_\delta \twoheadrightarrow EU$. It follows that $F\hat{f}_\delta$ (restricted) is a homeomorphism from EV_δ onto ET_δ with inverse $F\hat{g}_\delta$. By Theorem (2.3), there exists a proper map $\hat{h}_\delta: T_0 \to T_\delta$ such that $F\hat{h}_\delta \mid ET_0 = F\hat{f}_\delta \mid ET_0$. Now, let $g_\delta: T_0 \to K$ be defined by $g_\delta(x) = \hat{g}_\delta(h_\delta(x))$ for all $x \in T_0$. Note that if $e \in ET_0$, then

$$Fg_\delta(e) = F(\hat{g}_\delta \hat{h}_\delta)(e) = (F\hat{g}_\delta)(F\hat{h}_\delta)(e) = (F\hat{g}_\delta)(F\hat{f}_\delta)(e) = e.$$

We shall now verify that $\underline{g} = \{g_\delta \mid \delta \in \Delta\}$ is a fundamental net from T_0 to X in (T_0, K). To this end, let U be a closed neighborhood of X. We wish to find an index $\delta_0 \in \Delta$ such that if $\delta \geq \delta_0$, then $g_\delta \simeq_p g_{\delta_0}$ in U. Clearly there is no loss of generality in supposing that $U \in \Delta$. Now, since $X \in SUV^\infty$ there exist a closed neighborhood V of X lying in the interior of U, a tree T, proper maps $\bar{f}: V \twoheadrightarrow T$ and $\bar{g}: T \to U$, and a proper homotopy $H: V \times I \to U$ such that H_0 is the inclusion and $H_1 = \overline{g f}$. We may suppose $V \in \Delta$, and we take $\delta_0 = V$. Supposing now that $\delta \geq \delta_0$, we have that $g_\delta(T_0) \cup g_{\delta_0}(T_0) \subset V$ and that Fg_δ and Fg_{δ_0} agree on ET_0. By this latter statement and Theorem (2.3), $\bar{f} g_\delta \simeq_p \bar{f} g_{\delta_0}$

in T. Since $\overline{g}\,\overline{f} = H_1 \simeq_p i_V$ in U, we have $g_\delta \simeq_p \overline{g}\,\overline{f} g_{\delta_0}$ in U and $g_{\delta_0} \simeq_p \overline{g}\,\overline{f} g_{\delta_0}$ in U. Thus

$$g_\delta \simeq_p \overline{g}\,\overline{f} g_\delta \simeq_p \overline{g}\,\overline{f} g_{\delta_0} \simeq_p g_{\delta_0},$$

all in U, completing the proof that $\underline{g}$ is a fundamental net.

Now suppose $\delta \in \Delta$. Then, if $e \in ET_0$, $F(f^* g_\delta)(e) = (Ff^*)(Fg_\delta(e)) = (Ff^*)(e) = e$. Hence, by Theorem (2.3), $f^* g_\delta \simeq_p i_{T_0}$. It follows that $\underline{fg} \simeq_p \underline{i}_{T_0}$ in (T_0, T_0).

It remains only to show that $\underline{gf} \simeq_p \underline{i}_X$ in (K, K). Let U be a closed neighborhood of X in K. It is necessary to find $\delta_0 \in \Delta$ and a closed neighborhood V of X in U such that for all $\delta \geq \delta_0$, $g_\delta f^*|V \simeq_p i_V$ in U. Again, there is no loss of generality in supposing that $U \in \Delta$. Let $V = V_U$ and choose $\delta_0 \in \Delta$ such that if $\delta \geq \delta_0$, then $g_\delta(T_0) \subset V$. Recall that the diagram

$$\begin{tikzcd} V \arrow[rr, "\hat{f}_U"] \arrow[dr, "H_1^U"'] & & T_U \arrow[dl, "\hat{g}_U"] \\ & U & \end{tikzcd}$$

is commutative. Now, suppose $\delta \geq \delta_0$. If $e \in EV$, then

$$F(\hat{f}_U g_\delta f^*|V)(e) = (F\hat{f}_U)(Fg_\delta)(Ff^*|V)(e) = (F\hat{f}_U)(Fg_\delta)(e) = F\hat{f}_U(e).$$

Hence, by Theorem (2.3), $\hat{f}_U \simeq_p \hat{f}_U g_\delta f^*|V$ in T_U. Thus $i_V \simeq_p \hat{g}_U \hat{f}_U \simeq_p \hat{g}_U \hat{f}_U g_\delta f^*|V \simeq_p g_\delta f^*|V$, all in U.

Now, using Theorem (3.1), the classification results of §2 can be immediately translated into results on SUV^∞ spaces. For convenience, we make a listing of those restatements here. (Recall that if $X \in SUV^\infty$, then X is locally compact and metrizable.)

(3.2) **Corollary.** *Suppose X_1 and X_2 are SUV^∞ spaces. Then $Sb_pX_1 = Sb_pX_2$ if and only if $EX_1 \approx EX_2$.*

(3.3) **Corollary.** *Suppose X_1 and X_2 are SUV^∞ spaces. Then $Sb_pX_1 \geq Sb_pX_2$ if and only if EX_2 embeds in EX_1.*

In particular, if $X \in SUV^\infty$, then $Sb_pX \geq Sb_p[0, \infty)$.

(3.4) **Corollary.** *Suppose X_1 and X_2 are SUV^∞ spaces. Then either $Sb_pX_1 \geq Sb_pX_2$ or $Sb_pX_2 \geq Sb_pX_1$.*

Finally, we return to the remark made following the first portion of the proof of Theorem (3.1). By this remark, if T_0 is a tree and $Sh_pX \leq Sh_pT_0$, then $X \in SUV^\infty$. Thus we have the following.

(3.5) **Corollary.** *Suppose X is a locally compact metrizable space. Then the following are equivalent.*

(i) $X \in SUV^\infty$.

(ii) *There exists a tree T_0 such that $Sh_pX = Sh_pT_0$.*

(iii) *There exists a tree T_0 such that $Sh_pX \leq Sh_pT_0$.*

4. **Other characterizations of SUV^∞ spaces.** Suppose M is a space and $f\colon S^k \times J^+ \to V \subset U \subset M$ is a proper map, where S^k is the k-sphere and J^+ denotes the discrete space of positive integers. Then f is said to be *properly S^k-inessential in U* if there exists a proper map $F\colon S^k \times J^+ \times I \to U$ such that

(1) for all $x \in S^k$ and $i \in J^+$, $F(x, i, 0) = f(x, i)$, and

(2) for all $i \in J^+$, $F \mid S^k \times \{i\} \times \{1\}$ is a constant map.

Now, if $X \subset M$, then X is said to have *property $k - SUV$ in M* if for each closed neighborhood U of X, there exists a closed neighborhood V of X lying in the interior of U such that each proper map from $S^k \times J^+$ into V is properly S^k-inessential in U. If n is a positive integer and X has property $k - SUV$ in M for $1 \leq k \leq n$, then X is said to have *property SUV^n* in M (cf. §3 of [1]). As expected, property SUV^n is not intrinsic; however, the proof that property SUV^∞ is invariant under embeddings into locally compact ANR's is easily seen to show also that property SUV^n is invariant under such embeddings. Hence, we write $X \in SUV^n$ if X embeds as a closed SUV^n subspace of some locally compact ANR. It follows easily that $X \in SUV^\infty$ implies $X \in SUV^n$ for all positive integers n. We now state a partial converse, corresponding to Proposition 3.4 of [1].

(4.1) **Theorem.** *Suppose n is a positive integer, X is a closed connected subset of the piecewise linear n-manifold M, and $X \in SUV^n$. Then $X \in SUV^\infty$.*

Proof. We may suppose that X is noncompact, for otherwise the result follows from Proposition 3.4 of [1]. Suppose U is a closed neighborhood of X in M. Denote U by V_{n+1}. By hypothesis, there exist closed neighborhoods $V_{n+1} \supset V_n \supset V_{n-1} \supset \cdots \supset V_1$ of X in M such that if $k \in \{1, 2, \cdots, n\}$, then each proper map from $S^k \times J^+$ into V_k is properly S^k-inessential in V_{k+1}. We may also suppose that V_1 is a connected piecewise linear manifold with boundary. We shall show that there exist a closed tree $T \subset V_1$, a map $f\colon V_1 \twoheadrightarrow T$, and a proper homotopy $H\colon V_1 \times I \to U$ such that H_0 is the inclusion and $H_1 = gf$, where $g\colon T \to U$ is the inclusion of T into U. It follows that X has property SUV^∞ in M, and hence that $X \in SUV^\infty$.

Let T_0 be a closed tree in V_1 embedded so that the inclusion of T_0 into

V_1 induces the identity map on $ET_0 = EV_1$. By Theorem (2.3), there exists a proper map $h: V_1 \to T_0$ such that if $e \in EV_1$, then $Fh(e) = e$. Let $T = h(V_1)$. Then, since V_1 is connected and h is proper, T is (topologically) a tree and T is a closed subset of U. Let $f: V_1 \twoheadrightarrow T$ be defined by $f(x) = h(x)$ for all $x \in V_1$.

It now remains to construct H. Let $Q = V_1 \times \{0\} \cup V_1 \times \{1\} \subset V_1 \times I$, and define $H': Q \to U$ by

$$H'(x, t) = \begin{cases} x & \text{if } t = 0 \\ f(x) & \text{if } t = 1. \end{cases}$$

Then H' is a proper map. Let $K^0 = \{v_1, v_2, v_3, \cdots\}$ denote the 0-skeleton of V_1. We begin the construction of H by extending H' to $H^0: Q \cup (K^0 \times I) \to U$ as described in the following paragraph.

Let $V_1 = \bigcup_{i=1}^{\infty} C_i$, where each C_i is a compact subpolyhedron of V_1 and $\emptyset = C_1 \subset C_2 \subset C_3 \subset \cdots$. If $i \in J^+$, let j_i be the largest positive integer such that there exists a path from v_i to $f(v_i)$ in $V_1 - C_{j_i}$. The existence of j_i for each $i \in J^+$ follows from the path-connectivity of V_1. Let $\alpha_i: I \to V_1$ be a path such that $\alpha_i(0) = v_i$, $\alpha_i(1) = f(v_i)$, and $\alpha_i(I) \cap C_{j_i} = \emptyset$. Define $H^0: Q \cup (K^0 \times I) \to U$ by

$$H^0(x, t) = \begin{cases} H'(x, t) & \text{if } (x, t) \in Q \\ \alpha_i(t) & \text{if } (x, t) = (v_i, t) \in K^0 \times I. \end{cases}$$

To show that H^0 is proper, it clearly suffices to show that if $i_0 \in J^+$, then C_{i_0} intersects only finitely many of the sets $\{\alpha_1(I), \alpha_2(I), \cdots\}$. Suppose, by way of contradiction, that this is not the case, and let $i_1 < i_2 < i_3 < \cdots$ be positive integers such that $\alpha_{i_k} \cap C_{i_0} \neq \emptyset$ for all $k \in J^+$. We may suppose (by taking a subsequence if necessary) that the sequence $\{v_{i_k}\}_{k=1}^{\infty}$ converges to $e \in EV_1$. But then the sequence $\{f(v_{i_k})\}_{k=1}^{\infty}$ also converges to e. This implies that there exists a positive integer N such that if $k \geq N$, then v_{i_k} and $f(v_{i_k})$ lie in the same component (hence path component) of $V - C_{i_0}$, thus contradicting the choice of $\alpha_{i_N}, \alpha_{i_{N+1}}, \cdots$.

Now, let K^i denote the i-skeleton of V_1, where $i \in \{0, 1, \cdots, n\}$. Suppose, inductively, that H' has been extended to a proper map $H^j: Q \cup (K^j \times I) \to U$ such that the image of H^j lies in V_{j+1}, where $j \in \{0, 1, \cdots, n-1\}$. Since proper maps from $S^{j+1} \times J^+$ into V_{j+1} are properly S^{j+1}-inessential in V_{j+2}, H^j can be extended to a proper map $H^{j+1}: Q \cup (K^{j+1} \times I) \to U$ such that the image of H^{j+1} lies in V_{j+2}. The extension $H^n: V_1 \times I \to U$ is the desired map H.

Suppose now that X is a locally compact separable metric space of finite dimension. Let d_X be the minimum of the set $\{q \in J^+ \mid X$ embeds in some piecewise linear manifold of dimension $q\}$. Since X embeds in Euclidean space of dimension $2 \dim X + 1$, d_X exists and $d_X \leq 2 \dim X + 1$.

(4.2) **Corollary.** *Suppose X is a finite-dimensional locally compact connected metric space, and $X \in SUV^{d_X}$. Then $X \in SUV^{\infty}$.*

We conclude this section by using a recent result of E. M. Brown to characterize the SUV^{∞} spaces within the class of connected locally finite simplicial complexes. If K is a connected locally finite simplicial complex, $[a]$ is an end of K, $n \in J^+ \cup \{\infty\}$, then $\underline{\pi}_n(K, \underline{a})$ denotes the nth proper homotopy group of K based at $[a]$ as defined in [5].

(4.3) **Theorem.** *Suppose K is a connected locally finite simplicial complex. Then $K \in SUV^{\infty}$ if and only if*

(i) *for each $n \in J^+$, $\pi_n(K) = 0$, and*

(ii) *for each $n \in J^+ \cup \{\infty\}$ and end $[a]$ of K, $\underline{\pi}_n(K, \underline{a}) = 0$.*

Proof. Suppose first that $K \in SUV^{\infty}$. Then there exists a tree T such that $Sh_p K = Sh_p T$. By Theorem 3.12 of [3], $K \cong_p T$. The result now follows from the easily verified fact that $\pi_n(T) = 0$ for all $n \in J^+$ and $\underline{\pi}_n(T, \underline{b}) = 0$ for all $n \in J^+ \cup \{\infty\}$ and end $[b]$ of T.

Now suppose that (i) and (ii) hold. Let T be a tree such that $ET \approx EK$. By Theorem (2.3), there exists a proper map $f: K \to T$ such that Ff (restricted) is a homeomorphism of EK onto ET. Then, if $n \in J^+$, $\pi_n(T) = 0$ and, if $n \in J^+ \cup \{\infty\}$ and $[b]$ is an end of T, $\underline{\pi}_n(T, \underline{b}) = 0$. Hence, by the "proper Whitehead theorem" of [5], f is a proper homotopy equivalence. Then $K \cong_p T$ and, by Theorem 3.12 of [3], $Sh_p K = Sh_p T$. By Theorem (4.1), $K \in SUV^{\infty}$.

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