

ERGODIC MEASURE PRESERVING TRANSFORMATIONS WITH QUASI-DISCRETE SPECTRUM

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ABSTRACT. It is shown that an ergodic measure preserving transformation with quasi-discrete spectrum is conjugate to: (a) the skew-product of an ergodic measure preserving transformation with discrete spectrum and a measurable family of totally ergodic measure preserving transformations with quasi-discrete spectrum; (b) a factor of the direct product of an ergodic measure preserving transformation with discrete spectrum and a totally ergodic measure preserving transformation with quasi-discrete spectrum. Sufficient conditions are given to insure that an ergodic measure preserving transformation with quasi-discrete spectrum is conjugate to the direct product of an ergodic measure preserving transformation with discrete spectrum and a totally ergodic measure preserving transformation with quasi-discrete spectrum.

0. Introduction. The structure of ergodic measure preserving transformations with discrete spectrum and totally ergodic measure preserving transformations with quasi-discrete spectrum has been well studied (cf. Halmos [4, pp. 46–50], Abramov [1], and Hahn and Parry [2]). The purpose of this paper is to reduce as much as possible the study of ergodic measure preserving transformations with quasi-discrete spectrum that are not totally ergodic to the above two cases.

We shall show (Theorem 4.4) that an ergodic measure preserving transformation T with quasi-discrete spectrum is conjugate to a skew product $S \times \{S_x\}$ where S has discrete spectrum and where S_x is totally ergodic with quasi-discrete spectrum and whose quasi-eigenfunctions are independent of x . Next (Theorem 4.5) if the torsion subgroup of the group of eigenvalues of T is the direct product of a complete group and a group of bounded order, then T is conjugate to the direct product $S_1 \times S_2$ where S_1 has discrete spectrum and S_2 is totally ergodic with discrete spectrum. An example (§5) is given to show that if the torsion subgroup of eigenvalues of T is not the direct product of a complete group and a group of bounded order, then in general a factorization as in Theorem 4.5 is not possible. Finally using the above results it is shown (Theorem 4.6) that an ergodic measure preserving transformation with quasi-discrete spectrum is conjugate to a factor of the direct product of an ergodic transformation with discrete spectrum and a

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totally ergodic transformation with quasi-discrete spectrum. It also follows from our results that an ergodic measure preserving transformation with quasi-discrete spectrum whose eigenvalues are all roots of unity has, in fact, discrete spectrum.

1. Definitions and notation. Let T be an ergodic measure preserving transformation acting on a probability space (Ω, \mathcal{F}, P) . We shall denote the group of all complex numbers with absolute value 1 by K . If \mathcal{F}' is an invariant sub- σ -algebra of \mathcal{F} (i.e. $T^{-1}(\mathcal{F}') \subseteq \mathcal{F}' \subseteq \mathcal{F}$), $\langle \mathcal{F}', K \rangle$ will denote the group of all \mathcal{F}' -measurable functions from Ω to K . (In $\langle \mathcal{F}', K \rangle$ we shall identify functions that are equal almost everywhere and the restriction of P to \mathcal{F}' will still be denoted by P .) On $\langle \mathcal{F}', K \rangle$ the homomorphism R is defined by $R(f) = (f \circ T) \cdot \bar{f}$. That T is ergodic means that the kernel of R is K .

The *quasi-eigenfunctions* of T with respect to \mathcal{F}' are defined by $G_n(T, \mathcal{F}') = \{f \in \langle \mathcal{F}', K \rangle : R^n(f) = 1\}$ and $G(T, \mathcal{F}') = \bigcup_{n=0}^{\infty} G_n(T, \mathcal{F}')$. The *quasi-eigenvalues* are defined by $H_n(T, \mathcal{F}') = R[G_n(T, \mathcal{F}')] = \{R^n(f) : f \in G_n(T, \mathcal{F}')\}$ and $H(T, \mathcal{F}') = R[G(T, \mathcal{F}')] = \{R^n(f) : f \in G(T, \mathcal{F}')\}$. The elements of $G_1(T, \mathcal{F}')$ and $H_1(T, \mathcal{F}')$ are called *eigenfunctions* and *eigenvalues* respectively.

T is said to have *quasi-discrete spectrum* (resp. *discrete spectrum*) with respect to \mathcal{F}' if $L_2(\Omega, \mathcal{F}', P) = \overline{L[G(T, \mathcal{F}')]}$ (resp. $\overline{L[G_1(T, \mathcal{F}')]}$) where $L[A]$ denotes the linear space spanned by A and $\overline{L[A]}$ denotes the closure of $L[A]$ in $L_2(\Omega, \mathcal{F}', P)$.

Measure preserving transformations T on (Ω, \mathcal{F}, P) and T' on $(\Omega', \mathcal{F}', P')$ are said to be *conjugate* if they induce isomorphic automorphisms on their induced measure algebras (cf. Halmos [4, pp. 42–45]).

Finally we say that a group C is the direct product of groups A and B , written $C = A \times B$, if $C = A \cdot B$ and $A \cap B = \{1\}$.

2. The quasi-eigenvalues of Z_m . Let m be a positive integer. Z_m will denote the dynamical system $(\Omega, \mathcal{F}, P, T)$ where $\Omega = \{0, \dots, m-1\}$, $\mathcal{F} = 2^\Omega$, $P(A) = |A|/m$ (where $|A|$ denotes the number of points in A), and $T(\omega) = \omega + 1 \pmod{m}$. We shall set $G(T, \mathcal{F}) = G(Z_m)$, etc., for simplicity. We may represent $\langle \mathcal{F}, K \rangle$ as $K^m = \{x = (x_1, \dots, x_m) : x_i \in K, 1 \leq i \leq m\}$. Then R is given by $y = R(x)$ where $y_i = x_{i+1} \bar{x}_i$ (with $x_{m+1} = x_1$). Thus it follows that the range of R is contained in $\mathcal{X} = \{x = (x_1, \dots, x_m) \in K^m : \prod_{i=1}^m x_i = 1\}$.

2.1. Lemma. $R(\mathcal{X}) = \mathcal{X}$.

Proof. Clearly $R(\mathcal{X}) \subseteq \mathcal{X}$ by what was said above. Let $y = (y_1, \dots, y_m) \in \mathcal{X}$. Define $x_2 = y_1 x_1$, $x_3 = y_1 y_2 x_1$, \dots , $x_m = y_1 \dots y_{m-1} x_1$ and take x_1 so that $\prod_{i=1}^m x_i = x_1^m y_1^{m-1} \dots y_{m-1} = 1$. Then $x = (x_1, \dots, x_m) \in \mathcal{X}$, $\bar{x}_i x_{i+1} = \bar{y}_1 \dots \bar{y}_{i-1} \bar{x}_1 y_1 \dots y_i x_1 = y_i$ if $i < m$, and $\bar{x}_m x_1 = \bar{y}_1 \dots \bar{y}_{m-1} \bar{x}_1 x_1 = y_m$ since

$y \in \mathcal{X}$. Thus $R(x) = y$, and hence $\mathcal{X} \subseteq R(\mathcal{X})$. \square

2.2. Lemma. $R[H(Z_m)] = H(Z_m)$.

Proof. Clearly $R[H(T, \mathcal{F}')] \subseteq H(T, \mathcal{F}')$ for any transformation. Let $f \in H(Z_m)$. Since $f \in R[G(Z_m)]$, $f \in \mathcal{X}$. By Lemma 2.1 there exist f' and $f'' \in \mathcal{X}$ such that $R(f'') = f'$ and $R(f') = f$. Hence $f'' \in G(Z_m)$, $f' = R(f'') \in H(Z_m)$, and $f = R(f') \in R[H(Z_m)]$. \square

Since K is a complete group (i.e. for every $x \in K$ and for every $n > 0$ there exists $y \in K$ such that $y^n = x$), the exact sequence $1 \rightarrow K \rightarrow G(Z_m) \xrightarrow{R} H(Z_m) \rightarrow 1$ splits, i.e. there is a homomorphism $Q: H(Z_m) \rightarrow G(Z_m)$ such that RQ equals the identity on $H(Z_m)$.

2.3. Lemma. $Q[H_1(Z_m)]$ is an orthonormal basis for $L_2(Z_m)$.

Proof. The vectors in $Q[H_1(Z_m)]$ are orthogonal since they are eigenvectors of the unitary operator $(x_1, \dots, x_m) \rightarrow (x_2, \dots, x_m, x_1)$ corresponding to distinct eigenvalues. They are normal since their modulus is identically 1, and they are a basis since there are m of them ($H_1(Z_m) = \{\exp(2\pi i k/m): 1 \leq k \leq m\}$) and $L_2(Z_m)$ is m -dimensional. \square

2.4. Lemma. Let $m = m_1 \cdots m_k$ with $m_i = p_i^{n_i}$ and $1 < p_1 < \cdots < p_k$ be the prime decomposition of m . Then $H(Z_m)$ is isomorphic to the direct product $H(Z_{m_1}) \times \cdots \times H(Z_{m_k})$.

Proof. Let T be the direct product of the dynamical systems Z_{m_i} . Then $T^j(x_1, \dots, x_k) = (x_1 + j \pmod{m_1}, \dots, x_k + j \pmod{m_k})$. Since m is the least common multiple of $\{m_1, \dots, m_k\}$, the points $\{T^j(x_1, \dots, x_k): 1 \leq j \leq m\}$ are distinct. Hence T is isomorphic to Z_m . Clearly $H_n(Z_{m_1}) \times \cdots \times H_n(Z_{m_k}) \subseteq H_n(T)$, and $H_n(Z_{m_i}) \cap H_n(Z_{m_j}) = \{1\}$ since m_i and m_j are relatively prime if $i \neq j$. It follows from Lemma 2.2 that $|H_n(Z_m)| = m^n$. Thus $|H_n(Z_{m_1}) \times \cdots \times H_n(Z_{m_k})| = \prod_{j=1}^k m_j^n = m^n = |H_n(T)|$ and hence $H_n(T) = H_n(Z_{m_1}) \times \cdots \times H_n(Z_{m_k})$. \square

We shall need the following result from number theory.

2.5. Lemma. Let p be a prime and n a positive integer. Then p divides $a_k = (-1)^k \binom{p^n-1}{k} - 1$ for all k between 0 and p^n-1 .

Proof by induction on k . Since $a_0 = 1 - 1 = 0$, p divides a_0 . Suppose p divides a_{k-1} and $k \leq p^n-1$. Then

$$a_k = (-1)^k p^n \binom{p^n-1}{k-1} / k + a_{k-1}.$$

By induction p divides a_{k-1} and since $p^n(p^{n-1})/k$ is an integer, k divides $p^n(p^{n-1})$. Since $k < p$ and p is a prime, $p \cdot k$ divides $p^n(p^{n-1})$. Thus p divides $p^n(p^{n-1})/k$ and hence p divides a_k . \square

2.6. Lemma. *Let p be a prime number, let n be a positive integer and $m = p^n$. Then $H(Z_m)$ is the set of all $x \in \mathcal{X}$ such that there exists a positive k with $x^{p^k} = 1$. Hence $H(Z_m)$ is isomorphic to the direct product of $m-1$ copies of the complete group $p^\infty = \{\exp(2\pi ir/p^s): r, s \geq 0\}$.*

Proof. Let $\mathfrak{M} = \{x \in \mathcal{X}: x^{p^k} = 1 \text{ for some } k > 0\}$. By Lemma 2.2, $H(Z_m) = \{x \in \mathcal{X}: R^k(x) = 1 \text{ for some } k > 0\}$. Suppose $R(x) = y$ with $x \in \mathcal{X}$ and $y \in \mathfrak{M}$ (say $y^{p^l} = 1$). Then $R(x^{p^l}) = [R(x)]^{p^l} = y^{p^l} = 1$. Therefore $x^{p^l} = \exp(2\pi ik/p^n) \cdot 1$ for some k , and hence $x^{p^{l+n}} = 1$. Thus $x \in \mathfrak{M}$. Since $1 \in \mathfrak{M}$, it follows that $H(Z_m) \subseteq \mathfrak{M}$.

Now suppose that $x \in \mathfrak{M}$ with $x^{p^l} = 1$. Let $y = R^k(x)$. Then $y_j = \prod_{i=0}^k [x_{j+i}]^{\alpha}$ where $\alpha = (-1)^{k+i} \binom{k}{i}$, as can be seen by induction with $j+i$ being interpreted mod m . Letting $k = m-1$ we have

$$y_j = \prod_{i=0}^{m-1} [x_{j+i}]^{b_i} \quad \text{where } b_i = (-1)^{m-1+i} \binom{m-1}{i}.$$

Since $x \in \mathcal{X}$, $x_{j+m-1} = \prod_{i=0}^{m-2} \bar{x}_{j+i}$. Therefore

$$(1) \quad y_j = \prod_{i=0}^{m-2} [x_{j+i}]^{c_i} \quad \text{where } c_i = (-1)^{m-1+i} \binom{m-1}{i} - 1.$$

Thus if $p \neq 2$, $y_j = \prod_{i=0}^{m-2} [x_{j+i}]^{a_i}$. Since, by Lemma 2.5, p divides a_i , we have

$$y_j^{p^{l-1}} = \prod_{i=0}^{m-2} [x_{j+i}]^{p^{l-1} a_i} = 1.$$

Hence $[R^{m-1}(x)]^{p^{l-1}} = 1$. Thus, by induction, $R^r(x) = 1$ for $r = \sum_1^l p^i - 1$. Hence $x \in H(Z_m)$ which implies $\mathfrak{M} \subseteq H(Z_m)$.

If $p = 2$, (1) becomes $y_j = \prod_{i=0}^{m-2} [x_{j+i}]^{-a_i-2}$, and since 2 divides $-a_i-2$, the proof proceeds as above. \square

2.7. Lemma. *Let $m > 0$. Then $H(Z_m)$ is a complete torsion group.*

Proof. By Lemma 2.4, $H(Z_m)$ is the direct product of groups isomorphic to $H(Z_p^k)$ where p is a prime. By Lemma 2.6, therefore, it is isomorphic to the direct product of groups p^∞ where p is a prime. Since these are complete torsion groups, so is $H(Z_m)$. \square

3. The quasi-eigenvalues of T . Let T be an ergodic measure preserving transformation acting on a probability space (Ω, \mathcal{F}, P) . Throughout this section

quasi-eigenfunction, etc., will be with respect to \mathcal{F}' and we will write $G = G(T, \mathcal{F}')$, etc., for simplicity. If A is a group, $\tilde{A} = \{a \in A: a^n = 1 \text{ for some } n > 0\}$ will denote the torsion subgroup of A .

3.1. Lemma. (a) $R[\tilde{H}] = \tilde{H}$. (b) \tilde{H} is a complete group. (c) $\tilde{H} \subseteq L[G_1]$.

Proof. Since R is a homomorphism and $R[H] \subseteq H$, we have $R[\tilde{H}] \subseteq \tilde{H}$. Let $f \in \tilde{H}_n$ and let \mathcal{F}_f be the σ -algebra generated by $\{f, R(f), \dots, R^n(f)\}$. Since the algebra of functions generated by $\{f, R(f), \dots, R^n(f)\}$ is invariant under composition with T and since each $R^k(f)$, being a root of unity, assumes only finitely many different values, it follows that \mathcal{F}_f is a finite invariant σ -algebra. Since T is ergodic, it follows that Z_n is a factor of T where n is the number of atoms in \mathcal{F}_f , i.e. there is a measure preserving transformation $\phi: \Omega \rightarrow Z_n$ such that $\phi^{-1}(2^{\mathbb{Z}n}) = \mathcal{F}_f$. There is therefore a quasi-eigenvalue g in $H(Z_n)$ such that $f = g \circ \phi$. It follows from Lemmas 2.2 and 2.7 that for every $n > 0$ there exist quasi-eigenvalues g' and g'' such that $R(g') = g$ and $g''^n = g$. Setting $f' = g' \circ \phi$ and $f'' = g'' \circ \phi$, we see that $f', f'' \in \tilde{H}$, $R(f') = f$, and $f''^n = f$. Thus \tilde{H} is complete and $R[\tilde{H}] = \tilde{H}$. Since, by Lemma 2.3, $g \in L[G_1(Z_n)]$, it follows that $\tilde{H} \subseteq L[G_1]$. \square

3.2. Lemma. There exists a sequence of groups H_n^* such that (a) $H_n^* \subseteq H_{n+1}^*$; (b) $\tilde{H} \cdot H_n = \tilde{H} \times H_n^*$.

Proof. Let $L_n = \tilde{H} \cdot H_n$. Then $\tilde{L}_n = \tilde{H}$. Since \tilde{H} is complete, there exists a group H_1^* such that $L_1 = \tilde{H} \times H_1^*$. Proceeding by induction $\tilde{H} \times H_{n-1}^* \subseteq L_n$, and $\tilde{L}_n = \tilde{H}$. Thus there exists H_n^* such that $H_{n-1}^* \subseteq H_n^*$ and $L_n = \tilde{H} \times H_n^*$. \square

The group $H^* = \bigcup_{n=1}^{\infty} H_n^*$ in general will not be invariant under R . The remaining lemmas of this section will establish sufficient conditions for the existence of an invariant H^* .

3.3. Lemma. $H_n/[H_{n-1} \cdot \tilde{H}_n]$ is torsion free.

Proof by induction on n . The case $n = 1, H_1/\tilde{H}_1$, is immediate. Suppose the lemma is true for $n \geq 1$ and let $f \in H_{n+1}$ with $f^m \in H_n \cdot \tilde{H}_{n+1}$. Let $g = R(f)$. Then $g \in H_n$ and $g^m = R(f^m) \in H_{n-1} \cdot \tilde{H}_n$. Thus by induction $g = g' \cdot g'' \in H_{n-1} \cdot \tilde{H}_n$. By Lemma 3.1(a) there exists $f'' \in \tilde{H}_{n+1}$ with $R(f'') = g''$. Let $f' = f \cdot \overline{f''}$. Since $R(f') = g \cdot \overline{g''} = g' \in H_{n-1}$, $f' \in H_n$. Thus $f = f' \cdot f'' \in H_n \cdot \tilde{H}_{n+1}$ which completes the proof. \square

Next we need a group theoretic lemma.

3.4. Lemma. Let G be an abelian group whose torsion subgroup \tilde{G} is the direct product of a complete group and a group of bounded order. Let H be a

subgroup of G such that $\tilde{G} \cap H = \{1\}$. If $G/[\tilde{G} \times H]$ is torsion free, then there exists a subgroup H^* such that $H \subseteq H^*$ and $G = \tilde{G} \times H^*$.

Proof. First note that the lemma, in the case $H = \{1\}$, is a standard result (cf., e.g. [5, p. 202]). Let $\pi: G \rightarrow G/H$ be the natural projection and suppose $\pi(g) \in \tilde{G}/H$, i.e. there exists a positive n such that $g^n \in H \subseteq \tilde{G} \times H$. Since $G/[\tilde{G} \times H]$ is torsion free, $g = \tilde{g} \cdot g' \in \tilde{G} \times H$. Thus $\pi(\tilde{g}) = \pi(g)$. Thus we have shown that $\pi[\tilde{G}] = \tilde{G}/H$. Since $\tilde{G} \cap H = \{1\}$, $\pi|_{\tilde{G}}$ is an isomorphism onto \tilde{G}/H . Thus by the first part there exists a subgroup $H' \subseteq G/H$ such that $G/H = (\tilde{G}/H) \times H'$. It is easy to verify that $H^* = \pi^{-1}(H')$ satisfies the conditions of the lemma. \square

3.5. Lemma. If \tilde{H}_1 is the direct product of a complete group and a group of bounded order, then there exist groups H_n^* such that

- (a) $H_{n-1}^* \subseteq H_n^*$;
- (b) $H_n = \tilde{H}_n \times H_n^*$;
- (c) $R[H_n^*] \subseteq H_{n-1}^*$.

Proof by induction on n . By Lemma 3.4 there exists a group H_1^* such that $H_1 = \tilde{H}_1 \times H_1^*$. Setting $H_0^* = \{1\}$ we have (a)–(c) for $n = 1$. Suppose now that we have constructed H_0^*, \dots, H_n^* . Let $L = H_{n+1} \cap R^{-1}[H_n^*]$. If $f \in \tilde{L}$, then $R(f) \in \tilde{H} \cap H_n^* = \{1\}$. Hence $\tilde{L} = \tilde{H}_1$. Suppose $f \in L \subseteq H_{n+1}$ and $f^m \in \tilde{L} \times H_n^* \subseteq H_n \cdot \tilde{H}_{n+1}$. By Lemma 3.3 there exist $f' \in H_n$ and $\tilde{f} \in \tilde{H}_{n+1}$ such that $f = f' \cdot \tilde{f}$. By induction there exist $f'' \in H_n^* \subseteq L$ and $\tilde{f}'' \in \tilde{H}_n$ such that $f' = f'' \cdot \tilde{f}''$. Thus $f = f'' \cdot (\tilde{f}'' \cdot \tilde{f})$. Since f and $f'' \in L$, $\tilde{f}'' \cdot \tilde{f} \in \tilde{L}$. Hence $f \in \tilde{L} \times H_n^*$. Thus $L/[\tilde{L} \times H_n^*]$ is torsion free, and hence by Lemma 3.4 there exists H_{n+1}^* such that $L = \tilde{L} \times H_{n+1}^*$ and $H_n^* \subseteq H_{n+1}^*$. If $f \in H_{n+1}$, then $R(f) = \tilde{g} \cdot g^*$ where $\tilde{g} \in \tilde{H}_n$ and $g^* \in H_n^*$. By Lemma 3.1 there exists $\tilde{f} \in H_{n+1}^*$ such that $R(\tilde{f}) = \tilde{g}$. Let $f' = f \cdot \tilde{f}$. It follows that $f' \in L$ and hence $f' = f'' \cdot \tilde{f}''$ where $f'' \in H_{n+1}^*$ and $\tilde{f}'' \in \tilde{H}_1$. Hence $H_{n+1} \subseteq \tilde{H}_{n+1} \cdot H_{n+1}^*$. Clearly $H_{n+1}^* \cap \tilde{H}_{n+1} = \{1\}$ and $R[H_{n+1}^*] \subseteq R[L] \subseteq H_n^*$ by definition. \square

4. The quasi-eigenfunctions of T . Since K is a complete group, the exact sequence $1 \rightarrow K \rightarrow G(T, \mathcal{F}') \xrightarrow{R} H(T, \mathcal{F}') \rightarrow 1$ splits, i.e. there exists a homomorphism $Q: H(T, \mathcal{F}') \rightarrow G(T, \mathcal{F}')$ such that RQ is the identity on $H(T, \mathcal{F}')$. Throughout this section Q will be such a homomorphism. We shall also suppose that we have a decomposition $H(T, \mathcal{F}') = \tilde{H}(T, \mathcal{F}') \times H^*(T, \mathcal{F}')$ as in Lemma 3.2 and will set $\Phi_n(T, \mathcal{F}') = Q[H_n(T, \mathcal{F}')]$, $\tilde{\Phi}_n(T, \mathcal{F}') = Q[\tilde{H}_n(T, \mathcal{F}')]$, etc.

4.1. Lemma. If $f \in \Phi_n(T, \mathcal{F}')$, then either $f \in \tilde{\Phi}(T, \mathcal{F}')$ or $f \perp \tilde{\Phi}(T, \mathcal{F}')$.

Proof by induction on n . If $f \in \Phi_1(T, \mathcal{F}')$ and $f \notin \tilde{\Phi}(T, \mathcal{F}')$, then $f \perp \tilde{\Phi}_1(T, \mathcal{F}')$ since eigenfunctions corresponding to distinct eigenvalues are orthogonal. Thus, by Lemma 3.1, $f \perp L[\tilde{\Phi}_1(T, \mathcal{F}')] \supseteq \tilde{\Phi}(T, \mathcal{F}')$. Suppose therefore that the lemma is true for $\Phi_n(T, \mathcal{F}')$, that $f \in \Phi_{n+1}(T, \mathcal{F}')$, and that $f \notin \tilde{\Phi}(T, \mathcal{F}')$. As above it remains to prove that $f \perp \tilde{\Phi}_1(T, \mathcal{F}')$. Let k be the smallest integer such that $R^k(f) \in \tilde{\Phi}(T, \mathcal{F}')$. Suppose first that $k \neq n+2$. By Lemma 3.1 there exists $\tilde{f} \in \tilde{\Phi}(T, \mathcal{F}')$ such that $R^k(\tilde{f}) = R^k(f)$. Let $f' = f \cdot \tilde{f}$. Since $R^k(f') = 1$, $f' \in \Phi_{k-1}(T, \mathcal{F}')$ and $f' \notin \tilde{\Phi}(T, \mathcal{F}')$. Therefore by induction $f' \perp \tilde{\Phi}(T, \mathcal{F}')$. Hence $f = f' \cdot \tilde{f} \perp \tilde{\Phi}(T, \mathcal{F}')$.

Suppose therefore that $\lambda = R^{n+1}(f)$ is not a root of unity ($R^{n+2}(f) = 1 \in \tilde{\Phi}(T, \mathcal{F}')$). If $\tilde{f} \in \tilde{\Phi}_1(T, \mathcal{F}')$, then $\lambda = R^{n+1}(b)$ where $b = f \cdot \tilde{f}$. Following Hahn and Parry [2, Theorem 1] we have

$$b \circ T^k = \prod_{j=0}^{n+1} [R^j(b)]^{a_j^{(k)}} = \exp \left(2\pi i \sum_{j=0}^{n+1} \binom{k}{j} a_j(\cdot) \right) = \exp(2\pi i p(k))$$

where $R^j(b) = \exp(2\pi i a_j(\cdot))$. Then $p(\cdot)$ is a polynomial of degree $n+1$ with leading coefficient a_{n+1} which is irrational since $\lambda = \exp(2\pi i a_{n+1})$ is not a root of unity. Thus Weyl's theorem (cf. [6, p. 326]) implies that $(1/N) \sum_{k=1}^N b \circ T^k \rightarrow 0$ as $N \rightarrow \infty$, and the ergodic theorem implies that $\langle f, \tilde{f} \rangle = E(b) = \lim (1/N) \sum_{k=1}^N b \circ T^k = 0$. Thus $f \perp \tilde{\Phi}_1(T, \mathcal{F}')$. \square

4.2. Lemma. Let $\tilde{\mathcal{F}}'$ and \mathcal{F}'^* be the σ -algebras generated by $\tilde{\Phi}(T, \mathcal{F}')$ and $\Phi^*(T, \mathcal{F}')$ respectively and let $\mathcal{F}'' = \tilde{\mathcal{F}}' \vee \mathcal{F}'^*$. Then $\tilde{\mathcal{F}}'$ and \mathcal{F}'^* are independent and $\tilde{\Phi}_1(T, \mathcal{F}') \times \Phi^*(T, \mathcal{F}')$ is an orthogonal basis for $L_2(\Omega, \mathcal{F}'', P)$.

Proof. It follows immediately from Lemma 4.1 that $\Phi^*(T, \mathcal{F}')$ is an orthonormal basis for $L_2(\Omega, \mathcal{F}'^*, P)$, and from Lemma 3.1 that $\tilde{\Phi}_1(T, \mathcal{F}')$ is an orthonormal basis of $L_2(\Omega, \tilde{\mathcal{F}}', P)$. It also follows from Lemma 4.1 that if $\tilde{f} \in \tilde{\Phi}_1(T, \mathcal{F}')$ and $f^* \in \Phi^*(T, \mathcal{F}')$, then $\tilde{f} \perp f^*$ unless $\tilde{f} = f^* = 1$. Thus if $\tilde{A} \in \tilde{\mathcal{F}}'$ and $A^* \in \mathcal{F}'^*$, using orthogonal expansions we have $P(A \cap A^*) = \langle \chi_{\tilde{A}}, 1 \rangle \langle \chi_{A^*}, 1 \rangle = P(\tilde{A}) \cdot P(A^*)$. Thus the lemma follows. \square

Since $\tilde{\mathcal{F}}'$ and \mathcal{F}'^* are independent, T on $(\Omega, \mathcal{F}'', P)$ is conjugate to a transformation T on $(X, \mathcal{F}'_X, P_X) \times (Y, \mathcal{F}'_Y, P_Y)$ where $\tilde{\mathcal{F}}'$ -measurable functions on Ω correspond to \mathcal{F}'_X -measurable functions on X , and the \mathcal{F}'^* -measurable functions on Ω correspond to \mathcal{F}'_Y -measurable functions on Y . Since $R[\tilde{H}(T, \mathcal{F}')] = \tilde{H}(T, \mathcal{F}')$, T induces a transformation S on (X, \mathcal{F}'_X, P_X) and a measurable family of transformations $\{S_x: x \in X\}$ on (Y, \mathcal{F}'_Y, P_Y) such that $T(x, y) = (S(x), S_x(y))$, i.e. T is the skew product $S \times \{S_x: x \in X\}$. We shall identify $(\Omega, \mathcal{F}'', P)$ with $(X, \mathcal{F}'_X, P_X) \times (Y \times \mathcal{F}'_Y, P_Y)$.

- 4.3. Lemma. (a) $G(S, \mathcal{F}'_X) = K \times \tilde{\Phi}(T, \mathcal{F}')$.
 (b) $G(S_x, \mathcal{F}'_Y) = K \times \Phi^*(T, \mathcal{F}')$ for almost all x .
 (c) For $f \in \Phi_n^*(T, \mathcal{F}')$, $f \circ S_x(y) = \tilde{g}(x) \cdot g^*(y) \cdot f(y)$ where $\tilde{g} \in G(S, \mathcal{F}'_X)$ and $g^* \in \Phi_{n-1}^*(T, \mathcal{F}')$.
 (d) $H_1(S_x, \mathcal{F}'_Y)$ has no nontrivial roots of unity.

Proof. (a) is immediate and we shall prove (b)–(d) by induction on n . Let $f \in \Phi_1^*(T, \mathcal{F}')$. Then $R(f) = g \in H_1^*(T, \mathcal{F}') \subseteq \tilde{H}(T, \mathcal{F}') \cdot H_1(T, \mathcal{F}')$. Thus $g = \lambda \tilde{g}$ and if $f \neq 1$, λ will not be a root of unity. Now $f \circ S_x(y) = f(T(x, y)) = (R(f))(x, y) \cdot f(y) = \lambda \cdot \tilde{g}(x) \cdot f(y)$. Hence f is an eigenvalue of S_x with eigenvalue $\lambda \cdot \tilde{g}(x)$. Suppose we have shown that $\Phi_n^*(T, \mathcal{F}') \subseteq G_n(S_x, \mathcal{F}'_Y)$. Let $f \in \Phi_{n+1}^*(T, \mathcal{F}')$. Then $g = R(f) \in H_{n+1}^*(T, \mathcal{F}') \subseteq \tilde{H}(T, \mathcal{F}') \cdot H_{n+1}(T, \mathcal{F}')$, say $g = \tilde{g} \cdot g'$. Therefore $f \circ S_x(y) = f(T(x, y)) = \tilde{g}(x) \cdot g'(x, y) \cdot f(y)$. Now $g' \in H_{n+1}(T, \mathcal{F}') \subseteq K \times \tilde{\Phi}_n(T, \mathcal{F}') \times \Phi^*(T, \mathcal{F}')$, say $g'(x, y) = \lambda \cdot \tilde{g}(x) \cdot g^*(y)$. Thus $R_x(f)(y) = (f \circ S_x) \cdot f = \lambda \tilde{g}(x) \cdot \tilde{g}(x) \cdot g^*(y)$. Since $g^* \in G_n(S_x, \mathcal{F}'_Y)$ by induction, we have $\Phi_{n+1}^*(T, \mathcal{F}') \subseteq G_{n+1}(S_x, \mathcal{F}'_Y)$. Since $\tilde{H}_1(S_x, \mathcal{F}'_Y) = \{1\}$, it follows from Lemma 4.1 that $G(S_x, \mathcal{F}'_Y) = K \times \Phi^*(S_x, \mathcal{F}'_Y)$. Since $\Phi^*(T, \mathcal{F}')$ is an orthonormal basis of $L_2(Y, \mathcal{F}'_Y, P_Y)$, it follows that $G(S_x, \mathcal{F}'_Y) = K \times \Phi^*(T, \mathcal{F}')$. \square

A transformation is said to be *totally ergodic* if T^k is ergodic for all $k > 0$. For an ergodic transformation this is equivalent to saying that there are no eigenvalues, other than 1, that are roots of unity. Thus the transformations S_x above are totally ergodic.

4.4. Theorem. Let T be an ergodic measure preserving transformation on (Ω, \mathcal{F}, P) that has quasi-discrete spectrum with respect to \mathcal{F} . Then T is conjugate to the skew product $S \times \{S_x : x \in X\}$ on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, P)$ where S has discrete spectrum with respect to \mathcal{F}_X and $H(S, \mathcal{F}_X) = \tilde{H}(T, \mathcal{F})$, and where, for each x , S_x is totally ergodic with quasi-discrete spectrum with respect to \mathcal{F}_Y whose quasi-eigenfunctions are independent of x .

The proof follows at once from Lemma 4.5. In [2] Hahn and Parry represent a totally ergodic measure preserving transformation T with quasi-discrete spectrum acting on a Lebesgue space as a minimal affine transformation $T(g) = a \cdot A(g)$ on a compact abelian group. In this case our S_x will be given by $S_x(g) = a_x \cdot A(g)$ where the translation a_x (but not the automorphism A) will depend on x .

4.5. Theorem. Let T be as in Theorem 4.4 and suppose that $\tilde{H}_1(T, \mathcal{F})$ is the direct product of a complete group and a group of bounded order. Then T is conjugate to the direct product $S_1 \times S_2$ on $(X \times Y, \mathcal{F}_X \times \mathcal{F}_Y, P)$ where S_1 has discrete spectrum with respect to \mathcal{F}_X and $H(S_1, \mathcal{F}_X) = \tilde{H}(T, \mathcal{F})$, and where S_2 is totally ergodic with quasi-discrete spectrum with respect to \mathcal{F}_Y .

Proof. By Lemma 3.5 we may choose $H^*(T, \mathcal{F})$ such that $R[H^*(T, \mathcal{F})] \subseteq H^*(T, \mathcal{F})$. Hence \mathcal{F}_Y = the σ -algebra generated by $Q[H^*(T, \mathcal{F})]$ is an invariant σ -algebra and $\{S_x: x \in X\}$ on (Y, \mathcal{F}_Y, P) in Theorem 4.4 reduces to S_2 on (Y, \mathcal{F}_Y, P) . \square

The transformation S_1 on (X, \mathcal{F}_X, P_X) is obtained from T on (Ω, \mathcal{F}, P) in a unique way. However the transformations S_2 on (Y, \mathcal{F}_Y, P_Y) will depend on the factorization $H = \tilde{H} \times H^*$. In general there will be many possible such factorizations with different H_1^* . Thus there will be many nonconjugate S_2 's to choose from and there appears to be no naturally preferable one.

We shall now show that we can always write the given transformation as a factor of a transformation that is a direct product as in Theorem 4.5.

4.6. Theorem. *Let T be an ergodic transformation on (Ω, \mathcal{F}, P) with quasi-discrete spectrum with respect to \mathcal{F} . Then there exist an ergodic transformation S_1 on (X, \mathcal{F}_X, P_X) with discrete spectrum with respect to \mathcal{F}_X , and a totally ergodic transformation S_2 on (Y, \mathcal{F}_Y, P_Y) with quasi-discrete spectrum with respect to \mathcal{F}_Y , and an invariant subgroup $G' \subseteq G(S_1, \mathcal{F}_X) \times G(S_2, \mathcal{F}_Y)$ such that T is conjugate to $S_1 \times S_2$ on $(X \times Y, \mathcal{F}', P_X \times P_Y)$ where \mathcal{F}' is the σ -algebra generated by G' .*

Proof. Let T be written as a skew product $S \times \{S_x: x \in X\}$ as in Theorem 4.4. Let L be the smallest complete subgroup of \tilde{K} containing $\tilde{H}_1(T, \mathcal{F})$ and let S' be the ergodic transformation on $(X', \mathcal{F}_{X'}, P_{X'})$ with discrete spectrum and eigenvalues L (cf. e.g. Halmos [4, p. 48, Corollary 1]). There is a natural homomorphism $\phi: (X', \mathcal{F}_{X'}, P_{X'}, S') \rightarrow (X, \mathcal{F}_X, P_X, S)$. The measurable family of measure preserving transformations $\{S_x: x \in X\}$ may therefore be extended to a measurable family $\{S_{x'}: x' \in X'\}$ by setting $S_{x'} = S_{\phi(x')}$. Lemma 4.3(c) shows that the elements of $\Phi^*(T, \mathcal{F})$ will be quasi-eigenfunctions for the skew product $T' = S' \times \{S_{x'}: x' \in X'\}$. Moreover T' is ergodic. (For suppose f is a nonconstant invariant function and suppose $\tilde{g} \in \Phi_1(S', \mathcal{F}_{X'})$ and $g^* \in \Phi^*(T, \mathcal{F})$ with $R(\tilde{g}) = \lambda$. We may take $f \perp 1$. Then $(f \cdot \tilde{g} \cdot g^*) \circ T'^k = f \cdot \tilde{g} \cdot \lambda^k \cdot g^* \circ T'^k$. Thus using Weyl's theorem and the ergodic theorem as in the proof of Lemma 4.1, we find that $E(f \cdot \tilde{g} \cdot g^*) = 0$. Hence $f \perp \Phi_1(S', \mathcal{F}_{X'}) \times \Phi^*(T, \mathcal{F})$, and since these functions generate $L_2(X' \times Y, \mathcal{F}_{X'} \times \mathcal{F}_Y, P_{X'} \times P_Y)$, f must be zero almost everywhere which is a contradiction.) Therefore T' has quasi-discrete spectrum on $(X' \times Y, \mathcal{F}_{X'} \times \mathcal{F}_Y, P_{X'} \times P_Y)$. Let \mathcal{F}' be the invariant σ -algebra generated by $\Phi_1(S, \mathcal{F}) \times \Phi^*(T, \mathcal{F})$. Then T on (Ω, \mathcal{F}, P) is clearly conjugate to T' on $(X' \times Y, \mathcal{F}', P_{X'} \times P_Y)$. But since $H_1(T', \mathcal{F}_{X'} \times \mathcal{F}_Y) = L$ is complete, T' on $(X' \times Y, \mathcal{F}_{X'} \times \mathcal{F}_Y, P_{X'} \times P_Y)$ is conjugate to a direct product by Theorem 4.5. \square

5. An example. In this section we shall construct an ergodic measure preserving transformation T with quasi-discrete spectrum such that $H_1(T)$ can be split: $H_1(T) = \tilde{H}_1(T) \times H_1^*(T)$, but $\tilde{H}_2(T)$ is not a direct factor of $H_2(T)$. Thus it will follow that T is not conjugate to a direct product as in Theorem 4.5. The proofs of various properties will only be sketched as they are similar to the known algebra constructions (cf. e.g. [5, pp. 202–205]).

Let T' be the ergodic measure preserving transformation on $(\Omega', \mathcal{F}', P')$ with discrete spectrum and with eigenvalues $H_1(T') = \tilde{K}$ (cf. [4, p. 48, Corollary 1]). Let T'' be the totally ergodic measure preserving transformation on $(\Omega'', \mathcal{F}'', P'')$ with quasi-discrete spectrum and with quasi-eigenvalues given by $H(T'') = H_2(T'') = \{(x, y): x \text{ and } y \in L_\alpha\}$ where α is irrational and $L_\alpha = \{\exp(2\pi i q \alpha): q \text{ is rational}\}$, with $R(x, y) = (y, 1)$ (cf. [1, §3]). A complex number z is identified with $(z, 1)$. Let $T = T' \times T''$ on (Ω, \mathcal{F}, P) . Let Q be a splitting of the exact sequence $1 \rightarrow K \rightarrow G(T) \xrightarrow{R} H(T) \rightarrow 1$. We will view $H(T', \mathcal{F}')$ and $H(T'', \mathcal{F}'')$ as subgroups of $H(T, \mathcal{F})$.

Let $2 < p_1 < p_2 < \dots$ where p_j are prime numbers and let r_j be positive integers. Let M be the subgroup of $H(T, \mathcal{F})$ generated by:

$$H_1(T'', \mathcal{F}'') = \{(x, 1): x \in L_\alpha\}, \quad \{\exp(2\pi i/p_j^{r_j}): j \geq 1\},$$

$$\{\exp(2\pi i/p_j^{r_j+1})(1, \exp(2\pi i/p_j)): j \geq 1\}.$$

Let \mathcal{G} be the σ -algebra generated by $Q(M)$. Clearly $R(M) \subseteq M$, and therefore \mathcal{G} is invariant under T . It can be seen by direct computation that $H_1(T, \mathcal{G}) = H_1(T, \mathcal{G}) \times H_1^*(T, \mathcal{G})$ where $H_1(T, \mathcal{G})$ is generated by $\{\exp(2\pi i/p_j^{r_j}): j \geq 1\}$ and $H_1^*(T, \mathcal{G}) = H_1(T'', \mathcal{F}'')$. Also we have $H_2(T, \mathcal{G}) = H_2(T, \mathcal{G}) \cdot M$. It can be seen that if two does not divide n then $H_2(Z_n) \approx H_1(Z_n) \times H_1(Z_n)$ with $R(x, y) = (y, 1)$. Then it follows that the elements $\exp(2\pi i/p_j^{r_j}) \in H_1(T, \mathcal{G})$ are of zero height in $H_2(T, \mathcal{G})$. Thus it follows as in [5, pp. 204–205] that $\tilde{H}_2(T, \mathcal{G})$ is not a direct factor of $H_2(T, \mathcal{G})$. Hence T cannot be factored as in Theorem 4.5.

6. Remarks. We mention two of the problems that we have not yet been able to solve.

If two ergodic measure preserving transformations with quasi-discrete spectrum have equivalent systems of quasi-eigenvalues in the sense of Abramov (cf. [1, §2]), are they conjugate? The difficulty is that the linear relations in H may not be preserved under the equivalence homomorphism. In the case covered by Theorem 4.6 this does not really matter, but in general we do not know the answer.

Does every factor of an ergodic measure preserving transformation with quasi-discrete spectrum have quasi-discrete spectrum? This is known in the case of totally ergodic transformations (cf. Hahn and Parry [3]).

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