

## ON STRUCTURE SPACES OF IDEALS IN RINGS OF CONTINUOUS FUNCTIONS

BY

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**ABSTRACT.** A ring of continuous functions is a ring of the form  $C(X)$ , the ring of all continuous real-valued functions on a completely regular Hausdorff space  $X$ .

With each ideal  $I$  of  $C(X)$ , we associate certain subalgebras of  $C(X)$ , and discuss their structure spaces.

We give necessary and sufficient conditions for two ideals in rings of continuous functions to have homeomorphic structure spaces.

**Introduction.** For a subset  $A$  of  $C(X)$ , we define  $rA$  to be  $\{f + c \mid f \in A \text{ and } c \in R\}$ . ( $R$  denotes the set of all real numbers, and we make the usual identification between the real number  $c$  and the function which maps every  $x \in X$  onto  $c$ .) We denote by  $A^u$  the closure of  $A$  in the uniform topology.

With each ideal  $I$  in  $C(X)$ , we associate four subalgebras of  $C(X)$ ,  $I$  itself,  $I^u$ ,  $rI$ , and  $r(I^u)$ . (In [4],  $rI$  and  $r(I^u)$  were denoted by  $(I)$  and  $(I^u)$ , respectively.) In this paper, we characterize the maximal ideals of  $I^u$ ,  $rI$ , and  $r(I^u)$  (the maximal ideals of  $I$  were characterized in [4]) and then endow these sets of maximal ideals with the hull-kernel topology. We then investigate the resulting structure spaces.

In §1 we show that the prime and maximal ideals of  $rI$  are the intersections of  $rI$  respectively with the prime and maximal ideals of  $C(X)$ . This allows us to establish homeomorphisms between structure spaces of  $rI$  and modifications of structure spaces of  $C(X)$ . We also show that the structure space of  $rI$  is (homeomorphic to) the one-point compactification of the structure space of  $I$ .

In §2 we discuss the prime and maximal ideals of the algebras  $I^u$  and  $r(I^u)$ . We show that  $I$  and  $I^u$  have the same structure space and that  $rI$  and  $r(I^u)$  do also. In view of the fact that  $r(I^u)$  is (isomorphic to) a ring of continuous functions (see [4, 5.6]), it is thus established that every ideal in  $C(X)$  is a real ideal in a subalgebra of  $C(X)$  which has the same structure space as a ring of continuous functions. Results in §§1 and 2 generalize certain results in [5].

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The significance of homeomorphic structure spaces of two rings of continuous functions is well known; namely  $C(X)$  and  $C(Y)$  have homeomorphic structure spaces if and only if  $C^*(X)$  and  $C^*(Y)$  are isomorphic. (These are the subrings of bounded functions in  $C(X)$  and  $C(Y)$  respectively.) In §3 we establish necessary and sufficient conditions for two ideals in rings of continuous functions to have homeomorphic structure spaces.

In [2] and [3], the authors considered structure spaces of certain subalgebras of  $C(X)$  called "algebras on  $X$ ." In §4, we discuss the relationships between the properties of an algebra on  $X$  and the properties of the algebras considered in this paper.

**Preliminaries and notations.** For each  $f \in C^*(X)$  there is a unique extension  $\hat{f}$  from  $\beta X$  into  $R$ . For each  $f \in C(X)$ , there is a unique extension  $f^*$  from  $\beta X$  into the two-point compactification of  $R$ . (See [1, 7.5] and [4, 2.4] for more about  $f^*$ .) If  $f \in C^*(X)$ , then  $\hat{f} = f^*$ .

For any commutative ring, a collection of prime ideals endowed with the hull-kernel topology is called a *structure space* of the ring. The collection of all prime maximal ideals is referred to as *the* structure space of the ring. It is well known that  $\beta X$  is (homeomorphic to) the structure space of either  $C(X)$  or  $C^*(X)$  and that  $\nu X$  is the structure space of real maximal ideals of  $C(X)$ . (The reader is referred to 7.10 and 7.11 in [1] and [4, 2.3] for more about structure spaces.) The natural mapping  $M \rightarrow M^*$ , where  $M^* = M^\mu \cap C^*(X)$ , is a homeomorphism of the structure space of  $C(X)$  onto the structure space of  $C^*(X)$ . (See [1, 7.11].)

Associated with each ideal  $I$  in  $C(X)$  is a space  $X(I)$  and a fixed maximal ideal in  $C(X(I))$  denoted by  $F(I)$  with the property that  $I^\mu$  is isomorphic to  $F(I)$ . (See [4, 5.6].)

If  $f \in C(X)$ , we shall let  $(f)$  denote the principal ideal generated by  $f$ , and we let  $Z(f) = \{x \in X | f(x) = 0\}$ . If  $g \in C(X)$ , then  $f \vee g$  denotes the pointwise maximum of  $f$  and  $g$ .

A subring  $A$  of  $C(X)$  is called a *subalgebra* of  $C(X)$  if  $A$  is closed under multiplication by constants. The structure space (of prime maximal ideals) of  $A$  will be denoted by  $\mu A$ .

The symbol  $\cong$  is used to denote a homeomorphism between spaces, and  $\approx$  is used to denote an isomorphism between rings.

For the convenience of the reader we now list the main symbols associated with an ideal  $I$  of  $C(X)$ .

- $I^\mu$  = the uniform closure of  $I$ ;
- $I^* = I^\mu \cap C^*(X)$ ;
- $rI = \{f + c | f \in I \text{ and } c \in R\}$ ;
- $r(I^\mu) = \{f + c | f \in I^\mu \text{ and } c \in R\}$ ;

$\mu I$  = the structure space of  $I$ ;

$\Delta I = \bigcap \{Z(f) \mid f \in I\}$ ;

$M(I) = \{M \mid M \text{ is a maximal ideal in } C(X) \text{ and } M \supseteq I\}$ ;

$P(I) = \{P \mid P \text{ is a prime ideal in } C(X) \text{ and } P \supseteq I\}$ ;

$X(I)$  = a space with the property that  $C(X(I)) \approx r(I^\mu)$ ;

$F(I)$  = the real ideal in the space  $X(I)$  isomorphic to  $I^\mu$ .

1. The structure space of  $rI$ . It is known [4, 3.6] that the maximal ideals of an ideal  $I$  are precisely the intersections of  $I$  with the maximal ideals of  $C(X)$  not containing  $I$ . We now characterize the maximal ideals of the algebra  $rI$ .

1.1 Remark. Evidently,  $I$  itself is a real maximal ideal of  $rI$ .

1.2 Lemma. *If  $M$  is a maximal ideal in  $C(X)$ , then  $M \cap rI$  is a maximal ideal in  $rI$ .*

**Proof.** It is easy to show that for a proper ideal  $J$  of  $C(X)$ ,  $J \supseteq I$  if and only if  $J \cap rI = I$ . Thus if  $M \supseteq I$ ,  $M \cap rI$  equals  $I$  and is a maximal ideal in  $rI$ . Now suppose  $M \not\supseteq I$ . Then there exist  $m \in M$  and  $i \in I$  with  $m + i = 1$ . Assume  $M \cap rI \subsetneq K \subseteq rI$ , for some ideal  $K$  of  $rI$ . Let  $k \in K \setminus M$ . Then there exist  $m' \in M$  and  $b \in C(X)$  so that  $bk + m' = 1$ . Thus  $i = ibk + im' \in K$ . But  $m = -i + 1 \in M \cap rI \subseteq K$ ; whence  $1 \in K$  and  $K = rI$ .

We now proceed to establish the converse of 1.2.

1.3 Lemma. *If  $K$  is a proper prime ideal in  $rI$ , then  $K = P \cap rI$  for some prime ideal  $P$  in  $C(X)$ . Furthermore,  $P$  is unique if  $K \neq I$ .*

**Proof.** From the multiplicative semigroup  $G = (rI) \setminus K$ . From Zorn's lemma, it follows that there exists a prime ideal  $P$  in  $C(X)$  so that  $G \cap P = \emptyset$  and  $P$  is maximal with respect to this property. (See [1, 0.16].) We claim that  $K = P \cap rI$ . It is easy to see that  $K \supseteq P \cap rI$ . Now if  $P \supseteq I$ , then  $K = I = P \cap rI$ , so assume  $P \not\supseteq I$ . We must show that  $K \subseteq P$ . To this end, let  $k \in K$  and assume  $k \notin P$ . Let  $J$  denote  $P + k \cdot I$ , an ideal in  $C(X)$  which contains  $P$  properly. Hence  $J \cap G \neq \emptyset$ , and there exists  $p + ki \in G$ , where  $p \in P$  and  $i \in I$ . It follows that  $p \in rI$  and hence  $p \in K$ . But then  $p + ki \in K$ , a contradiction. The uniqueness is evident.

1.4 Theorem. *If  $K$  is a maximal ideal in  $rI$ , then  $K = M \cap rI$  for some maximal ideal  $M$  in  $C(X)$ . Furthermore, if  $K \neq I$ , then  $M$  is unique.*

**Proof.** By [4, 3.3]  $K$  is prime, and hence  $K = P \cap rI$ , for some prime ideal  $P$  in  $C(X)$ . Letting  $M$  be the maximal ideal containing  $P$ , the result follows.

We now consider some natural mappings between structure spaces of  $rI$  and  $C(X)$ . We shall let  $\mathcal{P}$  denote the space of prime ideals of  $C(X)$ ,  $\mathcal{P}(rI)$  the space

of prime ideals of  $\mathcal{P}$  and  $\mathcal{P}(I)$  the set of all prime ideals in  $C(X)$  which contain  $I$ . (Of course,  $\mathcal{P}$  and  $\mathcal{P}(I)$  are endowed with the hull-kernel topology.)

1.5 Lemma.  $\mathcal{P} \setminus \mathcal{P}(I)$  is homeomorphic to  $\mathcal{P}(I) \setminus \{I\}$ .

**Proof.** Consider the natural mapping  $P \rightarrow P \cap I$  and denote it by  $\phi$ . Clearly  $\phi$  is one-to-one and onto. Let  $s \in I$  and consider  $E(s) = \{K \mid K \in \mathcal{P}(I) \setminus \{I\} \text{ and } s \in K\}$ , a basic closed set in  $\mathcal{P}(I) \setminus \{I\}$ . Then  $\phi^{-1}(E(s)) = \{P \in \mathcal{P} \setminus \mathcal{P}(I) \mid s \in P\}$  a basic closed set in  $\mathcal{P} \setminus \mathcal{P}(I)$ . On the other hand, if  $f \in C(X)$ , and  $E(f) = \{P \in \mathcal{P} \setminus \mathcal{P}(I) \mid f \in P\}$ , a basic closed set in  $\mathcal{P} \setminus \mathcal{P}(I)$ , then  $\phi(E(f)) = \{P \cap I \mid P \in \mathcal{P} \setminus \mathcal{P}(I) \text{ and } f \in P\}$ , a closed set in  $\mathcal{P}(I) \setminus \{I\}$ .

We now extend the mapping  $\phi$  to obtain a mapping onto all of  $\mathcal{P}(I)$ . The preimage of  $I$  is  $\mathcal{P}(I)$  shrunk to a point. Specifically, we let  $\mathcal{P}' = [\mathcal{P} \setminus \mathcal{P}(I)] \cup \{\alpha\}$  where a neighborhood of  $\alpha$  is a set of the form  $\{\alpha\} \cup [W \setminus \mathcal{P}(I)]$  where  $W$  is open in  $\mathcal{P}$  and  $W \supseteq \mathcal{P}(I)$ . (For  $P \in \mathcal{P} \setminus \mathcal{P}(I)$ , a neighborhood of  $P$  is a neighborhood in the relative topology.)

1.6 Theorem. Let  $\psi: \mathcal{P}' \rightarrow \mathcal{P}(I)$  be defined by  $\psi(P) = P \cap I$  if  $P \neq \alpha$ , and  $\psi(\alpha) = I$ . Then  $\psi$  is a homeomorphism of  $\mathcal{P}'$  onto  $\mathcal{P}(I)$ .

**Proof.** Clearly  $\psi$  is one-to-one and onto  $\mathcal{P}(I)$ , and by virtue of 1.5, it suffices to show that  $\psi$  is continuous at  $\alpha$  and  $\psi^{-1}$  is continuous at  $I$ . To this end, let  $V$  be a basic neighborhood of  $\psi(\alpha)$  in the space  $\mathcal{P}(I)$ . Then  $V = \sim\{K \in \mathcal{P}(I) \mid s \in K\}$  for some  $s = i + c \in I$ . Since  $I \in V$ ,  $c \neq 0$ . Let  $W = \sim\{P \in \mathcal{P} \mid s \in P\}$ , an open set in  $\mathcal{P}$ .

It is easily seen that  $W \supseteq \mathcal{P}(I)$ , for if  $P \supseteq I$ , and  $P \not\subseteq W$ , it would follow that  $s \in P$ , a contradiction. Thus the set  $\{\alpha\} \cup [W \setminus \mathcal{P}(I)]$  is open in  $\mathcal{P}'$ , and it follows that its image under  $\psi$  is contained in  $V$ .

We now show that  $\psi^{-1}$  is continuous at  $I$ . Let  $\{\alpha\} \cup [W \setminus \mathcal{P}(I)]$  be a neighborhood of  $\alpha$  in  $\mathcal{P}'$ , with  $W = \sim\{P \in \mathcal{P} \mid P \supseteq J\}$  for some ideal  $J$  in  $C(X)$ . Since  $W \supseteq \mathcal{P}(I)$ , it follows that  $I + J = C(X)$ , and hence there exist  $i \in I$  and  $j \in J$  so that  $i + j = 1$ . Letting  $V$  denote the open set  $\sim\{K \in \mathcal{P}(I) \mid j \in K\}$  ( $j \in I$ ) it follows that  $\psi^{-1}(V) \subseteq \{\alpha\} \cup [W \setminus \mathcal{P}(I)]$ .

1.7 Remark. If we shrink  $M(I)$  to a point in the space of maximal ideals in  $C(X)$ , then the resulting space will be homeomorphic to  $\mu(I)$ . (The argument would be the same as 1.5 and 1.6 above.) It then follows that  $\mu(I)$  is homeomorphic to  $(\mu I)^*$ , the one-point compactification of  $\mu I$ . (See [4, 3.9].)

1.8 Remark. If we denote the set of all real maximal ideals containing  $I$  by  $R(I)$  (possibly  $R(I) = \emptyset$ ), then the structure space of real ideals of  $\mathcal{P}$  is homeomorphic to  $(\nu X \setminus R(I)) \cup \{\alpha\}$  where a neighborhood of  $\alpha$  is a set of the form  $\{\alpha\} \cup [W \setminus R(I)]$  for some  $W$  open in  $\nu X$  with  $W \supseteq R(I)$ .

Of course, the algebra  $rI$  will not in general be (isomorphic to) a ring of continuous functions. Indeed since  $I$  is a real ideal in  $rI$ , it follows from [4, 5.7] that  $rI$  is a ring of continuous functions if and only if  $I$  is uniformly closed.

In view of the fact that  $(f_1 + c_1) \vee (f_2 + c_2) = (f_1 \vee f_2) + (c_1 \vee c_2)$  where  $f_j \in C(X)$  and  $c_j$  are real numbers, it follows that  $rI$  is a  $\Phi$ -algebra (see [5]) if and only if  $f \vee g \in I$  whenever  $f$  and  $g$  are members of  $I$ . In particular,  $rI$  is a  $\Phi$ -algebra whenever  $I$  is absolutely convex in  $C(X)$ . (An ideal  $I$  is said to be *absolutely convex* in  $C(X)$  if  $f \in I$  and  $g \in C(X)$  with  $|g| \leq |f|$  imply that  $g \in I$ .)

2. The algebras  $I^\mu$  and  $r(I^\mu)$ . We now proceed to characterize the maximal ideals of  $I^\mu$  and  $r(I^\mu)$ . We recall that in [4, 5.6] it was shown that  $r(I^\mu)$  is (isomorphic to) a ring of continuous functions. (Indeed,  $r(I^\mu)$  is in a sense the smallest ring of continuous functions in which  $I$  is an ideal.)

2.1 Remark.  $I^\mu$  is a real maximal ideal in  $r(I^\mu)$ .

2.2 Lemma. If  $s \in r(I^\mu)$  with  $s(x) \geq \delta > 0$  for all  $x \in X$ , then  $(1/s) \in r(I^\mu)$ .

**Proof.** Suppose  $s = f + c$  where  $f \in I^\mu$  and  $c \in R$ , and  $s(x) \geq \delta > 0$  for all  $x \in X$ . It is easily seen that  $c \neq 0$ , and we then have  $1/s = -f/c(s) + 1/c$ . We claim that  $f/(f + c) \in I^\mu$ . To see this, let  $\epsilon > 0$  be given, and consider  $\epsilon' = \delta \cdot \epsilon$ . Then there exists  $i \in I$  with  $|f - i| < \epsilon'$ , whence

$$\left| \frac{f}{f+c} - \frac{i}{f+c} \right| < \frac{\epsilon'}{|f+c|} \leq \frac{\epsilon'}{\delta} = \epsilon.$$

2.3 Lemma. If  $M$  is a maximal ideal in  $C(X)$  with  $M \not\supseteq I$ , then  $M \cap r(I^\mu)$  is a maximal ideal in  $r(I^\mu)$ .

**Proof.** The argument is the same as 1.2.

2.4 Remark. If  $M \supseteq I$ , it is possible that  $M \cap r(I^\mu) \subsetneq I^\mu$ , and hence  $M \cap r(I^\mu)$  is not maximal (for example, if  $I$  is a hyperreal maximal ideal and  $M = I$ ).

2.5 Remark. It is possible to have prime ideals in  $r(I^\mu)$  which are not of the form  $P \cap r(I^\mu)$  for  $P$  prime in  $C(X)$ . As a simple example, the ideal  $I^\mu$  itself may not be of this form. We can say, however

2.6 Lemma. If  $K$  is a prime ideal in  $r(I^\mu)$  and  $K \not\supseteq I$ , then  $K = P \cap r(I^\mu)$  for a unique prime ideal  $P$  in  $C(X)$ .

**Proof.** The argument is essentially the same as 1.3.

2.7 Theorem. Let  $K$  be a maximal ideal of  $r(I^\mu)$ . If  $K \not\supseteq I$ , then  $K = M \cap r(I^\mu)$  for a unique maximal ideal  $M$  in  $C(X)$ . If  $K \supseteq I$ , then  $K = I^\mu$ .

**Proof.** The first part of the theorem is evident. For the second part, suppose  $K \supseteq I$ , and let  $k \in K$ , say  $k = f + c$  where  $f \in I^\mu$  and  $c \in R$ . Assume  $c > 0$ . Then for some  $i \in I$ ,  $|f - i| < c/3$ . Since  $i \in K$ , it follows that  $f + c - i \in K$ . But  $(f + c - i)(x) > 2c/3$  for all  $x \in X$ , and this contradicts 2.2. Similarly, if  $c < 0$  we arrive at a contradiction, so we must have  $c = 0$  and  $K \subseteq I^\mu$ .

We now consider the maximal ideals of the algebra  $I^\mu$ . We shall make use of the fact that  $r(I^\mu)$  is isomorphic to  $C(X(I))$  and that the isomorphism takes  $I^\mu$  onto  $F(I)$ .

**2.8 Theorem.** *Let  $K$  be a maximal ideal in  $I^\mu$ . Then  $K = M \cap I^\mu$  for some unique maximal ideal  $M$  in  $C(X)$  with  $M \not\supseteq I$ .*

**Proof.** Let  $\xi$  denote the isomorphism of  $r(I^\mu)$  onto  $C(X(I))$ . Since  $K$  is a maximal ideal in  $I^\mu$ , it follows that  $\xi(K) = F(I) \cap M'$  where  $M'$  is maximal in  $C(X(I))$ . But then  $M' = \xi[M \cap r(I^\mu)]$  (clearly  $M' \neq \xi(I^\mu)$ ) and it follows that  $K = M \cap I^\mu$ . Now, assume  $M \supseteq I$ , and let  $f \in I^\mu \setminus K$ . From the maximality of  $K$ , it follows that  $K + f \cdot I^\mu = I^\mu$ , whence  $f = k + fg$  for some  $k \in M \cap I^\mu$  and  $g \in I^\mu$ . But this implies that  $k = f(1 - g) \in M$ , and hence  $1 - g \in M$ . Since  $M \supseteq I$ ,  $g \in M^\mu$ , a contradiction.

**2.9 Corollary.** *The ideal  $I$  in  $I^\mu$  is not contained in a maximal ideal of  $I^\mu$ .*

**Proof.** Follows directly from 2.8 above.

**2.10 Lemma.** *If  $M$  is maximal in  $C(X)$  and  $M \not\supseteq I$ , then  $M \cap I^\mu$  is a maximal ideal in  $I^\mu$ . If  $M \supseteq I$ , then  $M \cap I^\mu$  is not a maximal ideal in  $I^\mu$ .*

**Proof.** The argument for the first part is the same as 1.2. The second part follows from 2.9.

**2.11 Corollary.**  $\mu I$  is homeomorphic to  $\mu(I^\mu)$ .

**Proof.** The natural mapping is a homeomorphism.

**2.12 Remark.** It is easily seen that the natural mapping  $M \cap rI \rightarrow M \cap r(I^\mu)$ , for  $M \not\supseteq I$ , and  $I \rightarrow I^\mu$  is a homeomorphism of  $\mu(rI)$  onto  $\mu(r(I^\mu))$ , and  $\mu(r(I^\mu))$  is the structure space of a ring of continuous functions. We thus have that every ideal in a ring of continuous functions is a real ideal in an algebra whose structure space is homeomorphic to a structure space of a ring of continuous functions.

**2.13 Remark.**  $(rI)^\mu = r(I^\mu)$ .

**Proof.** Let  $s \in (rI)^\mu$ . Then there is a sequence  $\langle s_n \rangle$  in  $rI$ , say  $s_n = i_n + c_n$  where  $i_n \in I$  and  $c_n \in R$ , which converges to  $s$ . We claim that the sequence  $\langle c_n \rangle$  is a Cauchy sequence. To see this, consider  $\epsilon > 0$ . There exists a positive

integer  $N$ , so that  $|s(x) - (i_n(x) + c_n)| < \epsilon/2$  for all  $n > N$  and all  $x \in X$ . Let  $n$  and  $m$  be any positive integers greater than  $N$ , and let  $t \in X$  so that  $i_n(t) = i_m(t) = 0$ . We then have  $|s(t) - c_n| < \epsilon/2$  and  $|s(t) - c_m| < \epsilon/2$ , from which it follows that  $|c_m - c_n| < \epsilon$ . Let  $c$  denote  $\lim_{n \rightarrow \infty} c_n$ . It then follows that the sequence  $\langle i_n \rangle$  converges to a function  $f \in I^u$ , and  $s = f + c$ . Conversely, if  $s = f + c \in r(I^u)$  then, given  $\epsilon > 0$ , there exists  $i \in I$  so that  $|f - i| < \epsilon$ . But  $|f - i| = |(f + c) - (i + c)|$ , and  $i + c \in rI$ .

The above remark generalizes [5, 3.8].

**2.14 Remark.** The algebras  $rI$  and  $r(I^u)$  cannot be (ring-) isomorphic unless they are identical. To see this, it suffices to observe that any isomorphism would take  $I$  onto  $I^u$  which would imply their equality by [4, 4.8].

**3. Structure spaces of ideals.** The structure space of an ideal is not usually a compact space. Indeed

**3.1 Remark.** For any ideal  $I$ , the following are equivalent.

- (1)  $\mu I$  is compact.
- (2)  $M(I)$  is open (and closed) in  $\beta X$ .
- (3)  $I$  is the principle ideal generated by an idempotent.

**Proof.** The equivalence of (1) and (2) follows from the fact that  $\mu I \cong \beta X \setminus M(I)$  (see [4, 3.9]).

(2)  $\Rightarrow$  (3) If  $M(I)$  is open, then  $\sim M(I) = \{M | M \supseteq J\}$  for some ideal  $J$  in  $C(X)$ . It follows that  $I \cap J = \{0\}$  and  $I + J = C(X)$ , and hence there exist  $i \in I$  and  $j \in J$  with  $i + j = 1$ . Thus  $I = (i)$  and  $i^2 = i$ .

(3)  $\Rightarrow$  (2) If  $I = (i)$  and  $i^2 = i$ , let  $J = (i - 1)$ . Then  $\sim M(I) = \{M | M \supseteq J\}$ .

However, the structure space of  $I$  is always locally compact, and its one-point compactification is homeomorphic to  $\mu(r(I^u))$ , a Stone-Čech compactification ( $\beta[X(I)]$ ). (It is also true that any  $\beta X$  is the one-point compactification of a  $\mu I$ ; simply take  $I$  to be a maximal ideal in  $C(X)$ .) Thus  $\mu I$  is in a sense a large space, in that it lacks only one point from being a Stone-Čech compactification.

We now wish to show to what extent the structure space of  $I$  determines  $I$ ; specifically, when are  $\mu I$  and  $\mu J$  homeomorphic?

We begin with a mapping from the set of ideals in  $C(X)$  into the set of ideals of  $C^*(X)$ . For an ideal  $I$  in  $C(X)$ , we denote  $I^u \cap C^*(X)$  by  $I^*$ . Note that this  $*$  mapping is the usual one from the set of maximal ideals in  $C(X)$  onto the set of maximal ideals in  $C^*(X)$ . (See [4, 2.4].)

**3.2 Lemma.**  $I^* = \{f \in C^*(X) | Z(\hat{f}) \supseteq M(I)\}$ .

**Proof.** Follows from [4, 5.1] and the fact that  $f^* = \hat{f}$  for bounded functions.

**3.3 Remark.** The  $*$  mapping is onto the set of full ideals of  $C(\beta X)$ . (An ideal is said to be *full* in  $C(Y)$  if it is of the form  $\{f \in C(Y) | Z(f) \supseteq G\}$  for some closed set  $G$  in  $Y$ .)

3.4 Lemma.  $\mu I$  is homeomorphic to  $\mu(I^*)$ .

**Proof.** The mapping  $*$  is a homeomorphism of the structure space of  $C(X)$  onto the structure space of  $C^*(X)$  [1, 7.11], and clearly  $M(I)$  will be mapped onto the set of maximal ideals in  $C^*(X)$  which contain  $I^*$ .

3.5 Lemma.  $I^*$  is isomorphic to the set of continuous real-valued functions on  $\mu(\tau I)$  which vanish at  $I$ .

**Proof.** Let  $M_I$  denote  $\{g \in C(\mu(\tau I)) \mid g(I) = 0\}$  and for each  $f \in I^*$ , let  $\bar{f}$  be the function on  $\mu(\tau I)$  defined by  $\bar{f}(M \cap \tau I) = \hat{f}(M)$  for  $M \not\supseteq I$  and  $\bar{f}(I) = 0$ . By virtue of the homeomorphism of  $\mu(\tau I)$  into  $\beta X$ , it follows that  $\bar{f}$  is continuous at  $M \cap \tau I$  for any  $M \not\supseteq I$ . To see that  $\bar{f}$  is continuous at  $I$ , consider  $\epsilon > 0$ . Let  $W = \hat{f}^{-1}(-\epsilon, \epsilon)$ , an open set in  $\beta X$ , and hence  $W = \sim\{M \mid M \supseteq J\}$  for some ideal  $J$  in  $C(X)$ . For any  $M \supseteq I$ ,  $\hat{f}(M) = f^*(M) = 0$  by [4, 5.1], from which it follows that there exist  $i \in I$  and  $j \in J$  with  $i + j = 1$ . Consider  $U = \sim\{K \in \mu(\tau I) \mid j \in K\}$ , a neighborhood of  $I$  in  $\mu(\tau I)$ . Clearly, for any  $M \cap (\tau I) \in U$ ,  $M \in W$ , and hence  $\bar{f}(M) \in (-\epsilon, \epsilon)$ .

For any  $g \in M_I$ , define  $\hat{f}: \beta X \rightarrow R$  by  $\hat{f}(M) = g(M \cap \tau I)$  for  $M \not\supseteq I$  and  $\hat{f}(M) = 0$  for  $M \supseteq I$ . Then  $f \in I^*$  and  $\bar{f} = g$ . ( $f$  is the restriction of  $\hat{f}$  to  $X$ .)

Using the fact that  $\hat{\phantom{x}}$  is an isomorphism of  $C^*(X)$  onto  $C(\beta X)$ , it follows easily that the mapping  $f \rightarrow \bar{f}$  is an isomorphism of  $I^*$  onto  $M_I$ .

3.6 Theorem.  $\mu I \cong \mu J$  if and only if  $I^* \approx J^*$ .

**Proof.** If  $\mu I \cong \mu J$ , then their one-point compactifications are homeomorphic. Thus,  $\mu(\tau I) \cong \mu(\tau J)$ , and this homeomorphism takes the point  $I$  onto the point  $J$ . It follows that  $M_I$  and  $M_J$  (as in the notation of the proof of 3.5) are isomorphic; whence  $I^* \approx J^*$  by 3.5.

Conversely if  $I^* \approx J^*$ , then  $\mu(I^*) \cong \mu(J^*)$ , which, by 3.4, yields the required result.

3.7 Corollary. If  $I^\mu \approx J^\mu$ , then  $I^* \approx J^*$ .

**Proof.** By 2.11, the hypothesis implies that  $\mu I \cong \mu J$ . The result then follows from 3.6 above.

3.8 Corollary.  $\mu I \cong \mu J$  if and only if  $[F(I)]^* \approx [F(J)]^*$ .

**Proof.** Evidently the mapping  $I^\mu \rightarrow F(I)$  described in §5 of [4] preserves bounded functions, and so does its inverse. Thus  $I^* \approx [F(I)]^*$  and the result follows by 3.6.

3.9 Remark. It is possible for  $\mu I \cong \mu J$  without  $I^\mu \approx J^\mu$ . For example, let  $X$  be any realcompact space which is not  $\beta$ -fixed (see [4, 6.12]) and let  $H$  be a



homeomorphism of  $\beta X$  onto itself which takes a real ideal  $M_1$  onto a hyperreal ideal  $M_2$ . Then  $[\beta X \setminus \{M_1\}] \cong [\beta X \setminus \{M_2\}]$ , i.e.  $\mu M_1 \cong \mu M_2$ , but certainly  $(M_1)^\mu$  and  $(M_2)^\mu$  are not isomorphic.

**3.10 Remark.** Theorem 3.6 tells us that the structure space of  $I$  in general tells us very little about the algebraic properties of the ideal  $I$ , since vastly different ideals can have the same uniform closures. Even with real ideals, homeomorphism of structure spaces does not necessarily imply isomorphism of the ideals. For example, consider  $M = \{f \in C(N) \mid f(1) = 0\}$  and  $M' = \{f \in C(\Sigma) \mid f(1) = 0\}$  where  $N$  is the space of natural numbers and  $\Sigma$  is the space of Exercise 4M in [1]. Then  $\mu M$  and  $\mu M'$  are homeomorphic (since  $\beta N \cong \beta \Sigma$ ) but certainly  $M$  and  $M'$  cannot be isomorphic (since  $C(N)$  and  $C(\Sigma)$  are not isomorphic).

Homeomorphism of real structure spaces of ideals does not seem to tell much about the algebraic structure of the ideals either. For example, let  $Q$  denote the space of rationals and let  $X$  denote the rationals with  $\{0\}$  open. Then  $M = \{f \in C(Q) \mid f(0) = 0\}$  and  $M' = \{f \in C(X) \mid f(0) = 0\}$  have the same structure space of real ideals.

**4. Algebras on  $X$ .** In [2], the author defines an *algebra on  $X$*  to be a subalgebra  $A$  of  $C(X)$  with the following properties: (i)  $A$  contains the constant functions. (ii)  $A$  is uniformly closed. (iii) If  $F$  is closed in  $X$  and  $x \in X \setminus F$ , then there exists  $f \in A$  so that  $f(x) \neq 0$  and  $f(F) = 0$ . ( $A$  separates points and closed sets.) (iv) If  $f \in A$  with  $Z(f) = \emptyset$ , then  $1/f \in A$ . ( $A$  is closed under inversion.)

Of the four algebras considered here  $(I, I^\mu, rI, rI^\mu)$ , none will be an algebra on  $X$  in general. We now discuss in a series of remarks and examples these algebras with respect to properties (iii) and (iv) above, since the first two properties involve only trivial considerations.

**4.1 Remark.**  $rI$  is closed under inversion in  $C(X)$ .

**Proof.** Let  $f + c \in rI$  ( $f \in I$  and  $c \in R$ ) with  $Z(f + c) = \emptyset$ . Clearly  $c \neq 0$ , and we have  $1/(f + c) = g + 1/c$  where  $g = ((-1/c)/(f + c)) \cdot f \in I$ .

We note that the above remark generalizes [5, 3.2].

In general,  $r(I^\mu)$  is not closed under inversion in  $C(X)$ . This may seem somewhat surprising in view of the fact that  $r(I^\mu)$  is (isomorphic to) a ring of continuous functions.

**4.2 Example.** Let  $M$  be a hyperreal maximal ideal in  $C(X)$ . Then there exists a unit  $b \in M^\mu$  [1, 7.9(b)]. If  $1/b \in r(M^\mu)$ , we would have  $1/b = m + c$ ,  $m \in M^\mu$  and  $c \in R$ , whence  $1 = b \cdot m + b \cdot c \in M^\mu$ , a contradiction.

**4.3 Remark.** An ideal  $I$  determines the topology on  $X$  (separates points and closed sets) if and only if  $I$  is a free ideal.

**Proof.** Assume  $I$  is a free ideal, and let  $K$  be closed in  $X$  with  $x \in X \setminus K$ . There exists  $f \in I$  with  $f(x) \neq 0$  and there exists  $g \in C(X)$  with  $g(x) \neq 0$  and  $g(K) = 0$ . Thus  $f \cdot g \in I$  which separates  $x$  from  $K$ . If  $I$  is a fixed ideal, then no point in  $\Delta I$  can be separated from a closed set not containing that point.

The algebra  $\tau I$  clearly cannot determine the topology on  $X$  if  $\Delta I$  has at least two points, and  $\tau I$  does determine the topology on  $X$  if  $\Delta I = \emptyset$ . If  $\Delta I$  consists of precisely one point, then  $\tau I$  may or may not determine the topology.

**4.4 Example.** Let  $X$  be the real line and  $I = \{f \in C(X) \mid f(0) = 0 \text{ and } f \text{ is eventually zero}\}$ . Let  $K$  be the closed set  $\{1, 2, 3, 4, \dots\}$ . Then  $0 \notin K$ , but no function in  $\tau I$  can separate 0 from  $K$ .

**4.5 Remark.** The isomorphic image  $\tau \bar{I}$  of  $\tau I$  under the natural isomorphism of  $\tau(I^u)$  onto  $C(X(I))$  (see [4, 5.4]) determines the topology on  $X(I)$ .

**Proof.** Since  $\Delta \bar{I} = \{F\}$ , we have the ambiguous case described above. Let  $x \in X(I)$  and  $U$  an open set in  $X(I)$ , with  $x \in U$ . If  $x$  is different from the point  $F$ , then  $x \in X \setminus (F \cap X)$ , and hence there is an  $f \in I$  with  $f(x) \neq 0$ . Let  $g \in C(X(I))$  with  $g(x) \neq 0$  and  $g(\sim U) = 0$ . Then  $\bar{f} \cdot g$  separates  $x$  from  $U$ . If  $x$  is  $F$  itself, then  $U = \{F\} \cup [V \cap (X \setminus F)]$  where  $V$  is open in  $\beta X$  and  $V \supseteq F$ . By the normality of  $\beta X$ , we can find an open set  $W$  and a function  $\hat{h} \in C(\beta X)$  so that  $F \subseteq W \subseteq \text{cl}_{\beta X} W \subseteq V$  and  $\hat{h}(W) = 0$  and  $\hat{h}(\sim V) = 1$ . Also, there exists  $\hat{k} \in C(\beta X)$  with  $\hat{k}(F) = 0$  and  $\hat{k}(W) = 1$ , and hence  $\hat{h} = \hat{h} \cdot \hat{k}$ . As usual we denote the restrictions of  $\hat{h}$  and  $\hat{k}$  to  $X$  by  $h$  and  $k$  respectively. We consider the functions  $\bar{h}, \bar{k}$  on the space  $X(I)$ . It is easily seen that  $\bar{h} = \bar{h} \cdot \bar{k}$  and that  $\bar{h}(F) = 0$ . Hence  $\bar{h} \in mF(I)$ , whence  $h \in I$ . Since  $\bar{h}(F) = 0$  and  $\bar{h}(\sim U) = 1$ , the result is established.

**4.6 Remark.**  $\tau \bar{I}$  is closed under inversion in  $C(X(I))$ . (The proof is the same as Remark 4.1.) If  $\tau \bar{I}$  is uniformly closed, then  $\tau \bar{I} = C(X(I))$  by 2.13, and only in this case can  $\tau \bar{I}$  be an algebra on  $X(I)$ .

If  $A$  is an algebra on  $X$ , then the structure space of maximal ideals  $\mu A$  (denoted by  $H(A^*)$  in [2] and [3]) is a compactification of  $X$ . Among its properties are

- (i)  $A \cap C^*(X) = C(\mu A)$ ;
- (ii)  $A = E(\mu A)$  = the ring of continuous functions from  $\mu A$  into the two-point compactifications of the reals. (See [2, 1.2].)

Of course, results such as the above do not hold in general for the algebras considered here.

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