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GOLDIE-LIKE CONDITIONS ON JORDAN MATRIX RINGS(1)

BY

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ABSTRACT. In this paper Goldie-like conditions are put on a Jordan matrix ring $J = H(R_n, \gamma_a)$ which are necessary and sufficient for R to be a *-prime Goldie ring or a Cayley-Dickson ring. Existing theory is then used to obtain a Jordan ring of quotients for J.

I. Introduction. Our basic reference for this paper is [5]. The Coordinatization Theorem [5, p. 137] states that any Jordan algebra over a field of characteristic $\neq 2$ with three or more connected idempotents which sum to 1 is isomorphic to a Jordan matrix algebra $H(R_n, \gamma_a)$ of symmetric elements with respect to a canonical involution γ_a. It is with this and Goldie's theorem for semiprime rings [4, p. 270] in mind that we approach the problem of quotient rings of Jordan rings and obtain a quotient ring which is of the type given in the Second Structure Theorem [5, p. 179].

II. Preliminaries and statement of main result. Let J be a Jordan ring whose multiplication is denoted by " $\cdot$ ". For each $a \in J$ there is the U-operator given by $U_a(b) = 2a \cdot (a \cdot b) - a^{\cdot 2} \cdot b$. J is said to be prime provided that, if A and B are ideals of J and $U_A(B) = \{U_a(b): a \in A, b \in B\} = 0$, then $A = 0$ or $B = 0$ [8]. $Q \subseteq J$ is said to be a quadratic ideal provided Q is an additive subgroup of $(J, +)$ and $U_Q(J) \subseteq Q$.

If we let R be a nonassociative ring with characteristic $\neq 2$ such that $\frac{1}{2} \in R$ and * is an involution on R, then the set of symmetric elements under a canonical involution (see [5, p. 125]), $J = H(R_n, \gamma_a)$, is a Jordan ring for $n \geq 3$ with product $c \cdot b = \frac{1}{2}(cb + bc)$ if and only if either R is associative or $n = 3$ and R is alternative with its set of symmetric elements H contained in its nucleus N [5, p. 127].

Let $a = \operatorname{diag}\{a_1, \cdots, a_n\}$ and the a_i's are symmetric invertible elements in the nucleus of R. The canonical involution γ_a acts on the $n \times n$ matrix X to give $a^{-1}X^{*t}a$ where X^{*t} denotes the matrix obtained by applying * to each of the entries of X and then taking the transpose.

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General assumption. In this paper: γ_a is a canonical involution; $J = (R_n, \gamma_a)$, $n \geq 2$, is a Jordan ring; $\frac{1}{2} \in R$ an alternative ring with involution * such that $H \subseteq N$.

If $z \in R$ and $\{e_{ij}\}$ is the standard set of matrix units, then we shall write z_{ij} for $ze_{ij} + a_j^{-1}z^*a_ie_{ji}$, $i \neq j$, in J. Although we shall have occasion to talk about matrices over R of more general form, we shall attempt to eliminate the confusion by reserving the subscripts i and j for this purpose, unless the matrix is written out. One can easily see that the elements of J are merely sums of elements of the form z_{ij}, $i \neq j$, and elements of the form ba_ie_{ii} where $b \in H$. We shall use J_{ij}, $i \neq j$, for $\{x_{ij} : x \in R\}$, and if $A \subseteq J$ then $A_{ij} = A \cap J_{ij}$. A quadratic ideal Q is said to be an *ij-quadratic ideal*, $i \neq j$, provided $Q_{ij} \neq 0$.

The next two definitions are basic for our Goldie-like conditions.

Definition. A nonempty set $\{Q_k\}$ of distinct nonzero quadratic ideals will be called a direct system provided that, if $\{Q_n : n \in I_1\}$ and $\{Q_l : l \in I_2\}$ are finite subsets of $\{Q_k\}$, where $I_1 \cap I_2 = \emptyset$, then the quadratic ideal generated by ΣQ_n intersected with the quadratic ideal generated by ΣQ_l is 0. The direct system is said to be infinite if $\{Q_k\}$ contains infinitely many quadratic ideals.

Definition. If Q is a quadratic ideal in J, then the quadratic ideal A generated by $\{z_{ij} \in J_{ij} : U_{z_{ij}}(Q_{ij}) = 0\}$ is defined to be the *ij*-annihilators of Q, $ij\text{-ann}(Q)$, provided $U_{A_{ij}}(Q_{ij}) = 0$, $i \neq j$; otherwise $ij\text{-ann}(Q) = 0$.

Our Jordan analogue for the associative ring of quotients is given as follows:

Definition. The Jordan ring J' is said to be a Jordan ring of quotients for the Jordan ring J provided:

(i) there exists an isomorphism $f: J \to J'$ (we shall consider J as a Jordan subring of J');

(ii) every regular element a in J (i.e., U_a is injective as it acts on J) is invertible in J';

(iii) every element of J' is of the form $U_a^{-1}(b)$ for a, b in J with a regular in J.

Main Theorem. *Let R be an alternative ring with involution * such that $H \subseteq N$ and $\frac{1}{2} \in R$ and let $J = H(R_n, \gamma_a)$, $n \geq 2$, be a Jordan matrix ring. Then J is prime, satisfies* ACC *on ij-annihilators, and contains no infinite direct system of ij-quadratic ideals if and only if either R is a $*$-prime Goldie ring or $n = 2, 3$ and R is a Cayley-Dickson ring. Moreover, in this case, the Jordan ring of quotients J' for J is $J' = H(R'_n, \gamma_a)$ where R' is the ring of quotients for R so that R' is a $*$-simple Artinian ring or a Cayley-Dickson algebra.*

III. Only if. Throughout this section we shall assume that J satisfies the conditions of the Main Theorem.

An ideal A of R is said to be a $*$-ideal if $A^* = A$. R is said to be a $*$-prime provided that, if A and B are $*$-ideals and $AB = 0$, then either $A = 0$ or $B = 0$.

If A is a $*$-ideal then $A \cap J$ is an ideal of J. Thus the primeness of J gives the $*$-primeness of R so that by [1] R contains a prime ideal P such that $P \cap P^* = 0$ (we shall use the existence of this prime ideal throughout). Also by [1], this implies that R is a Cayley-Dickson ring or R is associative. Thus we assume that R is associative, so that $U_a(b) = aba$ for all $a, b \in J$.

Theorem 3.1. *If J is prime and contains no infinite direct system of ij-quadratic ideals and R is associative then R contains no infinite direct sum of one sided ideals.*

Proof. Suppose ΣL_k is an infinite direct sum of left ideals in R and let P be a prime ideal such that $P \cap P^* = 0$. Then there are infinitely many L_k's not contained in P or infinitely many L_k's not contained in P^*. We assume $L_k \not\subseteq P$ for all k. Thus $(L_k^* + P^*)(L_k + P) \neq 0$ by the primeness of P so that $B_k = (L_k^* + P^*) \cap (L_k + P) \neq 0$. Let $0 = \Sigma b_k$ such that $b_k = 0$ for all but finitely many k's and $b_k \in B_k$, so that $b_k = l_k + p_k = l_k'^* + p_k'^*$ where $l_k, l_k' \in L_k$ and $p_k, p_k' \in P$. If $P = 0$, then $b_k = 0$ for all k since ΣL_k is a direct sum. If $P \neq 0$ then $0 = P^*0 = P^*\Sigma b_k$, so that $P^* l_k = 0$. Thus $l_k \in P$ and hence $b_k \in P$. By a similar argument $b_k \in P^*$ so that in any case $b_k = 0$. Therefore ΣB_k is a direct sum of nonzero subgroups of $(R, +)$.

If we set $Q_k = \{ba_ie_{ii} + ba_je_{ij} + b^*a_ie_{ji} + b'a_je_{jj}\colon b, b' \in H \cap B_k \text{ and } b \in B_k\}$, it is easy to check that Q_k is a quadratic ideal. Now let $\{Q_n\}_{n \in I_1}$ and $\{Q_m\}_{m \in I_2}$ be two finite disjoint subsets of $\{Q_k\}$. Let Q_{I_1} be the quadratic ideal generated by $\Sigma_{n \in I_1} Q_n$ and Q_{I_2} be the quadratic ideal generated by $\Sigma_{m \in I_2} Q_m$. Let $C = \Sigma_{n \in I_1} L_n$ and $D = \Sigma_{m \in I_2} L_m$ so that $C + D$ is a direct sum of left ideals, and we may perform similar constructions using C and D to obtain Q_C and Q_D, respectively, as we did to construct Q_k using L_k. We then have $Q_{I_1} \subseteq Q_C$ and $Q_{I_2} \subseteq Q_D$ and $Q_{I_1} \cap Q_{I_2} \subseteq Q_C \cap Q_D = 0$ since

$$[(C^* + P^*) \cap (C + P)] \cap [(D^* + P^*) \cap (D + P)] = 0.$$

Thus we see that $\{Q_k\}$ is an infinite direct system of ij-quadratic ideals.

We now turn our attention to considering ij-annihilators. For a nonempty subset S of R, we shall use $A_L(S)$, $A_R(S)$ for the left, right annihilator of S, respectively. If $L = A_L(S)$ and $T = A_R(L)$, we see that $A_L(T)$ is L. We would like to use T and L to give some insight into ij-annihilators.

If V is either a left or right ideal of R then

$$Q_{ji}(V) = \{xe_{ii} + a_i^{-1}v^*a_je_{ij} + ve_{ji} + ye_{jj}\colon v \in V,\ x \in a_i^{-1}(V \cap H), \text{ and } y \in Ha_j\}, \tag{1}$$

and

(2) $$Q'_{ij}(V) = \{xe_{ii} + ve_{ij} + a_j^{-1}v^*a_ie_{ji} + ye_{jj}:$$
$$v \in V,\ x \in (V \cap H)a_i, \text{ and } y \in a_j^{-1}H\}$$

are quadratic ideals of J. One should notice that $xe_{ii} = a^{-1}be_{ii}$, $b \in V \cap H$, is in J_{ii} as described above since $x = (a_i^{-1}ba_i^{-1})a_i$ and $a_i^{-1}ba_i^{-1} \in H$. Also, if $A_L(T) = L$ and $A_R(L) = T$ then

$$U_{[Q'_{ij}(L)]_{ij}}([Q_{ji}(T)]_{ij}) = 0.$$

Is $Q'_{ij}(L)$ the ij-annihilator of $Q_{ji}(T)$?

Lemma 3.2. *Let R be a prime associative ring and W an additive subgroup of $(R, +)$. Then $A(W) = \{a \in R : awRwa = 0$ for all $w \in W\} = A_L(W) \cup A_R(W)$.*

Proof. Let $a \in A(W)$ and suppose $wa \neq 0$ for some $w \in W$, so that $aw = 0$. Suppose $aw' \neq 0$ for some $w' \in W$, but $a(w + w') = 0$ or $(w + w')a = 0$, both of which lead to a contradiction.

Thus in the $*$-prime case, we see that $A(W) = \{a \in R : awRwa = 0$ for all $w \in W\}$ where W is an additive subgroup of $(R, +)$ is equal to $\bigcup_{i=1}^4 K_i(W)$ where

$$K_1(W) = \{a \in R\colon aW \subseteq P \text{ and } aW \subseteq P^*\}$$
$$K_2(W) = \{a \in R\colon aW \subseteq P \text{ and } Wa \subseteq P^*\}$$
$$K_3(W) = \{a \in R\colon Wa \subseteq P \text{ and } Wa \subseteq P^*\}$$
$$K_4(W) = \{a \in R\colon Wa \subseteq P \text{ and } aW \subseteq P^*\}.$$

Lemma 3.3. *Let R be a $*$-prime associative ring and $J = H(R_n, \gamma_a)$. Let B be a quadratic ideal of J and W be the set of elements of R which occur as ji-components of elements of B_{ij} (using the standard matrix units). A necessary and sufficient condition that B has a nonzero ij-annihilator is that the quadratic ideal Q generated by $[A(W)]_{ij} = \{x_{ij} : x \in \bigcup K_i(W)\}$ is nonzero and $Q_{ij} = [A(W)]_{ij}$. Moreover, if this condition is satisfied then $Q = ij$-ann(B).*

Proof. By the definition of ij-annihilators, it suffices to show that $[A(W)]_{ij} = \{x_{ij} \in J_{ij} : U_{x_{ij}}(B_{ij}) = 0\}$.

Let B and W be as stated above so that $B_{ij} = \{w_{ji} : w \in W\}$.

Suppose $U_{x_{ij}}(B_{ij}) = 0$ and $x_{ij} = xe_{ij} + a_j^{-1}x^*a_ie_{ji}$ so that $0 = U_{x_{ij}}U_{w_{ji}}(y_{ij}) = xwywxe_{ij} + a_j^{-1}(xwywx)^*a_ie_{ji}$ for all $y \in R$, $w \in W$ and hence $x \in \bigcup K_i(W)$.

Now suppose $x \in \bigcup K_i(W)$ and $x_{ij} = xe_{ij} + a_j^{-1}x^*a_ie_{ji}$. $U_{x_{ij}}(B_{ij}) = \{xwxe_{ij} + a_j^{-1}(xwx)^*a_ie_{ji} : w \in W\}$. Since $x \in \bigcup K_i(W)$, one of the following is the case:

$$xw,\ w^*x^* \in P; \quad xw,\ x^*w^* \in P; \quad wx,\ x^*w^* \in P; \quad wx,\ w^*x^* \in P.$$

In any case, $xwx \in P \cap P^* = 0$ so that $U_{x_{ij}}(B_{ij}) = 0$.

Remark. Lemma 3.3 tells us that a necessary condition for B to have a nonzero ij-annihilator is that $\bigcup K_i(W)$ be an additive subgroup of $(R, +)$, but in this case one can show that $\bigcup K_i(W) = K_q(W)$ for some $1 \leq q \leq 4$ by showing that if σ is the permutation (1 2 3 4) then

(a) $\bigcup K_i(W) = K_j(W) \bigcup K_{j\sigma}(W) \cup K_{j\sigma^2}(W)$, for one $j = 1$ or 3, and

(b) $K_j(W) \subseteq K_{j\sigma}(W)$ or $K_{j\sigma}(W) \subseteq K_j(W)$ for $1 \leq j \leq 4$.

The proof of this Remark is straightforward.

Lemma 3.4. *Let T be a nonzero right ideal of R; $L = A_L(T)$; and $T' = TP + TP^*$. Then*

(i) *if R is prime then $[ij\text{-ann}(Q_{ji}(T))]_{ij} = [Q'_{ij}(L)]_{ij}$ and*

(ii) *if R is $*$-prime (not prime) then $[ij\text{-ann}(Q_{ji}(T'))]_{ij} = [Q'_{ij}(L)]_{ij}$.*

Proof. (i) Since R is prime, the right annihilator of T is zero. Thus $\bigcup K_i(T) = K_1(T) = L$, and, by Lemma 3.3, $[Q'_{ij}(L)]_{ij} = [ij\text{-ann}(Q_{ji}(T))]_{ij}$.

(ii) Since R is $*$-prime (not prime), $P \neq 0$ so that $T' \neq 0$.

We shall show $\bigcup K_i(T') = L$. Clearly $L \subseteq \bigcup K_i(T')$. Let $a \in K_i(T')$ so that $aTPa = 0 = aTP^*a$. $aTPa = 0$ implies $a \in P^*$ or $aT \subseteq P^*$ and $aTP^*a = 0$ implies $a \in P$ or $aT \subseteq P$. Thus $a \in L$ and $\bigcup K_i(T') = L$, so that, by Lemma 3.3, $[ij\text{-ann}(Q_{ji}(T'))]_{ij} = [Q'_{ij}(L)]_{ij}$.

Theorem 3.5. *If J satisfies ACC on ij-annihilators and R is a $*$-prime associative ring then R satisfies ACC on left annihilator ideals.*

Proof. In Lemma 3.4, we showed that if $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ is an ascending chain on left annihilator ideals then we may form an ascending chain, $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ of ij-annihilators such that $(B_k)_{ij} = [Q'_{ij}(L_k)]_{ij}$. Since the chain of ij-annihilators terminates and B_k is generated by $(B_k)_{ij}$, we see that the chain of $[Q'_{ij}(L_k)]_{ij}$ terminates so that the chain $L_1 \subseteq L_2 \subseteq L_3 \subseteq \cdots$ terminates.

Summarizing results to this point we have:

Theorem 3.6. *Let $J = H(R_n, \gamma_a)$, $n \geq 2$, be a prime Jordan ring containing no infinite direct system of ij-quadratic ideals and satisfying ACC on ij-annihilators and let R be an alternative ring with characteristic $\neq 2$, $\frac{1}{2} \in R$, such that the set of symmetric elements, H, of R is contained in the nucleus N of R. Then R is a $*$-prime associative Goldie ring or n is either 2 or 3 and R is a Cayley-Dickson ring.*

By definition of a Cayley-Dickson ring, if Z is the center of R (i.e., $Z = \{z \in N : zx = xz$ for all $x \in R\}$) and Z' is the field of quotients of Z, then

$R' = Z' \otimes_Z R$ is a Cayley-Dickson algebra. E. Kleinfeld [6] has shown that if R is a prime alternative (not associative) ring, then $N = Z$. If R is a Cayley-Dickson ring with involution $*$ such that $H \subseteq N$ then $H = Z$. This is due to the fact that, if we let R' be the Cayley-Dickson algebra associated with R and extend $*$ on R to $*$ on R' by $(z^{-1} \otimes r)^* = z^{*-1} \otimes r^*$ for $z \in Z$ and $r \in R$, then R' is a Cayley-Dickson algebra with its symmetric elements invertible in its nucleus, so that $*$ is a standard involution (see [5]). This gives us that $x^*x = xx^*$ for all $x \in R$, and that $H = Z = N$.

IV. **If.** We first consider the case when R is a Cayley-Dickson ring. Let R' be the associated Cayley-Dickson algebra and consider R as a subring of R' and $*$ extended to R'. We shall write $z^{-1}r$ for $z^{-1} \otimes r$. Thus $R' = Z'R$ where Z' is as above. If $J' = H(R'_n, \gamma_a)$ then J' is a Z' algebra and $J \subseteq J'$.

Let $J = H(R_n, \gamma_a)$, $n = 2$ or 3 where R is a Cayley-Dickson ring with $H \subseteq N$ and $\frac{1}{2} \in R$. Let b_{ij} and x_{ji} be elements in J_{ij}, $i \neq j$, where $b_{ij} = be_{ij} + b^* a_j^{-1} a_i e_{ji}$ and $x_{ji} = x^* a_j a_i^{-1} e_{ij} + xe_{ji}$. Then since a_i's are in the center $Z(R)$, we have that $U_{b_{ij}}(x_{ji}) = bxbe_{ij} + b^*x^*b^*a_j^{-1}a_ie_{ji}$. From this it is clear that $U_{b_{ij}}(Z'x_{ji})$ is a one dimensional subspace of J' provided $bxb \neq 0$. But if $R' = Z'R$ then $bR'b \neq 0$ for $b \neq 0$, so that $bRb \neq 0$ for $b \neq 0$. Thus $U_{b_{ij}}(J'_{ij})$ is a subspace of J' of dimension greater than or equal to one for $b_{ij} \neq 0$.

Theorem 4.1. *Let $J = H(R_n, \gamma_a)$, $n = 2$ or 3, be a Jordan matrix ring. If R is a Cayley-Dickson ring with involution such that $H \subseteq N$ then J contains no infinite direct system of ij-quadratic ideals, $i \neq j$.*

Proof. Suppose the theorem is false. That is, J contains an infinite direct system of ij-quadratic ideals, $i \neq j$, say $\{Q_k\}$. Picking nonzero $q_k \in Q_k$ for each k, we obtain the system $\{U_{q_k}(J'_{ij})\}$ of nonzero Z' subspace of J'. By the finite dimensionality of J' over Z', we see for some choice of q_k and q_m's not equal to q_k that

$$U_{q_k}(J'_{ij}) \cap \left[\sum_m U_{q_m}(J'_{ij})\right] \neq 0.$$

Thus

$$0 \neq U_{q_k}(z^{-1}y_{ij}) = \sum_m U_{q_m}(z_m^{-1}y_{ij}^{(m)})$$

for some choice of $z, z_m \in Z$ and $y_{ij}, y_{ij}^{(m)} \in J_{ij}$. Therefore setting π equal to the product of the z_m's we have

$$0 \neq U_{q_k}(\pi y_{ij}) = \sum_m (U_{q_m}(\pi z_m^{-1} y_{ij}^{(m)})).$$

However, $\pi z_m^{-1} y_{ij}^{(m)}$ and πy_{ij} are elements in J. This contradicts the assumption that $\{Q_k\}$ is an infinite direct system.

Theorem 4.2. *Let $J = H(R_n, \gamma_a)$, $n = 2$ or 3, be a Jordan matrix ring. If R is a Cayley-Dickson ring with involution such that $H \subseteq N$ then J satisfies* ACC *on ij-annihilators, $i \neq j$.*

Proof. Let $Q_i = ij\text{-ann}(B_i)$ and $0 \neq Q_1 \subseteq Q_2 \subseteq Q_3 \subseteq \cdots$ be an ascending chain of ij-annihilators in J. Let $Q_i' = Z'Q_i$. The chain $Q_1' \subseteq Q_2' \subseteq Q_3' \subseteq \cdots$ terminates since it is a chain of subspaces in a finite dimensional space. Thus there exists a $Q_m' = \bigcup Q_i'$.

Now, we show that $Q_m = \bigcup Q_i$. It suffices to show $Q_m \cap J_{ij}$ contains $(\bigcup Q_i) \cap J_{ij}$ since Q_i is generated by $Q_i \cap J_{ij}$. Let $x \in (\bigcup Q_i) \cap J_{ij}$ so that x is in Q_m'. That is, for some s and t in Z and y in $Q_m \cap J_{ij}$ $x = s^{-1}ty$. Thus $U_x(B_m \cap J_{ij}) = U_{s^{-1}ty}(B_m \cap J_{ij}) = (s^{-1}t)^2 U_y(B_m \cap J_{ij}) = 0$ and we see that x is an element in J_{ij} which annihilates $B_m \cap J_{ij}$ so that x is in $ij\text{-ann}(B_m) = Q_m$.

The primeness of $J = H(R_n, \gamma_a)$, $n = 2$ or 3, when R is a Cayley-Dickson ring follows from the fact that $H(R_n', \gamma_a) = Z'J$.

We now turn our attention to the case when R is a $*$-prime associative Goldie ring. The primeness of J follows from the involution primeness of R_n under γ_a [5, p. 129].

Until stated otherwise we will make the following assumption which will lead us to a contradiction. We shall assume that J contains an infinite direct system, $\{Q_k\}$, of ij-quadratic ideals for some $i \neq j$, and R is a $*$-prime Goldie ring.

Since R is a $*$-prime Goldie ring with involution R is both left and right Goldie so that R/P and R/P^* are each both left and right Goldie [4, p. 268]. Thus R/P is a left and right order in the complete matrix ring, D_w, over a division ring D.

Let $\{f_{bb'}\}$ be the standard set of matrix units in D_w. Every element in D_w may be written as $\Sigma a_{bb'} f_{bb'}$ where $a_{bb'}$ is an element in the centralizer of $\{f_{bb'}\}$. We shall consider the coefficients, $a_{bb'}$, to be lexicographically ordered according to their subscripts. We shall say that $(a_{bb'}) \in D_w$ has l-zeros if the first l coefficients are zero and if $l \neq w^2$ then the $(l+1)$st one is not zero.

We shall consider R as being a subring of the direct sum $R/P + R/P^*$. Let g and g_* be the projections

$$g\colon R/P + R/P^* \to R/P, \qquad g_*\colon R/P + R/P^* \to R/P^*.$$

Also let

$$M_{ij}(Q_k) = \{g(x)\colon x \in R \text{ and } x_{ij} \in Q_k\} \quad \text{and} \quad M_{ij}^*(Q_k) = \{g_*(x)\colon x \in R \text{ and } x_{ij} \in Q_k\}.$$

Since R/P and R/P^* are prime Goldie rings we may consider each as a subring of a matrix ring over a division ring, so that $M_{ij}(Q_k)$ and $M^*_{ij}(Q_k)$ are sets of matrices and it makes sense to talk about the zeros of their elements. It should be pointed out that in light of this situation the elements of J are matrices whose entries are ordered pairs of matrices since $R \subseteq R/P + R/P^*$.

Lemma 4.3. *If each* $Q_k \cap J_{ij}$ *contains a nonzero element* $x^{(k)} = x_k a_j e_{ij} + x^*_k a_i e_{ji}$ *such that* $g(x_k a_j)$ *has l-zeros,* $l \neq w^2$ *where viewed as a matrix in* R/P, *then* J *contains an infinite direct system of ij-quadratic ideals* $\{A_p\}$ *such that each* $A_p \cap J_{ij}$ *contains a nonzero element* $y^{(p)} = y_p a_j e_{ij} + y^*_p a_i e_{ji}$ *such that* $g(y_p a_i)$ *has at least* $(l+1)$*-zeros. Moreover, if the* $x^{(k)}$*'s have the property that* $g_*(x_k a_j) = 0$ *then the* $y^{(p)}$*'s have the property that* $g_*(y_p a_i) = 0$.

Proof. If we let A_p be the quadratic ideal generated by $Q_p + Q_{p+1}$ for odd integers then $\{A_p\}$ is an infinite direct system of ij-quadratic ideals of J. We shall use the existence of $x^{(p)}$ and $x^{(p+1)}$ to construct $y^{(p)}$ in A_p.

Let p be an odd integer and let $x^{(p)}$ and $x^{(p+1)}$ be as in the statement of the lemma. Suppose the $(l+1)$st position is the one corresponding to the pair (r, s) so that

$$g(x_p a_j) = (a_{hh'}) = \begin{cases} 0 & \text{if } h < r \text{ or if } h = r \text{ but } h' < s, \\ a_{rs} \neq 0, & \\ ? & \text{otherwise,} \end{cases}$$

and

$$g(x_{p+1} a_j) = (b_{hh'}) = \begin{cases} 0 & \text{if } h < r \text{ or if } h = r \text{ but } h' < s, \\ b_{rs} \neq 0, & \\ ? & \text{otherwise.} \end{cases}$$

By the Faith-Utumi theorem, the centralizer D of $\{f_{hh'}\}$ contains a left and right order I such that $\Sigma I f_{hh'} \subseteq R/P$. Thus every element in D may be written as $c^{-1}d = uv^{-1}$ for some $c, d, u, v \in I$ so that $a_{rs} = c^{-1}d = uv^{-1}$ for c, d, u, v in I and $b_{rs} = c_0^{-1}d_0 = u_0 v_0^{-1}$ for c_0, d_0, u_0, v_0 in I. Since I is a left and right order in D, by Ore's theorem [4, p. 262] there exist nonzero elements x, x_0, y, y_0 in I such that $xd = x_0 d_0$ and $uy = u_0 y_0$. Now, let m and q be elements of R such that

$$g(m) = (m_{hh'}) = \begin{cases} 0 & \text{if } h \neq s \text{ or } h' \neq r, \\ vyxc & \text{if } h = s \text{ and } h' = r, \end{cases}$$

and

$$g(q) = (q_{hh'}) = \begin{cases} 0 & \text{if } h \neq s \text{ or } h' \neq r, \\ v_0 y_0 x_0 c_0 & \text{if } h = s \text{ and } h' = r. \end{cases}$$

Since $g(R) = R/P$, such an m and q exist in R.

Let $m_{ji} = a_i^{-1} m^* a_j e_{ij} + m e_{ji}$ and $q_{ji} = a_i^{-1} q^* a_j e_{ij} + q e_{ji}$. Here e_{ij} and e_{ji} are elements in the set of matrix units in R_n. Thus

$$\begin{aligned} U_{x^{(p)}}(m_{ji}) &= (x_p a_j e_{ij} + x_p^* a_i e_{ji})(a_i^{-1} m^* a_j e_{ij} + m e_{ji})(x_p a_j e_{ij} + x_p^* a_i e_{ji}) \\ &= (x_p a_j) m (x_p a_j) e_{ij} + x_p^* m^* a_j x_p^* a_i e_{ji}. \end{aligned}$$

But

$$(x_p a_j) m (x_p a_j) = g(x_p a_j) g(m) g(x_p a_j) + g_*(x_p a_j) g_*(m) g_*(x_p a_j)$$

and

$$\begin{aligned} g(x_p a_j) g(m) g(x_p a_j) &= (a_{bb'})(m_{bb'})(a_{bb'}) = (a_{bb'}) \left(\sum_r m_{sr} a_{rb'} \right)_{sb'} \\ &= \left(\sum_s a_{bs} \left(\sum_r m_{sr} a_{rb'} \right) \right)_{bb'} = \left(\sum_s \sum_r a_{bs} m_{sr} a_{rb'} \right)_{bb'} \\ &= \begin{cases} 0 & \text{if } b < r \text{ or if } b = r \text{ but } b' < s \\ uyxd & \text{if } b = r \text{ and } b' = s \\ ? & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly

$$U_{x^{(p+1)}}(q_{ji}) = (x_{p+1} a_j) q (x_{p+1} a_j) e_{ij} + x_{p+1}^* q^* a_j x_{p+1}^* a_i e_{ji}$$

where

$$(x_{p+1} a_j) q (x_{p+1} a_j) = g(x_{p+1} a_j) g(q) g(x_{p+1} a_j) + g_*(X_{p+1} a_j) g_*(q) g_*(x_{p+1} a_j)$$

and

$$g(x_{p+1} a_j) g(q) g(x_{p+1} a_j) = \begin{cases} 0 & \text{if } b < r \text{ or } b = r \text{ and } b' < s, \\ u_0 y_0 x_0 d_0 & \text{if } b = r \text{ and } b' = s, \\ ? & \text{otherwise.} \end{cases}$$

$U_{x^{(p)}}(m_{ji})$ is a nonzero element in Q_p and $U_{x^{(p+1)}}(q_{ji})$ is a nonzero element in Q_{p+1} so that $U_{x^{(p)}}(m_{ji})$ minus $U_{x^{(p+1)}}(q_{ji})$ is not equal to zero since $Q_p \cap Q_{p+1} = 0$. $g(x_p a_j) g(m) g(x_p a_j) - g(x_{p+1} a_j) g(q) g(x_{p+1} a_j)$ is an element of D_w with at least $(l+1)$-zeros, since $uyxd = u_0 y_0 x_0 d_0$. Let

$$y^{(p)} = U_{x^{(p)}}(m_{ji}) - U_{x^{(p+1)}}(q_{ji}).$$

$y^{(p)}$ is a nonzero element in $A_p \cap J_{ij}$ such that $g(y_p a_j)$ has at least $(l+1)$-zeros, where y_p is taken to be $x_p a_j m x_p - x_{p+1} a_j q x_{p+1}$ so that $y^{(p)} = y_p a_j e_{ij} + y_p^* a_j e_{ji}$.

This completes the proof of the lemma, since the last statement of the lemma is clear from the construction.

We use Lemma 4.3 to show that there is a contradiction built in the assumption stated above.

Theorem 4.4. *Let* $J = H(R_n, \gamma_a)$ *where* R *is associative and* $n \geq 2$. *If* R *is* $*$*-prime Goldie then* J *does not contain an infinite direct system of ij-quadratic ideals for all* $i \neq j$.

Proof. Suppose that the theorem is false. We may assume J contains an infinite direct system $\{Q_k\}$ such that each Q_k contains an $x^{(k)} = x_k a_j e_{ij} + x_k^* a_i e_{ji}$ such that $g(x_k a_j)$ has l-zeros, $l \neq w^2$, for all integral values of k, since this or the corresponding statement using $g_*(x_k a_j)$ is true for some infinite subset of $\{Q_k\}$.

By Lemma 4.3, J contains an infinite direct system $\{Q_k^{(1)}\}$ such that each $Q_k^{(1)} \cap J_{ij}$ contains a nonzero element $y^{(k)} = y_k a_j e_{ij} + y_k^* a_i e_{ji}$ such that $g(y_k a_j)$ has m-zeros where $m > l$.

Continuing by induction and Lemma 4.3, J contains an infinite direct system $\{Q_k^{(r)}\}$, $r \leq w^2$, such that $Q_k^{(r)} \cap J_{ij}$ contains an element $c^{(k)} = c_k a_j e_{ij} + c_k^* a_i e_{ji}$ $\neq 0$ such that $g(c_k a_j) = 0$. But since $c^{(k)} \neq 0$, it must be the case that $g_*(c_k a_j)$ $\neq 0$.

Similar to what we did above, we may assume that $g_*(c_k a_j)$ has m-zeros for each k. Then we may go through an argument similar to the one just completed to obtain an infinite direct system $\{Q_k^{(r)}\}$, $r \leq 2w^2$ such that $Q_k^{(r)} \cap J_{ij}$ contains an element $d^{(k)} = d_k a_j e_{ij} + d_k^* a_i e_{ij} \neq 0$ such that $g_*(d_k a_j) = 0$. But the last sentence of Lemma 4.3 tells us that this construction may be done so that $g(d_k a_j) = 0$. This is impossible.

We now show that R $*$-prime Goldie implies that J satisfies ACC on ij-annihilators.

Lemma 4.5. *If* $K_q(W_1) \subseteq K_q(W_2) \subseteq \cdots \subseteq K_q(W_m) \subseteq \cdots$ *while* $q = 1, 2, 3, 4$ *and the* W_m*'s are additive subgroups of* $(R, +)$, *then* $K_q(W_i) = K_q(\Sigma_{j \geq i} W_j)$. (*The notation used here is that which was introduced in the proof of Lemma* 3.2.)

Proof. If $q = 1$ or 3 then we are talking about left or right annihilators in R, and hence the lemma is true for $q = 1$ or 3. Also, a slight adaptation of the following proof yields a proof for these cases.

Suppose $q = 2$. Let a be an element in $K_2(W_i)$ so that $aW_j \subseteq P$ and $W_j a \subseteq P^*$ for all $j \geq i$, since $K_2(W_i) \subseteq K_2(W_j)$ for $j \geq i$. Thus $a(\Sigma_{j \geq i} W_j) \subseteq P$ and $(\Sigma_{j \geq i} W_j)a$ $\subseteq P^*$ so that a is an element in $K_2(\Sigma_{j \geq i} W_j)$. Therefore $K_2(W_i)$ is contained in $K_2(\Sigma_{j \geq i} W_j)$. Clearly if a is an element in $K^2(\Sigma_{j \geq i} W_j)$ then a is an element in $K_2(W_i)$ since $aW_i \subseteq a(\Sigma_{j \geq i} W_j) \subseteq P$ and $W_i a \subseteq (\Sigma_{j \geq i} W_j)a \subseteq P$ and $W_i a \subseteq (\Sigma_{j \geq i} W_j)a \subseteq$ P^*. Thus $K_2(W_i) = K_2(\Sigma_{j \geq i} W_j)$.

The proof for $q = 4$ is similar and therefore it is omitted.

Theorem 4.6. *Let* $J = H(R_n, \gamma_a)$, $n \geq 2$, *where* R *is associative and* $*$*-prime. If* R *is Goldie then* J *satisfies* ACC *on ij-annihilators.*

Proof. From the Remark of §III, it suffices to show R satisfies ACC on sets of the form $\bigcup K_i(T)$ where T is an additive subgroup of $(R, +)$. Thus it suffices to show R satisfies ACC on sets of the form $K_q(W)$ where W is an additive subgroup of $(R, +)$.

Suppose we have such a chain $K_q(W_1) \subseteq K_q(W_2) \subseteq \cdots \subseteq K_q(W_m) \subseteq \cdots$. By Lemma 4.5, we may assume that $W_1 \supseteq W_2 \supseteq \cdots \supseteq W_m \supseteq \cdots$.

By ACC on left and right annihilators in R/P, we have the following chains in R/P terminating

$$A_R(W_i/P) \subseteq A_R(W_{i+1}/P), \qquad A_L(W_i/P) \subseteq A_L(W_{i+1}/P)$$

and correspondingly in R/P^*

$$A_R(W_i/P^*) \subseteq A_R(W_{i+1}/P^*), \qquad A_L(W_i/P^*) \subseteq A_L(W_{i+1}/P^*).$$

Hence the following chains terminate in R:

$$\{a \in R\colon a + P \in A_R(W_i/P)\} \subseteq \{a \in R\colon a + P \in A_R(W_{i+1}/P)\},$$

$$\{a \in R\colon a + P \in A_L(W_i/P)\} \subseteq \{a \in R\colon a + P \in A_L(W_{i+1}/P)\},$$

$$\{a \in R\colon a + P^* \in A_R(W_i/P^*)\} \subseteq \{a \in R\colon a + P^* \in A_R(W_{i+1}/P^*)\},$$

$$\{a \in R\colon a + P^* \in A_L(W_i/P^*)\} \subseteq \{a \in R\colon a + P^* \in A_L(W_{i+1}/P^*)\},$$

But the chain $K_q(W_i) \subseteq K_q(W_{i+1})$ is the intersection of corresponding terms of two chains which terminate. Hence we have that the chain $K_q(W_i) \subseteq K_q(W_{i+1})$ terminates.

V. Quotients. In order to complete the proof of the Main Theorem, the last statement is all that needs to be shown.

In the case when R is a $*$-prime Goldie ring this follows from [2].

Let R be a Cayley-Dickson ring and let R' be the Cayley-Dickson algebra associated with R. Extend the involution on R to the standard involution on R' and extend γ_a on R_n to γ_a on R'. We need to show that every element in $J' = H(R'_n, \gamma_a)$, $n = 2, 3$, has the form $U_a^{-1}(b)$ for some a and b in J with a regular in J, and that regular elements in J are invertible in J'.

Since every element in R' is of the form $z^{-1}b$ for $0 \neq z$ in the center $Z = Z(R)$ and b in R, given an element t in $H(R'_n, \gamma_a)$ we may express it as an $n \times n$ matrix $(c_{kk'})$ where each entry $c_{kk'}$ is of the form $z_{kk'}^{-1} b_{kk'}$ for $z_{kk'} \neq 0$ in $Z(R) = H(R)$ and $b_{kk'}$ in R. We shall exhibit a y and a w such that $t = U_w^{-1}(y)$. Let π be the product of $z_{kk'}$'s so that $\pi^2 z_{kk'}^{-1}$ is in $H(R)$ and let w be the diagonal matrix $\pi e_{11} + \cdots + \pi e_{nn}$. Let $y = (\pi^2 z_{kk}^{-1}, b_{kk'})$. Then y is in $H(R_n, \gamma_a)$ since

$$\gamma_a(y) = \gamma_a(\pi^2 z_{kk}^{-1}, b_{kk'}) = \pi^2 \gamma_a(z_{kk}^{-1}, b_{kk'}) = \pi^2(z_{kk}^{-1}, b_{kk'}) = (\pi^2 z_{kk}^{-1}, b_{kk'}) = y.$$

Clearly y and w have the desired property.

Finally, let x be a regular element of J. Let t be an arbitrary element in J'. By what was shown above, $t = U_{w^{-1}}(y)$ for w in $Z(R)$ and y in J. Here we are considering J' as a Z'-algebra where Z' is the field of quotients for $Z(R)$ and identifying z^{-1} with $z^{-1}1$ for 1 in J'. Hence

$$U_x(t) = U_x U_{w^{-1}}(y) = U_x(w^{-2}y) = w^{-2}U_x(y) \neq 0$$

since x is regular in J. Thus U_x is 1-1 on the finite dimensional Z'-algebra J'. Recall R' is 8-dimensional over Z'. Therefore, U_x is a 1-1 linear transformation on a finite dimensional vector space and hence U_x is onto so that 1 is in $U_x(J')$ and x is invertible.

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