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λ CONNECTED PLANE CONTINUA

BY

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ABSTRACT. A continuum M is said to be λ connected if any two distinct points of M can be joined by a hereditarily decomposable continuum in M. Recently this generalization of arcwise connectivity has been related to fixed point problems in the plane. In particular, it is known that every λ connected nonseparating plane continuum has the fixed point property. The importance of arcwise connectivity is, to a considerable extent, due to the fact that it is a continuous invariant. To show that λ connectivity has a similar feature is the primary purpose of this paper. Here it is proved that if M is a λ connected continuum and f is a continuous function of M into the plane, then $f(M)$ is λ connected. It is also proved that every semiaposyndetic plane continuum is λ connected.

Introduction. A nondegenerate metric space that is both compact and connected is called a *continuum*. It is known that every plane continuum that has a hereditarily decomposable boundary and does not separate the plane has the fixed point property [1]. Recently the author [4] proved that every arcwise connected nonseparating plane continuum has a hereditarily decomposable boundary. Hence all arcwise connected nonseparating plane continua have the fixed point property. In [7] it is pointed out that the author's theorem remains true if the word "arcwise" is replaced by "λ". In fact, in [7] it is proved that a plane continuum that does not have infinitely many complementary domains is λ connected if and only if its boundary does not contain an indecomposable continuum.

This paper is primarily concerned with the following questions:

(1) What other theorems about arcwise connected continua also hold for λ connected continua?

(2) Are there general properties, other than arcwise connectivity for plane continua, that imply λ connectivity?

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Here, in response to the first question, we prove that every planar continuous image of a λ connected continuum is λ connected.

To deal with the second question, we must consider general properties that do not imply arcwise connectivity. Examples have been given [13, Example 4], [10, Example 6], and [3, Example 1], which indicate that, for plane continua, arcwise connectivity is not a consequence of the aposyndetic property defined by F. Burton Jones. Jones' property can be generalized as follows.

Definition. A continuum M is said to be *semiaposyndetic* if for each pair of distinct points x and y in M there exists a subcontinuum F of M such that the sets $M - F$ and the interior of F relative to M each contain a point of $\{x, y\}$.

Although not all semiaposyndetic plane continua are arcwise connected, these properties are related. If X is a semiaposyndetic plane continuum and for any positive real number ϵ there are at most a finite number of complementary domains of X of diameter greater than ϵ, then X is arcwise connected [6]. For another arc theorem involving semiaposyndesis see [5]. In the last section of this paper it is proved that all semiaposyndetic plane continua are λ connected.

Throughout this paper S^2 is a 2-sphere. The closure and the boundary of a given set Z are denoted by $\mathrm{Cl}\, Z$ and $\mathrm{Bd}\, Z$ respectively. The union of the elements of Z is denoted by $\mathrm{St}\, Z$.

Definitions. Let X be a continuum in S^2. A continuum L in X is said to be a *link* in X if L is either the boundary of a complementary domain of X or the limit of a convergent sequence of complementary domains of X. A continuum T in X is said to be a *2-link* in X if T is the union of two (not necessarily distinct) links in X. An indecomposable subcontinuum I of X is said to be *terminal* in X if there exists a composant C of I such that each subcontinuum of X that meets both C and $X - I$ contains I.

Preliminary results.

Theorem 1. *If X is a continuum in S^2 and I is an indecomposable subcontinuum of X that is contained in the union of finitely many links in X, then every subcontinuum of X that contains a nonempty open subset of I contains I.*

Proof. Assume I is contained in the union of α (α is a natural number) links in X. Suppose Z is a collection of $\alpha + 1$ disjoint circular regions in S^2 such that each element of Z intersects I. There exist points c and d belonging to distinct elements of Z such that $\{c, d\}$ is contained in a complementary domain of X. The theorem now follows directly from [4, Theorem 1 (proof)].

Theorem 2. *Suppose M is a hereditarily decomposable continuum and f is a continuous function of M into S^2. Then no indecomposable subcontinuum of $f(M)$ is terminal in $f(M)$.*

Proof. Assume there exists an indecomposable subcontinuum I of $f(M)$ that is terminal in $f(M)$. There exists a composant C of I such that each subcontinuum of $f(M)$ that meets both C and $f(M)-I$ contains I. Note that since $f(M)$ is decomposable, I is a proper subcontinuum of $f(M)$. Let p be a point of $f^{-1}(C)$ and let Z be the p-component of $f^{-1}(I)$.

For each positive integer n, let G_n be an open set in M such that (1) $Z \subset G_n$, (2) $\mathrm{Bd}\, G_n \cap f^{-1}(I) = \emptyset$, (3) $M - G_n$ is not the empty set, and (4) the distance from each point of G_n to Z is less than n^{-1}. For each n, let Y_n be the p-component of $\mathrm{Cl}\, G_n$. Note that, for each n, the continuum Y_n meets $\mathrm{Bd}\, G_n$ [12, Theorem 50, p. 18] and consequently $f(Y_n)$ is a continuum in $f(M)$ that meets both C and $f(M)-I$. Hence for each n, the continuum $f(Y_n)$ contains I. It follows from the continuity of f that $f(Z)$ is I. Since Z is hereditarily decomposable, this is a contradiction. Hence no indecomposable subcontinuum of $f(M)$ is terminal in $f(M)$.

Theorem 3. *If X is a continuum in S^2 and no finite collection of links in X contains an indecomposable continuum in its union, then X is λ connected.*

Proof. Let p_1 and q_1 be distinct points of X. We shall construct a hereditarily decomposable continuum Q in X that contains p_1 and q_1.

Let A be an arc in S^2 from p_1 to q_1 ordered by $<$.

Notation. For points x and y of A $(x \leq y)$ let $[x, y]$, $[x, y)$, $(x, y]$, and (x, y) denote the sets $\{z \in A \mid x \leq z \leq y\}$, $\{z \in A \mid x \leq z < y\}$, $\{z \in A \mid x < z \leq y\}$, and $\{z \in A \mid x < z < y\}$ respectively.

Let $\{G_n\}$ be the countable collection consisting of all complementary domains of X. If no element of $\{G_n\}$ intersects A, then A is in X and the construction is trivial. Assume this is not the case. Suppose without loss of generality that $G_1 \cap A \neq \emptyset$.

1. **The construction of a continuum Z_1 in $(X \cup A) - G_1$ from p_1 to q_1.** Let H_0 be a link in X whose intersection with A is p_1 (if no such link exists let H_0 be $\emptyset$). Let L_0 be a link in X that meets A only at q_1 (if no such link exists let L_0 be $\emptyset$). The links H_0 and L_0 will be referred to as selected 2-links.

With H_0, L_0, the arc $[p_1, q_1]$, and the complementary domain G_1 as sets of reference, we now construct either a continuum K_1 or a set T_1 as follows.

If $H_0 \cap L_0 \neq \emptyset$, define $K_1 = H_0 \cup L_0$.

Assume that H_0 and L_0 are disjoint sets. Let x_1 and y_1 be the first and last points respectively of $[p_1, q_1] \cap \mathrm{Bd}\, G_1$ (with respect to the order of A). Let W_1 be the collection consisting of all links in X that meet $[p_1, x_1] \cup [y_1, q_1]$. Let V_1 be the collection of all 2-links L in X such that L is the union of two (not necessarily distinct) elements of W_1.

If L is an element of V_1 that is the union of elements H and K (not necessarily distinct) of W_1, then we call any point set $\{x, y\}$ such that x belongs to $H \cap ([p_1, x_1] \cup [y_1, q_1])$ and y belongs to $K \cap ([p_1, x_1] \cup [y_1, q_1])$ a *basic set* in L.

Suppose no element of V_1 meets both $[p_1, x_1) \cup H_0$ and $[y_1, q_1] \cup L_0$ and no element of V_1 meets both $[p_1, x_1] \cup H_0$ and $(y_1, q_1] \cup L_0$. Define

$$K_1 = [p_1, x_1] \cup \text{Bd } G_1 \cup [y_1, q_1].$$

In the last part of this proof, Bd G_1 will be referred to as a selected 2-link.

Assume that V_1 does not have this property.

Suppose there exists an element E of V_1 in $X - \{p_1, q_1\}$ that meets both H_0 and L_0. Define $K_1 = H_0 \cup E \cup L_0$. Later E will be referred to as a selected 2-link.

Assume no element of V_1 misses $\{p_1, q_1\}$ and meets both H_0 and L_0.

Suppose there exists an element of V_1 that meets both H_0 and $[y_1, q_1]$. Let d be the least upper bound (with respect to $<$) of the set of points $[y_1, q_1]$ that can be joined to H_0 by an element of V_1. Since the limit of a convergent sequence of links is a link, there is an element of V_1 that joins d to H_0 [12, Theorem 59, p. 24]. If possible, select an element H of V_1 joining d to p_1 and define $K_1 = H \cup [d, q_1]$. If no such element exists, select an element E of V_1 missing p_1 that joins d to H_0 and define $K_1 = H_0 \cup E \cup [d, q_1]$.

Assume that no element of V_1 meets both H_0 and $[y_1, q_1]$.

Suppose an element of V_1 meets L_0 and $[p_1, x_1]$. Let a be the first point of $[p_1, x_1]$ that can be joined to L_0 by an element of V_1. If possible, select an element L of V_1 that joins a to q_1 and define $K_1 = [p_1, a] \cup L$. If no such element exists, select an element E of V_1 missing q_1 that joins a to L_0 and define $K_1 = [p_1, a] \cup E \cup L_0$.

Assume that no element of V_1 meets both L_0 and $[p_1, x_1]$. Let a_1 be the first point of $[p_1, x_1]$ that can be joined to $[y_1, q_1]$ by an element of V_1. There exist a continuum L_1 and a point q_2 of $[y_1, q_1]$ such that (1) L_1 is a selected element of V_1 that joins q_2 to a_1, and (2) q_2 is the first point of $[y_1, q_1]$ that can be joined to a_1 by an element of V_1.

Let d_1 be the last point of $[y_1, q_1]$ that can be joined to $[p_1, x_1]$ by an element of V_1. There exist a continuum H_1 and a point p_2 such that (1) H_1 is a selected element of V_1 that joins p_2 to d_1, and (2) p_2 is the last point of $[p_1, x_1]$ that can be so joined to d_1.

Suppose an element E of V_1 contains $\{a_1, d_1\}$. Define $K_1 = [p_1, a_1] \cup E \cup [d_1, q_1]$.

Assume $\{a_1, d_1\}$ is not contained in an element of V_1.

Suppose there exists an element E of V_1 that does not contain p_1 and meets both H_0 and H_1. Define $K_1 = H_0 \cup E \cup H_1 \cup [d_1, q_1]$.

Assume that each element of V_1 that meets both H_0 and H_1 contains p_1.

Suppose an element E of V_1 in $X - \{q_1\}$ meets L_0 and L_1. Define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup L_0$.

Assume that no element of V_1 in $X - \{q_1\}$ meets both L_0 and L_1.

Suppose an element of V_1 meets $[p_1, a_1]$ and H_1. Let a be the first point of $[p_1, a_1]$ that can be joined to H_1 by an element of V_1. Select an element E of V_1 that joins a to H_1. Define $K_1 = [p_1, a] \cup E \cup H_1 \cup [d_1, q_1]$.

Assume that no element of V_1 meets both $[p_1, a_1]$ and H_1.

Suppose an element of V_1 meets L_1 and $[d_1, q_1]$. Let d be the last point of $[d_1, q_1]$ that can be joined to L_1 by an element of V_1. Select an element E of V_1 that joins d to L_1. Define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup [d, q_1]$.

Assume that no element of V_1 meets both L_1 and $[d_1, q_1]$.

Suppose an element of V_1 meets H_0 and $[p_2, x_1]$. Let a be the first point of $[p_2, x_1]$ that can be joined to H_0 by an element of V_1. If possible, select an element H of V_1 that joins a to p_1 and define $K_1 = H \cup [p_2, a] \cup H_1 \cup [d_1, q_1]$. If no such element exists, select an element E of V_1 missing p_1 that joins a to H_0 and define $K_1 = H_0 \cup E \cup [p_2, a] \cup H_1 \cup [d_1, q_1]$.

Assume that no element of V_1 meets both H_0 and $[p_2, x_1]$.

Suppose an element of V_1 meets $[y_1, q_2]$ and L_0. Let d be the last point of $[y_1, q_2]$ that can be joined to L_0 by an element of V_1. If possible, select an element L of V_1 that joins d to q_1 and define $K_1 = [p_1, a_1] \cup L_1 \cup [d, q_2] \cup L$. If no such element exists, select an element E of V_1 missing q_1 that joins d to L_0 and define $K_1 = [p_1, a_1] \cup L_1 \cup [d, q_2] \cup E \cup L_0$.

Assume that no element of V_1 meets both $[y_1, q_2]$ and L_0.

Suppose an element of V_1 meets $[p_1, a_1]$ and $[p_2, x_1]$. Let a be the first point of $[p_1, a_1]$ that can be joined to $[p_2, x_1]$ by an element of V_1. There exist a continuum E and a point b such that (1) E is a selected element of V_1 that joins b to a, and (2) b is the first point of $[p_2, x_1]$ that can be so joined to a. Define $K_1 = [p_1, a] \cup E \cup [p_2, b] \cup H_1 \cup [d_1, q_1]$.

Assume no element of V_1 meets both $[p_1, a_1]$ and $[p_2, x_1]$.

Suppose an element of V_1 meets $[y_1, q_2]$ and $[d_1, q_1]$. Let d be the last point of $[d_1, q_1]$ that can be joined to $[y_1, q_2]$ by an element of V_1. There exist a continuum E and a point c such that (1) E is a selected element of V_1 that joins c to d, and (2) c is the last point of $[y_1, q_2]$ that can be so joined to d. Define $K_1 = [p_1, a_1] \cup L_1 \cup [c, q_2] \cup E \cup [d, q_1]$.

Assume no element of V_1 meets both $[y_1, q_2]$ and $[d_1, q_1]$.

If an element E of V_1 can be selected that meets L_1 and H_1, define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup H_1 \cup [d_1, q_1]$.

Assume no element of V_1 meets both L_1 and H_1.

Suppose an element of V_1 meets L_1 and $[p_2, x_1]$. Let a be the first point of $[p_2, x_1]$ that can be joined to L_1 by an element of V_1. Let E be a selected element of V_1 that joins a to L_1. Define $K_1 = [p_1, a_1] \cup L_1 \cup E \cup [p_2, a] \cup H_1 \cup [d_1, q_1]$.

Assume no element of V_1 meets both L_1 and $[p_2, x_1]$.

Suppose an element of V_1 meets $[y_1, q_2]$ and H_1. Let d be the last point of $[y_1, q_2]$ that can be joined to H_1 by an element of V_1. Let E be a selected element of V_1 that joins d to H_1. Define $K_1 = [p_1, a_1] \cup L_1 \cup [d, q_2] \cup E \cup H_1 \cup [d_1, q_1]$.

Assume no element of V_1 meets both $[y_1, q_2]$ and H_1. Define $T_1 = [p_1, a_1] \cup L_1 \cup H_1 \cup [d_1, q_1]$.

If at this stage of the construction K_1 is defined, then let $Z_1 = K_1$. If K_1 is not defined, then repeat the process, using the continua H_1, L_1, the arc $[p_2, q_2]$ and the complementary domain G_1 as reference sets to define K_2 or T_2. Note that no 2-link in X meets two distinct elements of $\{[p_1, a_1], (p_2, q_2), [d_1, q_1]\}$. Hence the resulting set, K_2 or T_2, meets T_1 only at the points p_2 and q_2. If K_2 is defined, let $Z_1 = K_2 \cup T_1$. If K_2 is not defined, repeat the process using H_2, L_2, $[p_3, q_3]$, and G_1 as reference sets.

If after this process is repeated n (finitely many) times K_n is defined, then let $Z_1 = K_n \cup \bigcup_{i=1}^{n-1} T_i$. If, for each positive integer n, the set T_n is defined, let $Z_1 = \mathrm{Cl} \bigcup_{i=1}^{\infty} T_i$. Note that Z_1 is a continuum in $S^2 - G_1$ that contains $\{p_1, q_1\}$.

2. **The properties of Z_1.** Let M_1 be the collection consisting of all selected 2-links that appear in the definition of Z_1. The continua that appear in the definition of Z_1 have the *linear separating property*; that is, for any two distinct continua L and E in $S^2 - \{p_1\}$ appearing in the definition of Z_1 (each of L and E is either a selected 2-link or an arc in $A \cap Z_1$ that is maximal with respect to not having an interior point in common with $\mathrm{Cl}(\mathrm{St}\, M_1)$), either L separates p_1 from $E - L$ or E separates p_1 from $L - E$ in Z_1 but not both. Note that only one continuum appearing in the definition of Z_1 contains p_1.

Note that Z_1 has the *linear linking condition* with respect to M_1; that is, no 2-link in X meets more than one component of $Z_1 - \mathrm{Cl}(\mathrm{St}\, M_1)$ and if J is a component of $Z_1 - \mathrm{Cl}(\mathrm{St}\, M_1)$ and r is an endpoint of J, there is at most one point v that is an endpoint of a component of $Z_1 - (J \cup \mathrm{Cl}(\mathrm{St}\, M_1))$ that can be joined to r by a 2-link in X.

3. **The second phase of the construction of Q.** If no element of $\{G_n\}$ intersects Z_1, define $Q = Z_1$.

Assume that some element of $\{G_n\}$ meets Z_1. Define $Q_1 = Z_1$. Suppose without loss of generality that $G_2 \cap Q_1 \neq \emptyset$. Since no 2-link in X meets more than one component of $Q_1 - \mathrm{Cl}(\mathrm{St}\, M_1)$, the domain G_2 intersects only one component (p_1^2, q_1^2) of $Q_1 - (\mathrm{Cl}(\mathrm{St}\, M_1) \cup \{p_1, q_1\})$. Using the construction methods of §1, define a continuum Z_2 such that $Q_2 = (Q_1 - (p_1^2, q_1^2)) \cup Z_2$ is a continuum in $S^2 - (G_1 \cup G_2)$ that contains $\{p_1, q_1\}$. Here we start with two selected 2-links in Z_1, the arc $[p_1^2, q_1^2]$, and the domain G_2 as sets of reference for the construction of Z_2. Note that each basic set in any 2-link that is selected in the construction of Z_2 is by definition in $[p_1^2, q_1^2] - G_2$.

Assume without loss of generality that the continua used to define Q_2 (i.e., all continua except $[p_1^2, q_1^2]$ appearing in the definitions of Q_1 and Z_2) have the linear separating property. Note that it may be necessary to delete an element of M_1 in the definition of Q_2 to get this property.

Let M_2 be the collection of selected 2-links used to define Q_2. The continuum Q_2 has the linear linking condition with respect to M_2.

4. **Continuing until no complementary domain meets the constructed set.** If no element of $\{G_n\}$ meets Q_2, define $Q = Q_2$. If this is not the case, we continue the process.

Assume the continuum Q_n is defined and M_n is the collection of all selected 2-links in Q_n. If Q_n is contained in X, define $Q = Q_n$. Assume Q_n intersects some element of $\{G_n\}$. There exists an integer i greater than n such that $Q_n \cap G_i \neq \emptyset$ and $Q_n \cap \bigcup_{j=1}^{i-1} G_j = \emptyset$. The domain G_i intersects only one component (p_1^{n+1}, q_1^{n+1}) of $Q_n - (\mathrm{Cl}(\mathrm{St}\, M_n) \cup \{p_1, q_1\})$. Using the construction methods of §1, define Z_{n+1} such that $Q_{n+1} = (Q_n - (p_1^{n+1}, q_1^{n+1})) \cup Z_{n+1}$ is a continuum in $S^2 - \bigcup_{j=1}^{i} G_j$ containing $\{p_1, q_1\}$. We can assume without loss of generality that the continua used to define Q_{n+1} have the linear separating property. The continuum Q_{n+1} has the linear linking condition with respect to M_{n+1} the collection of all selected 2-links used to define Q_{n+1}.

Suppose that for each positive integer n, a continuum Q_n is defined. Define Q to be the limit of $\{Q_n\}$.

Since in all cases Q is a continuum in X that contains $\{p_1, q_1\}$, the construction is complete. Now we must show that Q is hereditarily decomposable.

5. **Derived subcontinua of Q.** Suppose there exist an arc $[r, v]$ in A and links F_1, F_2, and F_3 contained in $X - [r, v)$ such that for each $i = 1, 2$, and 3, F_i is the limit of a convergent sequence $\{E_n^i\}$ of links in $X - \bigcup_{j=1}^{3} F_j$ with the following properties:

(1) For each n, the link E_n^i is contained in a selected 2-link L and a point v_n^i of a basic set in L is in $[r, v) \cap E_n^i$.

(2) The sequence $\{v_n^i\}$ converges to v.

We now show that one of F_1, F_2, and F_3 is contained in the union of the other two.

Assume that for $j = 1$, 2, and 3, there exists a point w_j of $F_j - \bigcup_{i \neq j} F_i$. There exist a point s of (r, v), a positive integer m, an element E of $\{E_n^3\}$, and circular regions U_1, U_2, and U_3 whose closures are disjoint containing w_1, w_2, and w_3 respectively such that (1) $\mathrm{Cl}(U_1 \cup U_2 \cup U_3)$ is in $S^2 - [r, v]$, (2) E meets both $[r, s)$ and U_3 and does not intersect $\mathrm{Cl}(U_1 \cup U_2)$, and (3) for each n greater than m,

$$(F_1 \cup E_n^1) \cap \mathrm{Cl}(U_2 \cup U_3) = (F_2 \cup E_n^2) \cap \mathrm{Cl}(U_1 \cup U_3)$$

$$= (F_3 \cup E_n^3) \cap \mathrm{Cl}(U_1 \cup U_2) = \varnothing.$$

Let X_1 be a continuum in $S^2 - E$ that is the union of an arc in (s, v), an element of $\{E_n^1\}$ in $X - \mathrm{Cl}(U_2 \cup U_3)$ that meets both (s, v) and U_1, and an element of $\{E_n^2\}$ in $X - \mathrm{Cl}(U_1 \cup U_3)$ that meets both (s, v) and U_2.

There exists a continuum X_2 in $S^2 - (E \cup X_1)$ that is the union of an arc in (s, v), an element of $\{E_n^1\}$ in $X - \mathrm{Cl}(U_2 \cup U_3)$ that meets both (s, v) and U_1, and an element of $\{E_n^2\}$ in $X - \mathrm{Cl}(U_1 \cup U_3)$ that meets both (s, v) and U_2.

There exists a point t of (s, v) such that no element of $\{E_n^3\}$ meets both (t, v) and $X_1 \cup X_2$. Let T be the continuum $F_1 \cup F_2 \cup X_1 \cup X_2 \cup \mathrm{Cl}(U_1 \cup U_2)$. There is an element H of $\{E_n^3\}$ in $S^2 - T$ that meets both (t, v) and U_3. The disjoint continua $F_1 \cup F_2$, X_1, and X_2 can be closely approximated by arcs in such a way that the existence of the continuum $E \cup H \cup \mathrm{Cl}\, U_3$ contradicts [12, Theorem 116, p. 247]. Hence one of F_1, F_2, and F_3 is contained in the union of the other two.

It follows that the collection of all links having the properties that F_1, F_2, and F_3 have with respect to $[r, v]$ when partially ordered by inclusion has at most two maximal elements. Hence the union of this collection is a 2-link F in X. Since r precedes v with respect to the order of A, we shall refer to F as a continuum in Q that is *derived from the left at* v. Similarly continua that are derived from the right at a point of A are 2-links in X. A continuum that is derived from the left or from the right at a point z of A will be referred to as a *derived continuum* in Q with *base point* z.

6. Definition of the collection C. If E is an element of M_i, for some i, that is not deleted at a later stage of the construction to get the linear separating property or if E is an arc in $A \cap Q$ that is maximal with respect to not having an interior point in common with $\bigcup \mathrm{Cl}(\mathrm{St}\, M_i)$, then E is said to be a continuum used to define Q. Let B denote the collection of all continua used to define Q. Let D denote the collection consisting of all derived continua in Q.

Define C to be the collection consisting of all continua Y that satisfy one of the following conditions:

(1) Y is the union of an element E of B with all elements of D that have their base points in E.

(2) There is an element F of D that does not have a base point in an element of B. The union of F with all elements of D that meet F is Y.

Note that $Q - \mathrm{St}\, C$ is a totally disconnected set in A.

For each continuum E belonging to B, there are at most two elements of D that have base points in E. Furthermore, for each element F of D at most one element of B contains a base point of F. Since the continua at all stages of the construction have the linear linking condition, each element of D meets at most one other element of D. It follows that for each element Y of C, there exist points r and t (not necessarily distinct) of A and a collection N, consisting of finitely many links in X and possibly one arc in $A \cap Q$ that is maximal with respect to not having an interior point in $\bigcup \mathrm{Cl}(\mathrm{St}\, M_i)$, such that each element of N meets $\{r, t\}$ and the union of N is Y. The set $\{r, t\}$ will be referred to as the *core set* of Y.

7. **An order relation in C in terms of separating.** Define the binary relation $\ll$ in C as follows. For distinct elements Y and Z of C, $Y \ll Z$ if Y contains p_1 or Y separates p_1 from the set $Z - Y$ in Q. Since the continua at all stages of the construction have the linear linking condition and the sets used to define these continua have the linear separating property, $\ll$ is an order relation in C (i.e., $\ll$ is transitive and satisfies the law of trichotomy). When checking the properties of $\ll$, it is helpful to note that for any two elements Y and Z of C, if $Y \ll Z$, then the p_1-component of $Q - Z$ meets $Y - Z$.

Note that if V, Y, and Z are elements of C such that $V \ll Y$ and $Y \ll Z$, then V and Z are mutually exclusive.

8. **All continua in Q are decomposable.** Suppose there exists an indecomposable continuum I in Q. Since I is not contained in the union of finitely many links and arcs in X and $Q - \mathrm{St}\, C$ is totally disconnected, there exist mutually exclusive elements V, Y, and Z of C such that $V \ll Y$, $Y \ll Z$, and each composant of I meets both V and Z. Note that since I is not separated by any of its proper subcontinua, Y cannot be an arc in A.

Suppose that the core set of Y consists of two distinct points r and t. The continuum Y separates V from Z in Q. Hence there exist six disjoint continua $\{P_i\}_{i=1}^6$ in $I - \{r, t\}$, each intersecting V, Z, and a link in X that meets $\{r, t\}$. Let R and U be disjoint circular regions in $S^2 - (V \cup Z \cup \bigcup_{i=1}^6 P_i)$ that are centered on r and t respectively. For each integer i $(1 \leq i \leq 6)$, let e_i be a point of P_i that can be joined to either r or t by a link in X and let R_i be a circular region centered on e_i in $S^2 - (V \cup Z \cup \bigcup_{j \neq i} P_j)$. For each i, since e_i

belongs to a link in X that meets $\{r, t\}$, there exists an arc J_i in a complementary domain of X that meets both R_i and a component of $R \cup U$.

It follows that some three of $\{J_i\}_{i=1}^{6}$ meet one component of $R \cup U$. Assume without loss of generality that J_1, J_2, and J_3 each meet R. The continua V, Z, P_1, P_2, and P_3 can be closely approximated by arcs in such a way that the existence of J_1, J_2, and J_3 contradicts [12, Theorem 116, p. 247].

Using an argument similar to the preceding, we can show that assuming the core set of Y consists of one point also involves a contradiction. Hence Q is hereditarily decomposable.

Continuity.

Theorem 4. *Suppose M is a hereditarily decomposable continuum, f is a continuous function of M into the plane, and R is the union of finitely many links in $f(M)$. Then R does not contain an indecomposable continuum.*

Proof. Assume there exists an indecomposable continuum I in R. According to Theorem 1, every subcontinuum of $f(M)$ that contains a nonempty open subset of I contains I. It follows that I is terminal in $f(M)$ [8, Theorem 2]. Since M is hereditarily decomposable, this contradicts Theorem 2. Hence R does not contain an indecomposable continuum.

Theorem 5. *If M is a λ connected continuum and f is a continuous function of M into the plane, then $f(M)$ is λ connected.*

Proof. Suppose p and q are distinct points of $f(M)$. Since M is λ connected, there exists a hereditarily decomposable continuum H in M that meets both $f^{-1}(p)$ and $f^{-1}(q)$. According to Theorems 3 and 4, $f(H)$ is λ connected. Hence there exists a hereditarily decomposable continuum in $f(H)$ that joins p and q. It follows that $f(M)$ is λ connected.

Comment. Theorem 5 cannot be generalized to include all continuous images in Euclidean 3-space. There are continuous images of the topologist's sine curve in E^3 that are not λ connected.

Aposyndesis.

Theorem 6. *If X is a semiaposyndetic plane continuum, then X is λ connected.*

Proof. Assume there exists an indecomposable continuum I that is contained in the union of finitely many links in X. Let x and y be distinct points of I. According to Theorem 1, every subcontinuum of X that contains a point of $\{x, y\}$ in its interior relative to X contains I. This contradicts the assumption that X is semiaposyndetic. It follows from Theorem 3 that X is λ connected.

Corollary. *Every aposyndetic plane continuum is λ connected.*

Example. An aposyndetic continuum in Euclidean 3-space need not be λ connected. To see this let M be the product of the pseudo arc [11] with itself. M is aposyndetic [9, Theorem 7] and not λ connected. In fact, M does not contain a hereditarily decomposable continuum. Since the pseudo arc is chainable, M is embeddable in Euclidean 3-space [2].

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