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GENERALIZED ALMOST PERIODICITY IN GROUPS(1)

BY

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ABSTRACT. A module of almost periodic functions on a group is closed with respect to a quite general seminorm. The new space of functions is characterized in terms of the internal properties of its members. This yields new characterizations of Besicovitch and Weyl almost periodic functions in a variety of group-theoretic settings. Eberlein's theorem that weakly almost periodic functions on the real line are Weyl almost periodic is extended to locally compact groups.

1. Introduction. Let $\alpha(G)$ be a module of von Neumann AP (= almost periodic) functions on a group G. Using a transformation L we define a seminorm $\| \ \|$ with which we close $\alpha(G)$, obtaining a larger space $\alpha_L(G)$ (see 2.2). Our procedure extends to groups the Besicovitch-Bohr procedure of closing the trigonometric polynomials on the real line [4, Chapter 2]. Due to the abstract way L is defined, our results include a wide variety of Besicovitch and Weyl-like AP functions on groups (see 2.3).

The space $(\alpha_L(G), \| \ \|)$ is a seminormed linear space and L is an invariant mean on $\alpha_L(G)$. The main theorem of the paper (4.9) characterizes the functions $f \in \alpha_L(G)$ in terms of their internal properties. We obtain two conditions: condition (A) is a rather standard almost periodicity condition (slightly disguised, see 4.10, 4.11). Condition (B) is that $|f(xt) - f(x)|h(x)$ must satisfy a kind of weak Fubini theorem for certain $h \in \alpha(G)$. When applied to the classical Besicovitch and Weyl AP functions on the real line our results are new (see 4.12).

In §5 we give a Bohr-like expression for the mean value of weakly AP functions on locally compact topological groups. Using this we extend Eberlein's result that on the real line weakly AP functions are Weyl AP. In most nonabelian cases we only conclude that they are Besicovitch AP.

In §3 we state some facts about modules which are used in §4.

2. Notation, definitions and examples.

2.1 **Notation.** Let G be a group. e denotes the identity of G and $AP(G)$ is the set of complex-valued von Neumann AP functions on G. Mf is the mean value of $f \in AP(G)$ and $f \times g$ is the convolution of $f, g \in AP(G)$. We shall say $\mathcal{S} \subset AP(G)$

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is an *invariant vector space of AP functions* if $\mathcal{S} \neq \emptyset$ and

(i) $f, g \in \mathcal{S}$, α, β complex $\Rightarrow \alpha f + \beta g \in \mathcal{S}$,

(ii) $f \in \mathcal{S}$, $x, y \in G \Rightarrow {}_x f_y \in \mathcal{S}$, where we define ${}_x f_y(t) = f(xty)$.

Using the terminology of van Kampen [18], we say $\mathcal{S}$ is a *module* if in addition to (i), (ii) we have

(iii) $f, g \in \mathcal{S} \Rightarrow fg \in \mathcal{S}$ (fg denotes pointwise multiplication),

(iv) $f \in \mathcal{S} \Rightarrow \overline{f} \in \mathcal{S}$ ($\overline{f}$ = conjugate of f),

(v) $\mathcal{S}$ is *closed, i.e.*, $f_n \in \mathcal{S}$, $\|f_n - f\|_\infty \xrightarrow[n]{} 0 \Rightarrow f \in \mathcal{S}$.

Here f is a complex-valued function on G and $\| \ \|_\infty$ denotes the supremum norm. We say a module is *nontrivial* iff it contains a nonzero function.

If $(G, \mathfrak{T})$ is a topological group, $AP(G, \mathfrak{T})$ is the set of $\mathfrak{T}$-continuous members of $AP(G)$. If $(G, \mathfrak{T})$ is a locally compact T_0 topological group (= LC group), then μ denotes left Haar measure on G. $L_{1,\mathrm{loc}}(G)$ is the set of μ-measurable complex-valued functions f on G such that $\int_E f\, d\mu$ exists and is finite for all compact $E \subset G$.

Let R, C denote, respectively, the set of real and complex numbers. If $\mathfrak{F}$ is a set of complex-valued functions, $\mathfrak{F}^r$ denotes its real-valued members. The symmetric differences of two sets A, B is denoted $A \triangle B$. Let $f\colon G \to C$. We define

$$E_1(\epsilon, f) = \{x \in G\colon \|{}_x f - f\|_\infty < \epsilon\},$$

$$E_2(\epsilon, f) = \{x \in G\colon \|f_x - f\|_\infty < \epsilon\}.$$

$\mathfrak{R} f$ denotes the real part of f and, if f is real-valued, f^+, f^- denote its positive and negative parts. Finally, by an *ϵ-mesh* in a metric space is meant a finite set of points of the space such that each point of the space is within ϵ of some member of the finite set.

2.2 **Definition.** Let $\mathfrak{a}(G)$ be a nontrivial module of AP functions on G. Let $\mathfrak{E}$ be a set of complex-valued functions on G satisfying

(E1) $f, g \in \mathfrak{E}$, $\alpha, \beta \in C \Rightarrow \alpha f + \beta g \in \mathfrak{E}$;

(E2) $f \in \mathfrak{E}$, $x, y \in G \Rightarrow {}_x f_y \in \mathfrak{E}$;

(E3) $f \in \mathfrak{E} \Rightarrow \overline{f} \in \mathfrak{E}$;

(E4) $f \in \mathfrak{E} \Rightarrow |f| \in \mathfrak{E}$;

(E5) $\mathfrak{E} \supset \mathfrak{a}(G)$;

(E6) $f \in \mathfrak{E}$, $g \in \mathfrak{a}(G) \Rightarrow fg \in \mathfrak{E}$.

Let $\mathfrak{D}$ be a set of extended real-valued functions on G and let $L\colon \mathfrak{D} \to [-\infty, \infty]$. Assume

(D1) $\mathfrak{D} \supset \mathfrak{E}^r$.

(D2) $f \in \mathfrak{D} \Rightarrow |f| \in \mathfrak{D}$.

(D3) $f, g \in \mathfrak{D} \Rightarrow f + g \in \mathfrak{D}$.

(L1) $0 \leq f \in \mathfrak{E}^r$, $x, y \in G \Rightarrow L({}_x f_y) = Lf$.

(L2) $\lambda \geq 0$, $\lambda f, f \in \mathfrak{D} \Rightarrow L(\lambda f) = \lambda Lf$.

(L3) $|f|, -|f| \in \mathfrak{D} \Rightarrow -L(-|f|) \leq L|f|$.

(L4) $f, g \in \mathfrak{D}, f \leq g \Rightarrow Lf \leq Lg$ ($f \leq g$ means that $f(t) \leq g(t)$ for all $t \in G$).

(L5) $f, g \in \mathfrak{D} \Rightarrow L(f+g) \leq Lf + Lg$ whenever the right side is well defined, *i.e.*, is $\neq \infty - \infty$.

(L6) $f \in \alpha(G^r) \Rightarrow Lf = Mf$.

(DL) For every $f \in \mathfrak{S}$, $h \in \alpha(G)^r$ one has $L_x|f(xt) - f(t)|h(x) \in \mathfrak{D}$, as a function of t, provided that either $L|f| < \infty$ or $h \equiv 1$.

Here we use the notation $L_x(g(x)) = Lg$. Define $\|f\| = L|f|$ for all $f \in \mathfrak{S}$. Given such a system $\alpha(G)$, $\mathfrak{S}$, $\mathfrak{D}$, L we then define $\alpha_L(G)$ to be the set of all $f \in \mathfrak{S}$ such that for every $\epsilon > 0$ there exists $g \in \alpha(G)$ satisfying $\|f - g\| < \epsilon$. We keep this definition of $\alpha_L(G)$ even if L fails to satisfy (DL) (cf. 2.3(g), (h)).

2.3 **Examples.** We use the fact that any set of the form $AP(G, \mathfrak{T})$—which includes $AP(G)$—is a nontrivial module. Actually these are the only nontrivial modules (3.3).

(a) Let $G = R$ be made a group under addition and let R have the usual topology, $\mathfrak{T}$. Let $\alpha(R) = AP(R, \mathfrak{T})$ be the Bohr AP functions.

Let

$$\mathfrak{S} = L_{1,\mathrm{loc}}(G); \tag{1}$$

$$\mathfrak{D} = \{f : f \text{ is a } \mu\text{-measurable function from } G \text{ to } (-\infty, \infty], f \geq g \text{ for some } g \in \mathfrak{S}^r\}. \tag{2}$$

Define L on $\mathfrak{D}$ by

$$Lf = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} f\, d\mu.$$

(L1) is satisfied because for $0 \leq f \in \mathfrak{S}^r$ we have

$$Lf_a = \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T+a}^{T+a} f\,d\mu \leq \overline{\lim_{T\to\infty}} \frac{2(T+|a|)}{2T}\, \frac{1}{2(T+|a|)}\int_{-(T+|a|)}^{T+|a|} f\,d\mu = Lf,$$

which includes the opposite inequality. To see that (DL) is satisfied take $f \in \mathfrak{S}$, $h \in \alpha(R)^r$ and define $\phi(t) = L|f_t - f(t)|h$. ϕ is measurable because we need only consider rational T in the definition of L. If $h \equiv 1$, $\phi \geq 0$ so $\phi \in \mathfrak{D}$. If $L|f| < \infty$, then $\phi \in \mathfrak{D}$ because

$$\phi(t) \geq -\|h\|_\infty [L|f| + |f(t)|].$$

The other conditions of 2.2 are easy to check. $\alpha_L(R)$ is the set of Besicovitch AP functions on R.

(b) Let G, $\alpha(G)$, $\mathfrak{S}$, $\mathfrak{D}$ be as in (a). For $f \in \mathfrak{D}$ define

$$Lf = \lim_{T\to\infty} \sup_{x\in G} \frac{1}{T}\int_{-T}^{T} f_x\, d\mu.$$

$L|f_t - f(t)|h$ is measurable in t because we need only consider rational T, x in the definition of L. $\alpha_L(R)$ is the set of Weyl AP functions on R.

(c) Let $(G, \mathfrak{T})$ be an LC group of the form $R^a \times Z^b \times F$, where $Z = \{0, \pm 1, \cdots\}$, F is a compact group and $a, b \geq 0$ are integers. Such groups include the compactly generated abelian LC groups [17, 9.8]. Let $\alpha(G) = AP(G, \mathfrak{T})$ and let $\mathfrak{E}, \mathfrak{D}$ be given by (1), (2), above. Define K_n on $\mathfrak{D}$ by

$$K_n f = \frac{1}{\mu[(-n, n)^{a+b} \times F]} \int_{(-n,n)^{a+b} \times F} f \, d\mu, \qquad n = 1, 2, \cdots.$$

Define L_1, L_2 on $\mathfrak{D}$ by

$$L_1 f = \overline{\lim_{n\to\infty}}\, K_n f, \qquad L_2 f = \lim_{n\to\infty} \sup_{x,y \in G} K_n({}_x f_y).$$

$L_2 |f_t - f(t)| b$ is a measurable function of t because

$$\sup_{x,y \in G} K_n[{}_x f_y] = \sup_{z \in R^a \times Z^b} K_n[f_{(z,e)}]$$

and one need only consider z's in the countable dense subset of $R^a \times Z^b$. $\alpha_{L_1}(G)$, $\alpha_{L_2}(G)$ are, respectively, the Besicovitch and Weyl AP functions on G (with respect to L_1, L_2).

(d) Let $(G, \mathfrak{T})$ be a σ-compact abelian LC group. Let $\alpha(G) = AP(G, \mathfrak{T})$ and let $\mathfrak{E}$ be the *bounded* members of $L_{1,\mathrm{loc}}(G)$. Let $\mathfrak{D}$ be given by (2). Let $\{V_n\}_1^\infty$ be a sequence of subsets of G such that $0 < \mu(V_n) < \infty$ for all n and such that for each $x \in G$

$$\lim_{n\to\infty} \frac{\mu(xV_n \,\Delta\, V_n)}{\mu(V_n)} = 0.$$

Define L on $\mathfrak{D}$ by

$$Lf = \overline{\lim_{n\to\infty}}\, \frac{1}{\mu(V_n)} \int_{V_n} f \, d\mu.$$

For the fact that $\{V_n\}_1^\infty$ exists and that (L6) is satisfied see [17, 18.10–18.14]. (L1) follows because the members of $\mathfrak{E}$ are bounded (cf. proof of 18.10 in [17]). $\alpha_L(G)$ is the set of *bounded* Besicovitch AP functions on G, with respect to L. The general Besicovitch AP functions on G are discussed in 4.14, below.

(e) Let G be an arbitrary group and $\alpha(G) = AP(G)$. Let $\mathfrak{E}$ be the set of all complex-valued functions on G and let $\mathfrak{D}$ be the set of all functions from G to $(-\infty, \infty]$. Set

$$\mathfrak{A} = \left\{(\{\alpha_r\}_{r=1}^n, \{a_r\}_{r=1}^n): a_r \in R,\ 0 < \alpha_r \in G, \textstyle\sum \alpha_r = 1, 1 \leq r \leq n; n \text{ is a positive integer}\right\}.$$

Define L on $\mathfrak{D}$ by

$$Lf = \inf \left\{ \sup_{x,y \in G} \sum_r \alpha_r f(x a_r y): (\{\alpha_r\}, \{a_r\}) \in \mathfrak{A} \right\}.$$

It is not hard to verify the conditions of 2.2. We call $\alpha_L(G)$ the *Følner-Weyl* AP functions on G. They are discussed in [13] and [7].

(f) Let G be an infinite group, $\mathfrak{a}(G) = AP(G)$ and $\mathfrak{S}$, $\mathfrak{D}$ as in (e), above. There exists a pairwise disjoint sequence $E_1, E_2, \cdots$ of symmetric subsets of G with the property that for every finite set $\{a_1, \cdots, a_n\} \subset G$

$$\bigcap_{1 \le i, j \le n} a_i E_h a_j \ne \emptyset, \qquad h = 1, 2, \cdots .$$

Let $\mathfrak{A}$ be as in (e). For $h = 1, 2 \cdots$ and $f \in \mathfrak{D}$ define

$$\overline{M}_h f = \inf_{A,B,D} \sup_{x,y} \sum_{i=1}^{n} \alpha_i f(x a_i y),$$

where $A = (\{\alpha_i\}, \{a_i\}) \in \mathfrak{A}$, $B = \{b_1, \cdots, b_k\} \subset G$, $D = \{d_1, \cdots, d_k\} \subset G$, and the supremum is to be taken over the nonempty set of those $x, y \in G$ for which $b_j x a_i y d_j \in E_h$ whenever $1 \le j \le k$, $1 \le i \le m$. Now define L on $\mathfrak{D}$ by $Lf = \overline{\lim}_{h\to\infty} \overline{M}_h f$. For the fact that the sequence E_h exists and that L satisfies the conditions of 2.2 see [14]. We call $\mathfrak{a}_L(G)$ the *Følner-Besicovitch* AP functions on G. $(\mathfrak{a}_L(G), \| \ \|)$ is complete and has properties very analogous to the usual Besicovitch AP functions considered in (a). See [14].

(g) Let $(G, \mathfrak{T})$ be an abelian LC group. Let $\mathfrak{a}(G) = AP(G, \mathfrak{T})$ and let $\mathfrak{S}$ be the *bounded* members of $L_{1,\mathrm{loc}}(G)$. Let $\mathfrak{D}$ be given by (2). Since G is amenable, there is a net $(V_d, d \in D, \ge)$ of subsets of G such that $0 < \mu(V_d) < \infty$ for all d and such that for all $x \in G$

$$\lim_{d\in D} \frac{\mu(xV_d \,\Delta\, V_d)}{\mu(V_d)} = 0$$

(cf. [15, p. 43] and [11]). Define L on $\mathfrak{D}$ by

$$Lf = \overline{\lim_{d\in D}} \frac{1}{\mu(V_d)} \int_{V_d} f \, d\mu.$$

The conditions of 2.2 are satisfied except possibly the requirement in (DL) that $L|f_t - f(t)|h$ be a measurable function of t. We return to this problem in 4.13. $\mathfrak{a}_L(G)$ is the set of *bounded* Besicovitch AP functions on G, with respect to L. The general Besicovitch AP functions on G are discussed in 4.14.

(h) Let $(G, \mathfrak{T})$ be an LC group. Let $\mathfrak{a}(G) = AP(G, \mathfrak{T})$ and let $\mathfrak{S}$, $\mathfrak{D}$ be given by (1), (2). Let $(V_d, d \in D, \ge)$ be a net of subsets of G such that $0 < \mu(V_d) < \infty$ for every d and such that for every $f \in \mathfrak{a}(G)$

$$\lim_{d\in D} \frac{1}{\mu(V_d)} \int_{V_d} f \, d\mu = Mf.$$

Such a net exists by 3.4 of [6]. Define L on $\mathfrak{D}$ by

$$Lf = \overline{\lim_{d\in D}} \sup_{x,y\in G} \frac{1}{\mu(V_d)} \int_{V_d} {}_x f_y \, d\mu.$$

Again the conditions of 2.2 are satisfied except possibly the measurability in t

of $L|f_t - f(t)|b$. We consider this problem in 4.13. $\alpha_L(G)$ is the set of Weyl AP functions on G, with respect to L.

3. **Preliminaries on modules.** Let $\{D^\lambda: \lambda \in \Lambda\}$ be a complete(2) set of inequivalent irreducible unitary finite-dimensional representations of the group G. We shall let s_λ denote the degree of D^λ and let $\mathfrak{M}^\lambda$ be the finite-dimensional subspace of $AP(G)$ spanned by $\{D^\lambda_{\rho\sigma}\}^{s_\lambda}_{\rho,\sigma=1}$. For $\Lambda_0 \subset \Lambda$ define $\Sigma_{\lambda\in\Lambda_0}\mathfrak{M}^\lambda$ to be smallest closed vector subspace of $AP(G)$ containing $\mathfrak{M}^\lambda$ for every $\lambda \in \Lambda_0$. Take $f \in AP(G)$. Maak [19, p. 141] defines the *summation module* $[f]$ *determined by* f by $[f] = \Sigma_{k=1}^\infty \mathfrak{M}^{\lambda_k}$, where the λ_k are indices of those representations occurring nontrivially in the Fourier series of a certain sequence of weight functions. We shall denote $\{\lambda_k\}_1^\infty$ by $\Lambda(f)$.

Let $g: G \to C$. We say $\{A_1, \cdots, A_n\}$ is an *ϵ-covering of G with respect to* g iff

$$G = \bigcup_{i=1}^n A_i \quad \text{and} \quad \sup_{a,b\in G} |g(axb) - g(ayb)| < \epsilon$$

whenever $x, y \in A_i$ for some i. One can show that $g \in AP(G)$ iff for every $\epsilon > 0$ G has an ϵ-covering with respect to g (this is the definition of $AP(G)$ in [19]). Let $f, g \in AP(G)$. We say g is *just as AP as* f iff for every $\epsilon > 0$ there exists $\delta > 0$ such that every δ-covering of G with respect to f is an ϵ-covering of G with respect to g. The following theorem (3.2) gives alternative descriptions of $[f]$. (i) is due to Maak [19, p. 143].

3.1 **Lemma.** *Let $f \in AP(G)$. There exists a smallest topology $\mathfrak{T}(f)$ for G such that $(G, \mathfrak{T}(f))$ is a topological group and f is $\mathfrak{T}(f)$-continuous. The sets*

$$V(\epsilon, f) = \left\{ x \in G: \sup_{a,b\in G} |f(axb) - f(ab)| < \epsilon \right\}, \qquad \epsilon > 0,$$

are a fundamental neighborhood system of e in $\mathfrak{T}(f)$.

3.2 **Theorem.** *Let $f \in AP(G)$.*

(i) *$[f]$ consists of those AP functions on G which are just as AP as f.*

(ii) $[f] = AP(G, \mathfrak{T}(f))$.

Proof of 3.1. Notice that $e \in V(\epsilon, f)^{-1} = V(\epsilon, f)$, $V(\epsilon, f)^2 \subset V(2\epsilon, f)$ and $yV(\epsilon, f)y^{-1} = V(\epsilon, f)$ for all $y \in G$. Thus $\{V(\epsilon, f)\}_{\epsilon>0}$ is a basic neighborhood system of e in a topology $\mathfrak{T}(f)$ such that $(G, \mathfrak{T}(f))$ is a topological group. If $x_n \to x(\mathfrak{T}(f))$, then $x_n x^{-1}$ is eventually in every $V(\epsilon, f)$ from which it follows that $|f(x_n) - f(x)|$ is eventually $< \epsilon$ for every $\epsilon > 0$. Hence f is $\mathfrak{T}(f)$-continuous.

To complete the proof we must show that if $f \in AP(G, \mathfrak{T})$ then $\mathfrak{T} \supset \mathfrak{T}(f)$. To this end it suffices to show that each $V(\epsilon, f)$ is a $\mathfrak{T}$-neighborhood of e. Take

(2) I.e., every irreducible (finite-dimensional) representation of G is equivalent to some D^λ [19, §30].

$\epsilon > 0$ and let $\{{}_{a_i}f_{b_i}\}_{i=1}^n$ be an ϵ-mesh in $(\{{}_af_b: a, b \in G\}, \| \|_\infty)$. *Let* U be a $\mathfrak{T}$-neighborhood of e such that

$$|{}_{a_i}f_{b_i}(x) - {}_{a_i}f_{b_i}(e)| < \epsilon \quad \text{for all } x \in U,\ 1 \le i \le n.$$

Now take any $x \in U$ and any $a, b \in G$. Take $i \in \{1, \cdots, n\}$ such that $\|{}_{a_i}f_{b_i} - {}_af_b\|_\infty < \epsilon$. Then

$$|f(axb) - f(ab)| \le |f(axb) - f(a_ixb_i)| + |f(a_ixb_i) - f(a_ib_i)| + |f(a_ib_i) - f(ab)| < 3\epsilon.$$

As $x \in U$, $a, b \in G$ are arbitrary, $U \subset V(3\epsilon, f)$. As $\epsilon > 0$ is arbitrary and U is a $\mathfrak{T}$-neighborhood of e, this proves the lemma.

Proof of 3.2. We use 3.1 and (i) to prove (ii). $g \in [f]$ iff [for every $\epsilon > 0$ there exists $\delta > 0$ such that $\sup_{a,b\in G} |f(aub) - f(avb)| < \delta$ implies $\sup_{a,b\in G}|g(aub) - g(avb)| < \epsilon$] iff [for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $v \in G$ $vV(\delta, f) \subset vV(\epsilon, g)$] iff $[\mathfrak{T}(f) \supset \mathfrak{T}(g)]$ iff $g \in AP(G, \mathfrak{T}(f))$.

3.3 **Theorem.** *Let* $\mathfrak{A}(G)$ *be an invariant vector space of AP functions on G. The following are equivalent*:

(i) $\mathfrak{A}(G)$ *is a nontrivial module*;

(ii) $\mathfrak{A}(G) = AP(G, \mathfrak{T})$ *for some topology* $\mathfrak{T}$;

(iii) $\mathfrak{A}(G)$ *is closed and* $f \in \mathfrak{A}(G)$ *implies* $[f] \subset \mathfrak{A}(G)$.

Proof. That (i) implies (ii) is known. Since we do not know where an explicit proof may be found, we outline a proof here: Let $A = \mathfrak{A}(G)$ and let T_x: $A \to A$ by $T_xf = {}_{x^{-1}}f$ for all $x \in G$. Let A have the topology induced by $\| \|_\infty$. Let $\overline{G}$ be the closure in A^A of $T(G)$ and let $\mathcal{P}$ be the restriction to $\overline{G}$ of the product topology for A^A. $(\overline{G}, \mathcal{P})$ is a compact T_2-topological group under composition and $T: G \to \overline{G}$ is a homomorphism with $\ker T = \{x \in G: f(x) = f(e)$ for all $f \in A\}$. The set of finite intersections of $\{V(\epsilon, f): f \in A, \epsilon > 0\}$ is a local neighborhood base at e for a topology $\mathfrak{T}$ such that $(G, \mathfrak{T})$ is a topological group and $A \subset AP(G, \mathfrak{T})$. T is $\mathfrak{T}$-continuous and the map of $(G, \mathfrak{T}) \to (T(G), \mathcal{P}/T(G))$ by $x \to T_x$ is open. For each $f \in A$ define $\overline{f}: \overline{G} \to C$ by $\overline{f}(R) = R^{-1}(f)(e)$ for all $R \in \overline{G}$. Then $\overline{f}(T_x) = f(x)$ when $x \in G$; also $\overline{f} \in C(\overline{G})$, the $\mathcal{P}$-continuous functions on $\overline{G}$. Letting $\overline{A} = \{\overline{f}: f \in A\}$ one can show by Stone-Weierstrass that $\overline{A} = C(\overline{G})$. Finally if $h \in AP(G, \mathfrak{T})$, h is constant on the cosets of $\ker T$ so we may define H on $T(G)$ by $H(T_x) = h(x)$. Since $x \to T_x$ is open, $H \in AP(T(G), \mathcal{P}/T(G))$. Let $\overline{H} \in AP(\overline{G}, \mathcal{P}) = C(\overline{G})$ extend H. $\overline{H}$ exists by direct proof or by [2]. But then there exists $f \in A$ such that $\overline{f} = \overline{H}$ from which it follows that $h = f \in A$. Hence $AP(G, \mathfrak{T}) \subset A$, completing the proof of (ii).

That (ii) implies (iii) follows from 3.2(ii).

Assume (iii) holds. Since $[0]$ = the constant functions, $\mathfrak{A}(G)$ is nontrivial. If $f \in \mathfrak{A}(G)$, then $\overline{f} \in [f] \subset \mathfrak{A}(G)$. Thus to prove (i) we need only show that if

$f, g \in \mathfrak{a}(G)$, then $fg \in \mathfrak{a}(G)$. But $f^2 \in [f]$, $g^2 \in [g]$, $(f+g)^2 \in [f+g]$, by 3.2, so that by combining these we get $fg \in \mathfrak{a}(G)$. This proves the theorem.

3.4 **Remarks.** (1) Let $\mathfrak{a}(G)$ be a nontrivial module. We use in the sequel the fact that $\mathfrak{a}(G)$ has the following closure properties (which follow easily from the above): $\mathfrak{a}(G)$ contains the constant functions; $f \in \mathfrak{a}(G)$ implies that $|f|$, $\Re f$, $(\Re f)^{\pm} \in \mathfrak{a}(G)$; $f, g \in \mathfrak{a}(G)$ implies that $f \times g$, $M|f_t - f(t)|g \in \mathfrak{a}(G)$, the latter as a function of t.

(2) Every closed invariant vector space of AP functions on G may be written in the form

$$\mathfrak{a}(G) = \sum_{\mathfrak{M}^\lambda \subset \mathfrak{a}(G)} \mathfrak{M}^\lambda$$

([19, p. 132]; Maak uses the term "module" differently than we). If G is abelian, each $\mathfrak{M}^\lambda$ is the one-dimensional subspace spanned by some $\lambda \in G^*$, the dual group of G. In this case the nontrivial modules on G are precisely the sets $\sum_{\lambda \in X^*} \mathfrak{M}^\lambda$, where X^* runs through the subgroups of G^* (cf. [1]). For example, let $\mathfrak{T}$ be the usual topology for R and let $\mathfrak{a}(R) = AP(R, \mathfrak{T})$ be the Bohr AP functions. If $0 \not\equiv h \in \mathfrak{a}(R)$, then $\Lambda(h)$ is the additive subgroup of R generated by the Fourier exponents of h; if $h \equiv 0$, then $\Lambda(h) = \{0\}$.

The following two lemmas will be used in §4. Recall the definition of E_1, E_2 in §2.1.

3.5 **Lemma.** *Let $g: G \to C$, $f \in AP(G)$. Take $j \in \{1, 2\}$. Suppose that for every $\epsilon > 0$ there exists $\delta > 0$ such that $E_j(\delta, f) \subset E_j(\epsilon, g)$. Then $g \in [f]$.*

Proof. Take, for example, $j = 2$. For any $h: G \to C$ and $\eta > 0$ notice that $\{h_{a_i}\}_{i=1}^n$ is an η-mesh in $(\{h_a: a \in G\}, \| \|_\infty)$ iff $G = \bigcup_{i=1}^n a_i E_2(\eta, h)$. It follows that $g \in AP(G)$. To show that $g \in [f]$ it suffices to show that g is $\mathfrak{T}(f)$-continuous. To this end it suffices to show that for any $h \in AP(G)$ the finite intersections of sets in $\{aE_2(h, \epsilon)a^{-1}: a \in G, \epsilon > 0\}$ are a base for the $\mathfrak{T}(h)$-neighborhood system of e. Since $V(\epsilon, h) \subset E_2(\epsilon, h)$ for all $\epsilon > 0$ and $aV(\epsilon, h)a^{-1} = V(\epsilon, h)$, it suffices to show that each $V(\epsilon, h)$ contains an appropriate finite intersection.

Take $\epsilon > 0$. Let $\{h_{a_i}\}_{i=1}^n$ be an ϵ-mesh in $(\{h_a: a \in G\}, \| \|_\infty)$. Take any $x \in \bigcap_{i=1}^n a_i E_2(\epsilon, h) a_i^{-1}$. Take arbitrary $a, b \in G$. Take i_0 such that $b \in a_{i_0} E_2(\epsilon, h)$. Then

$$|h(axb) - h(ab)| \leq |h(axb) - h(axa_{i_0})| + |h(axa_{i_0}) - h(aa_{i_0})| + |h(aa_{i_0}) - h(ab)|.$$

The first and third terms above are $\leq \|h_b - h_{a_i}\|_\infty < \epsilon$, since $b \in a_{i_0} E_2(\epsilon, h) = \{y \in G: \|h_y - h_{a_{i_0}}\|_\infty < \epsilon\}$. As $e, x \in a_{i_0} E_2(\epsilon, h) a_{i_0}^{-1}$, the second term is of the form $|h(ac) - h(ac')|$, where $c, c' \in a_{i_0} E_2(\epsilon, h)$. Hence $|h(axb) - h(ab)| < 4\epsilon$. It follows that $\bigcap_{i=1}^n a_i E_2(\epsilon, h) a_i^{-1} \subset V(4\epsilon, h)$, proving the lemma.

3.6 **Lemma.** *Let* $\Phi \in AP(G)$. *There exists a sequence of weight functions* $h_n \in [\Phi]$ *such that* $\|g \times h_n - g\|_\infty \xrightarrow[n]{} 0$ *for all* $g \in [\Phi]$. *We may require that*

$$h_n(x) = \sum_{\nu \in \Lambda(\Phi)} s^\nu \beta_n^\nu \sum_{\rho=1}^{s^\nu} D_{\rho\rho}^\nu(x), \qquad n = 1, 2, \cdots,$$

where the right side converges uniformly and each $\beta_n^\nu \geq 0$. *(By saying* h_n *is a weight function is meant that* $Mh_n = 1$ *and* $h_n \geq 0$, $n = 1, 2, \cdots$.)

Proof. Let g_n be the weight functions considered in [19, pp. 139–143, Theorems 3 through 6]. Define $h_n = g_n \times g_n$, $n = 1, 2, \cdots$. Evidently each h_n is a weight function and since

$$g_n \sim \sum_{\nu \in \Lambda(\Phi)} s^\nu \gamma_n^\nu \sum_{\rho=1}^{s^\nu} D_{\rho\rho}^\nu$$

the last assertion follows with $\beta_n^\nu = |\gamma_n^\nu|^2$.

From Theorem 4, p. 140 of [19] and the definition of $\Lambda(\Phi)$, we have $\lim_{n\to\infty} \beta_n^\nu = 1$ for all $\nu \in \Lambda(\Phi)$. For each $g \in [\Phi]$, $g \times h_n$ is majorized by g and bounded by $\|g\|_\infty$. Hence by Theorem 1, p. 136 of [19], $\|g \times h_n - g\|_\infty \xrightarrow[n]{} 0$. Finally $h_n = g_n \times g_n \in [\Phi]$ by 3.4(1) since $g_n \in [\Phi]$. This proves the lemma.

3.7 **Remark.** We shall use the fact that, each h_n being real-valued,

$$h_n = \sum_{\nu \in \Lambda(\Phi)} s^\nu \beta_n^\nu \sum_{\rho=1} \Re D_{\rho\rho}^\nu, \qquad n = 1, 2, \cdots.$$

4. **Characterization of** $\alpha_L(G)$. We continue the notation of §§2 and 3. Unless stated otherwise we assume that $\alpha(G)$, $\mathfrak{E}$, $\mathfrak{D}$, and L satisfy all the conditions (E1), (D1), (L1) and (DL) of 2.2. For $f \in \mathfrak{E}$ define Δ_f, Δ_f^r on G by $\Delta_f(x) = \|{}_xf - f\|$, $\Delta_f^r(x) = \|f_x - f\|$ for all $x \in G$. For $\epsilon > 0$, $f \in \mathfrak{E}$ define

$$LE_1(\epsilon, f) = \{x \in G: \Delta_f(x) < \epsilon\}, \qquad LE_2(\epsilon, f) = \{x \in G: \Delta_f^r(x) < \epsilon\}.$$

In the sequel we shall use freely the conditions (E1), (D1), (L1) and (DL) of 2.2 without always stating them explicitly.

4.1 **Lemma.** *Let* $f \in \mathfrak{E}$ *and* $\Delta_f \in \alpha(G)$. *Then both sides of*

$$(*) \qquad L_t L_x |f(xt) - f(t)| \leq L_x L_t |f(xt) - f(t)|$$

are well defined. If (*) *holds, then* $L|f| < \infty$.

Proof. The left side of (*) is well defined by (DL) and the right side is simply $M\Delta_f$. Assume (*) is true. Now $L|f| \leq L_x |f(xt) - f(t)| + |f(t)|$ by (L1), (L4), (L5) and (L6). If $L|f| = \infty$, then since $|f(t)| < \infty$ for all t, we have $L_x |f(xt) - f(t)| = \infty$ for all t. By (*)

$$\infty = L_t L_x |f(xt) - f(t)| \leq M\Delta_f,$$

a contradiction. Therefore $L|f| < \infty$.

4.2 **Remark.** Let $f \in \mathfrak{S}$ and $\Delta_f \in \mathfrak{a}(G)$. Let us show that the following condition is well defined:

(B) $L_t L_x |f(xt) - f(t)| \Re D^\lambda_{\rho\rho}(x) \leq L_x L_t |f(xt) - f(t)| \Re D^\lambda_{\rho\rho}(x)$ for all $\lambda \in \Lambda(\Delta_f)$ and $1 \leq \rho \leq s_\lambda$.

Taking $\lambda_0 \in \Lambda$ such that $D^{\lambda_0} \equiv 1$, notice that $\lambda_0 \in \Lambda(\Delta_f)$ (for example, by 3.2; or the definition of $[\Delta_f]$). Thus the inequalities of (B) include (*) of 4.1, which by 4.1 is well defined. From the truth of (*) we get that $L|f| < \infty$. Notice that $D^\lambda_{\rho\rho} \in \mathfrak{a}(G)$ when $\lambda \in \Lambda(f)$ by 3.3. Thus by (DL) the left sides of the other inequalities in (B) are well defined. The right side of (B) is simply $M(\Delta_f \Re D^\lambda_{\rho\rho})$.

4.3 **Lemma.** *Suppose $f \in \mathfrak{S}$, $\Delta_f \in \mathfrak{a}(G)$. Set $\Phi = \Delta_f$ in 3.6 and let b_n be as in that lemma. If* (B) *holds, then*

$$L_t L_x |f(xt) - f(t)| b_n(x) \leq L_x L_t |f(xt) - f(t)| b_n(x), \qquad n = 1, 2, \cdots. \tag{1}$$

Proof. The right side of (1) is $M(\Delta_f b_n)$ and the left side is well defined by (B), (4.1) and (DL). If $\Delta_f \equiv 0$, it turns out that the g_n of 3.6 are identically 1, whence $b_n \equiv 1$ and (1) is a restatement of (B). Thus we may suppose $\Delta_f \not\equiv 0$.

Fix n arbitrarily and set $K = b_n$. Take $\epsilon > 0$ and set

$$\delta = \epsilon / M\Delta_f. \tag{2}$$

By 3.7

$$\|K - P\|_\infty < \delta \tag{3}$$

where

$$P = \sum\nolimits^* \beta^\lambda_\rho \Re D^\lambda_{\rho\rho}, \qquad \beta^\lambda_\rho \geq 0, \tag{4}$$

and Σ^* denotes a finite sum. If $b \in \mathfrak{a}(G)$ is arbitrary,

$$\begin{aligned} L_x |f(xt) - f(t)| \Re b(x) &\geq L_x [- |f(xt) - f(t)| \, \|b\|_\infty] \quad \text{by (L4)} \\ &\geq - \|b\|_\infty L_x |f(xt) - f(t)| \quad \text{by (L2), (L3)} \\ &\geq - \|b\|_\infty [L|f| + |f(t)|]. \end{aligned}$$

Hence

$$L_t L_x |f(xt) - f(t)| \Re b(x) \geq -2\|b\|_\infty L|f| > -\infty.$$

This observation allows us to apply (L5) in the following chain of inequalities.

$$L_t L_x |f(xt) - f(t)| K(x)$$

$$\le L_t L_x |f(xt) - f(t)| \left[\sum{}^{*} \beta_\rho^\lambda \Re D_{\rho\rho}^\lambda(x) + \delta\right] \quad \text{by (3), (4), (L4)}$$

$$\le \sum{}^{*} \beta_\rho^\lambda L_t L_x |f(xt) - f(x)| \Re D_{\rho\rho}^\lambda(x) + \delta L_t L_x |f(xt) - f(t)|$$

$$\text{by (L5), (L2) since } \beta_\rho^\lambda \ge 0$$

$$\le \sum{}^{*} \beta_\rho^\lambda L_x L_t |f(xt) - f(t)| \Re D_{\rho\rho}^\lambda(x) + \delta L_x L_t |f(xt) - f(t)| \quad \text{by (B)}$$

$$= M(\Delta_f P) + \delta M \Delta_f$$

$$\le M[\Delta_f (K + \delta)] + \delta M \Delta_f \quad \text{by (3)}$$

$$= M(\Delta_f K) + 2\delta M \Delta_f$$

$$< L_x L_t |f(xt) - f(t)| K(x) + 2\epsilon \quad \text{by (2).}$$

As $\epsilon > 0$ is arbitrary, (1) follows, proving the lemma.

4.4 **Lemma.** *Take $f \in \mathfrak{S}$ and $\epsilon > 0$. If Δ_f^r is finite, then $E_2(\epsilon, \Delta_f^r) = LE_2(\epsilon, f)$; if Δ_f is finite, $E_1(\epsilon, \Delta_f) = LE_1(\epsilon, f)$.*

Proof. We consider Δ_f^r, Δ_f being similar. Take $u \in G$.

$$\|(\Delta_f^r)_u - \Delta_f^r\|_\infty = \sup_{x \in G} \big| \|f_{xu} - f\| - \|f - f_x\| \big|$$

$$\le \sup_{x \in G} \|f_{xu} - f_x\| \quad \text{by (L5) since } \Delta_f^r \text{ is finite}$$

$$= \|f_u - f\|.$$

Thus $LE_2(\epsilon, f) \subset E_2(\epsilon, \Delta_f^r)$. If $u \in E_2(\epsilon, \Delta_f^r)$, then

$$\sup_{x \in G} \big| \|f_{xu} - f\| - \|f_x - f\| \big| < \epsilon.$$

Taking $x = e$ gives $\|f_u - f\| < \epsilon$, whence $u \in LE_2(\epsilon, f)$. Therefore $E_2(\epsilon, \Delta_f^r) \subset LE_2(\epsilon, f)$, proving the lemma.

4.5 **Theorem.** *Let $f \in \mathfrak{S}$. Suppose*

(A) $$\Delta_f, \Delta_f^r \in \mathfrak{a}(G)$$

and that f satisfies (B) *of* 4.2. *Then $f \in \mathfrak{a}_L(G)$.*

$$M(\Delta_f b_n) = M_x[\Delta_f(x) b_n(x^{-1})] = \Delta_f \times b_n(e) \to \Delta_f(e) = 0$$

by 3.6. Take n so large that the left side, above, is $< \epsilon/2$ and set $K = b_n$. Applying 3.3, 3.6, 4.3 and (A) gives

$$(1)\qquad \begin{cases} K \in \mathfrak{A}(G),\ K \geq 0, \\ MK = 1,\ M(\Delta_f K) < \epsilon/2, \\ \text{(B) holds with } \Re D^\lambda_{\rho\rho} \text{ replaced by } K. \end{cases}$$

Write $f = u + iv$ where $u, v\colon G \longrightarrow R$. By (E1), (E2), (E3), (E6) and (1) $u_t K$, $v_t K \in \mathcal{E}^r$ for all $t \in G$. Define $\Phi_1(t) = L(u_t K)$, $\Phi_2(t) = L(v_t K)$ for all $t \in G$. The following string of inequalities shows that

$$(2)\qquad -\infty < \Phi_1(t) < \infty \quad \text{for all } t \in G.$$

They are justified by applying (L1), (L2), (L3), (L4) and recalling that by 4.1 and (B) $L|f| < \infty$.

$$-\infty < -\|K\|_\infty L|f| \leq -\|K\|_\infty L|u| = -L_x|u(xt)|\,\|K\|_\infty \leq -L_x|u(xt)|K(x)$$

$$\leq L_x[-|u(xt)|K(x)] \leq L(u_t K) = \Phi_1(t) \leq L|u_t|K \leq \|K\|_\infty L|u| \leq \|K\|_\infty L|f| < \infty.$$

Similarly Φ_2 is finite. (2) allows us to apply (L5) and (L4) to get

$$|\Phi_1(tx) - \Phi_1(t)| \leq L|u_{tx} - u_t|K \leq \|K\|_\infty \|f_x - f\| \quad \text{by (L1), (L2), (L4).}$$

Consequently, by Lemma 4.4, $E_2(\rho/\|K\|_\infty, \Delta^r_f) = LE_2(\rho/\|K\|_\infty, f) \subset E_2(\rho, \Phi_1)$ for all $\rho > 0$. By 3.5, $\Phi_1 \in [\Delta^r_f] \subset \mathfrak{A}(G)$. Similarly $\Phi_2 \in \mathfrak{A}(G)$. Set $\Phi = \Phi_1 + i\Phi_2$ so that $\Phi \in \mathfrak{A}(G)$.

Since $u(t)$ is finite for each $t \in G$, (L4), (L5) and (1) give $|L(u_t K) - u(t)| \leq L|u_t - u(t)|K$, that is, $|\Phi_1(t) - u(t)| \leq L|u_t - u(t)|K$. Similarly for Φ_2. Consequently

$$|\Phi(t) - f(t)| \leq L|u_t - u(t)|K + L|v_t - v(t)|K \leq 2L|f_t - f(t)|K,$$

by (L4). Therefore, by (1), (L2) and (L4)

$$\|\Phi - f\| = L_t|\Phi(t) - f(t)| \leq 2L_t L_x|f(xt) - f(t)|K(x)$$

$$\leq 2L_x L_t|f(xt) - f(t)|K(x) = 2M(\Delta_f K) < \epsilon.$$

Since $\Phi \in \mathfrak{A}(G)$ and $\epsilon > 0$ is arbitrary, this proves the theorem.

4.6 **Lemma.** *If* $f \in \mathfrak{A}_L(G)$, *then* $L|f| < \infty$ *and* $\Delta_f, \Delta^r_f \in \mathfrak{A}(G)$.

Proof. Take $\epsilon > 0$ and $g \in \mathfrak{A}(G)$ such that $\|f - g\| < \epsilon/2$. Since $L|f| = \|f\| \leq \|f - g\| + \|g\| \leq \epsilon/2 + \|g\|_\infty < \infty$, the first assertion is clear. To see that $\Delta_f \in \mathfrak{A}(G)$ define $G(u) = \|{}_u g - g\|$. Then

$$\|{}_t G - G\|_\infty = \sup_{x \in G} \big|\, \|{}_{tx} g - g\| - \|{}_x g - g\| \,\big| \leq \|{}_t g - g\| \leq \|{}_t g - g\|_\infty.$$

Hence $E_1(\eta, g) \subset E_1(\eta, G)$ for all $\eta > 0$. It follows from 3.5 that $G \in [g] \subset \mathfrak{A}(G)$. Also $\|{}_u f - f\| \leq \|{}_u f - {}_u g\| + \|{}_u g - g\| + \|g - f\|$ and similarly with f, g interchanged. This gives $|\Delta_f(u) - G(u)| \leq 2\|g - f\| < \epsilon$. Thus $\|\Delta_f - G\|_\infty \leq \epsilon$. As $\epsilon > 0$ is arbitrary and $\alpha(G)$ is closed, $\Delta_f \in \alpha(G)$. Similarly $\Delta_f^r \in \alpha(G)$. Q.E.D.

4.7 **Lemma.** *If* $0 \leq f$, f, $-f \in \mathfrak{L}$ *and if* f *is* $\| \ \|$*-approximatable by members of* $\alpha(G)^r$, *then* $Lf < \infty$ *and* $L(-f) = -Lf$.

Proof. Take $g_n \in \alpha(G)^r$ such that $\|f - g_n\| \xrightarrow[n]{} 0$. If $\|f - g_m\| < 1$, then $Lf = \|f\| \leq \|f - g_m\| + \|g_m\| \leq 1 + \|g_m\|_\infty < \infty$, proving the first assertion. Noting that $-\infty < -Lf \leq L(-f) \leq 0$, we get from (L4), (L5)

$$L(-g_n) \leq L|(-g_n) - (-f)| + L(-f). \tag{1}$$

Interchanging f and g_n and using the fact that $L(-f)$, $L(-g_n)$ are finite gives

$$|L(-f) - L(-g_n)| \leq L|f - g_n|. \tag{2}$$

Replacing $(-g_n)$ with g_n and $(-f)$ with f in (1) and then interchanging f and g_n gives, as before,

$$|Lf - Lg_n| \leq L|f - g_n| \tag{3}$$

since Lf, Lg_n are finite. From (2) and (3) we have

$$|L(-f) - (-Lf)| \leq |L(-f) - L(-g_n)| + |L(-g_n) - (-Lf)|$$

$$= |L(-f) - L(-g_n)| + |Lf - Lg_n| \leq 2L|f - g_n| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Here we use the fact that $L(-g_n) = M(-g_n) = -Mg_n = -Lg_n$. The lemma follows.

4.8 **Theorem.** *If* $f \in \alpha_L(G)$, *then for all* $h \in \alpha(G)^r$

$$L_t L_x |f(xt) - f(t)| h(x) \leq L_x L_t |f(xt) - f(t)| h(x). \tag{1}$$

Proof. By 4.6 the right side of (1) is $M(\Delta_f h)$ and, hence, well defined. Also by 4.6, $L|f| < \infty$ so by (DL) the left side of (1) is well defined for all $h \in \alpha(G)^r$.

We are given $h \in \alpha(G)^r$. Take $f_n \in \alpha(G)$ such that $\|f - f_n\| \xrightarrow[n]{} 0$. Let h^+, h^- be the positive and negative parts of h so that $h = h^+ - h^-$, $0 \leq h^\pm \in \alpha(G)^r$. We shall show that

$$L_t L |f_t - f(t)| h^\pm = \lim_n L_t L |f_{nt} - f_n(t)| h^\pm. \tag{2}$$

From (L4), (L5) and the fact that $h^\pm \geq 0$ we have

$$\begin{aligned} L|f_t - f(t)| h^\pm &\leq L|[f_t - f_{nt}] + [f_n(t) - f(t)]| h^\pm + L|f_{nt} - f_n(t)| h^\pm \\ &\leq \|h^\pm\|_\infty [\|f - f_n\| + |f_n(t) - f(t)|] + L|f_{nt} - f_n(t)| h^\pm. \end{aligned} \tag{3}$$

Operating with L_t on both sides of (3) gives

$$L_t L |f_t - f(t)| h^\pm \leq 2\|h^\pm\|_\infty \|f - f_n\| + L_t L |f_{nt} - f_n(t)| h^\pm. \tag{4}$$

Observe that $0 \leq L_t L|f_t - f(t)|b^{\pm} < 2\|b^{\pm}\|_\infty L|f| < \infty$ and similarly with f replaced by f_n. Thus if we interchange f and f_n in (4) and combine the two results we get

$$|L_t L|f_t - f(t)|b^{\pm} - L_t L|f_{nt} - f_n(t)|b^{\pm}| \leq 2\|b^{\pm}\|_\infty \|f - f_n\| \xrightarrow[n]{} 0.$$

This establishes (2).

Interchanging f and f_n in (3), combining the result with (3) and then operating with L_t gives

$$L_t |L|f_t - f(t)|b^{\pm} - L|f_{nt} - f_n(t)|b^{\pm}| \leq 2\|b^{\pm}\|_\infty \|f - f_n\| \xrightarrow[n]{} 0.$$

That operating with L_t is permissible follows from (D2), (D3), the fact that $-L|f_{nt} - f_n(t)|b^{\pm} \in \mathfrak{a}(G)$ as a function of t, and the fact that $L|f_t - f(t)|b^{\pm} \in \mathfrak{D}$ as a function of t (as was pointed out in the first paragraph of the proof). It follows that

(5) The function $t \longrightarrow L|f_t - f(t)|b^{\pm}$ is $\| \, \|$-approximatable by members of $\mathfrak{a}(G)^r$.

From the proof of 4.6 we see that $L|_x f_n - f_n| \xrightarrow[n]{} \Delta_f(x)$ uniformly in x. Hence, for arbitrary $\eta > 0$,

$$-\eta < \Delta_f(x)b^{\pm}(x) - L|_x f_n - f_n|b^{\pm}(x) < \eta$$

for all $x \in G$ when n is sufficiently large. Operating on this with L_x gives that

$$L_x L|_x f - f|b^{\pm}(x) = \lim_n L_x L|_x f_n - f_n|b^{\pm}(x). \tag{6}$$

The right sides of (2) and (6) are equal since $f_n, b^{\pm} \in \mathfrak{a}(G)$. Hence

$$L_t L_x |f(xt) - f(t)|b^{\pm}(x) = L_x L_t |f(xt) - f(t)|b^{\pm}(x). \tag{7}$$

For each t, $0 \leq L_x|f(xt) - f(t)|b^+(x) \leq \|b^+\|_\infty [L|f| + |f(t)|] < \infty$. Thus we may apply (L5) to get

$$L_t L_x |f(xt) - f(t)|b(x) \leq L_t [L_x |f(xt) - f(t)|b^+(x) + L_x |f(xt) - f(t)|(-b^-(x))].$$

By (5) and 4.7 $0 \leq L_t L_x |f(xt) - f(t)|b^+(x) < \infty$. Thus we may apply (L5) again to get

$$\begin{aligned} &L_t L_x |f(xt) - f(t)|b(x) \\ &\qquad \leq L_t L_x |f(xt) - f(t)|b^+(x) + L_t L_x [-|f(xt) - f(t)|b^-(x)]. \end{aligned} \tag{8}$$

Notice that, for fixed t,

$$\| |f_t - f(t)|b^- - |f_{nt} - f(t)|b^- \| \leq \|f - f_n\| \, \|b^-\|_\infty \xrightarrow[n]{} 0.$$

Thus for each fixed t $\pm|f_t - f(t)|b^- \in \mathfrak{a}_L(G)$. By 4.7,

$$(9) \qquad L_x[-|f(xt)-f(t)|h^-(x)] = -L_x|f(xt)-f(t)|h^-(x)$$

for each $t \in G$. As $L|f| < \infty$, (DL) implies that the left and hence the right side of (9) is in $\mathfrak{D}$ as a function of t. By (5) and 4.7,

$$(10) \qquad L_t[-L_x|f(xt)-f(t)|h^-(x)] = -L_tL_x|f(xt)-f(t)|h^-(x).$$

Combining (7), (8), (9), 10) and the fact that $\Delta \in \mathfrak{a}(G)$ (by 4.6) gives

$$L_tL_x|f(xt)-f(t)|h(x) \leq L_xL_t|f(xy)-f(t)|h^+(x) - L_xL_t|f(xt)-f(t)|h^-(x)$$

$$= M(\Delta_f h^+) - M(\Delta_f h^-) = M(\Delta_f h) = L_xL_t|f(xt)-f(t)|h(x).$$

This proves the theorem.

4.9 **Main Theorem.** *Let $\mathfrak{a}(G)$ be a nontrivial module of AP functions on G. Let $\mathfrak{S}$, $\mathfrak{D}$, $\mathfrak{L}$ be as in 2.2. Take $f \in \mathfrak{S}$.* Then *$f \in \mathfrak{a}_L(G)$ if and only if*

(A) $\Delta_f, \Delta_f^r \in \mathfrak{a}(G)$,

(B) $L_tL_x|f(xt)-f(t)|\Re D^\lambda_{\rho\rho}(x) \leq L_xL_t|f(xt)-f(t)|\Re D^\lambda_{\rho\rho}(x)$ *for all* $\lambda \in \Lambda(\Delta_f)$ *and* $1 \leq \rho \leq s_\lambda$.

Proof. This follows from 4.5, 4.6 and 4.8.

4.10 **Remarks.** (1) The requirement in (L1) that $L({}_xf) = Lf$ when $0 \leq f \in \mathfrak{S}^r$ is used only in the proof of 4.6 and in the proof of (6) in 4.8. The requirement that $L(f_y) = Lf$ is used directly in all of the above proofs except 4.7.

(2) Condition (A) is in some cases equivalent to other conditions which appear in AP function theory. For $f \in \mathfrak{S}$ let

$$O_1(f) = \{{}_xf: x \in G\}, \qquad O_2(f) = \{f_x: x \in G\}.$$

When G has a topology we say that a symmetric set $S \subset G$ is *relatively dense* iff there exists a compact set $K \subset G$ such that $Kz \cap S \neq \emptyset$ for all $z \in G$ (equivalently: iff there exists compact $F \subset G$ such that $zF \cap S \neq \emptyset$ for all $z \in G$ iff there exists compact $K_0 \subset G$ such that $G = SK_0$, etc.). On the real line this is equivalent to the usual definition of relative density. Notice that $E_i(\epsilon, f)$, $LE_i(\epsilon, f)$ are symmetric for $\epsilon > 0$, $i = 1, 2$. Consider the following conditions on a function of $f \in \mathfrak{S}$:

(A0) For every $\epsilon > 0$ there exists $w_1, \cdots, w_n, z_1, \cdots, z_m \in G$ such that

$$G = \bigcup_{i=1}^n w_i LE_2(\epsilon, f) = \bigcup_{j=1}^m LE_1(\epsilon, f) z_j.$$

(A1) $(O_i(f), \|\ \|)$ is totally bounded, $i = 1, 2$.

(A2) For every $\epsilon > 0$, $LE_i(\epsilon, f)$ is relatively dense and open, $j = 1, 2$.

4.11 Theorem. *Let $f \in \mathcal{E}$. Then*

(i) (A) $\Rightarrow$ (A1) $\Leftrightarrow$ (A0).

(ii) *Assume that* $(G, \mathcal{T})$ *is a topological group and that* $\alpha(G) \subset AP(G, \mathcal{T})$. *Then* (A) $\Rightarrow$ (A2) $\Rightarrow$ (A1) $\Leftrightarrow$ (A0).

(iii) *Assume that* $(G, \mathcal{T})$ *is a locally compact* (*or complete metric*) *topological group, that* $\alpha(G) = AP(G, \mathcal{T})$, *that every neighborhood of* e *generates* G (*or that* Δ_f, Δ_f^r *are finite*) *and that each* $LE_i(\epsilon, f)$ *is a borel set* ($i = 1, 2$). *Then* (A0) $\Rightarrow$ (A).

Proof. In 2.1 of [5] the equivalency of (A*i*), $0 \le i \le 2$, is shown when G is the real line (usual topology). Therefore we omit details here which resemble those of [5].

(i) That (A1) implies (A0) may be argued as in [5] and the converse is similar. Take $\epsilon > 0$. Let us show, for example, that (A) implies that there exists $w_1, \cdots, w_n \in G$ such that

$$G = \bigcup_{i=1}^{n} w_i LE_2(\epsilon, f). \tag{1}$$

Since $\Delta_f^r \in \alpha(G)$, $(\{(\Delta_f^r)_u : u \in G\}, \|\ \|_\infty)$ is totally bounded. Hence there exists $w_1, \cdots, w_n \in G$ such that

$$G = \bigcup_{i=1}^{n} w_i E_2(\epsilon, \Delta_f^r). \tag{2}$$

By 4.4 $E_2(\epsilon, \Delta_f^r) = LE_2(\epsilon, f)$ from which (1) follows.

(ii) Suppose that (A) is true. Then $E_2(\epsilon, \Delta_f^r)$ is relatively dense for all $\epsilon > 0$, because $\Delta_f^r \in AP(G)$. Since $\Delta_f^r \in AP(G, \mathcal{T})$, $\mathcal{T} \supset \mathcal{T}(\Delta_f^r)$ by 3.1. Hence, for every $\epsilon > 0$, $V(\epsilon, \Delta_f^r)$ is a $\mathcal{T}$-neighborhood of e. Also $V(\epsilon, \Delta_f^r) \subset E_2(\epsilon, \Delta_f^r)$ and, by 4.4, $E_2(\epsilon, \Delta_f^r) = LE_2(\epsilon, f)$. Hence $LE_2(\epsilon, f)$ is relatively dense and contains a $\mathcal{T}$-neighborhood of e for all $\epsilon > 0$. To see that $LE_2(\epsilon, f)$ is open note that for each $x \in LE_2(\epsilon, f)$ there exists $\epsilon_1 < \epsilon$ such that $x \in LE_2(\epsilon_1, f)$. Taking $0 < \epsilon_2 < \epsilon - \epsilon_1$, let $U \subset LE_2(\epsilon_2, f)$ be an open neighborhood of e. Then $x \in LE_2(\epsilon_1, f)U \subset LE_2(\epsilon, f)$. Since $x \in LE_2(\epsilon, f)$ is arbitrary, $LE_2(\epsilon, f)$ is open. Similarly one shows that $LE_1(\epsilon, f)$ is relatively dense and open.

To see that (A2) implies (A0) let us show, for example, that for each $\epsilon > 0$ there exists $z_1, \cdots, z_m \in G$ such that $G = \bigcup_{j=1}^m LE_1(\epsilon, f)z_j$. Take $\epsilon > 0$. Take compact $K \subset G$ such that $G = LE_1(\epsilon/2, f)K$. Since $U = LE_1(\epsilon/2, f)$ is an open neighborhood of e, there exists $z_1, \cdots, z_m \in K$ such that $\bigcup_{j=1}^m Uz_j \supset K$. Hence

$$\bigcup_{j=1}^{m} LE_1(\epsilon, f)z_j \supset \bigcup_{j=1}^{m} LE_1(\epsilon/2, f)Uz_j \supset LE_1(\epsilon/2, f)K = G.$$

(iii) By Baire's theorem (A0) implies that each $LE_2(\epsilon, f)$ is of second category. As $LE_2(\epsilon, f) \supset [LE_2(\epsilon/2, f)]^2$ and is borel, each $LE_2(\epsilon, f)$ contains a neighborhood of e. Arguing as in (ii) one deduces that each $LE_2(\epsilon, f)$ is open.

Since every neighborhood of e generates G,

$$\dot{G} = \bigcup_{n=1}^{\infty} LE_2(1, f)^n \subset \bigcup_{n=1}^{\infty} LE_2(n, f).$$

It follows that Δ_f^r is finite. By 4.4, $E_2(\epsilon, \Delta_f^r) = LE_2(\epsilon, f)$ for all $\epsilon > 0$. Hence, by (A0), $\Delta_f^r \in AP(G)$. Since each $E_2(\epsilon, \Delta_f^r)$ is open, $\mathfrak{T} \supset \mathfrak{T}(\Delta_f^r)$ (see proof of 3.5), whence $\Delta_f^r \in AP(G, \mathfrak{T}) = \mathfrak{a}(G)$. Similarly for Δ_f. This proves the theorem.

4.12 **Remark.** We now apply the main theorem to the classical Besicovitch AP functions on R. Let $\mathfrak{a}(R)$ be the set of Bohr AP functions on R and let L be as in 2.3(a). It is customary to denote L by $\overline{M}$. From 4.9, 4.11 and 3.4(2) we have $f \in L_{1,\text{loc}}(R)$ is Besicovitch AP if and only if

(A) $\Delta_f \in \mathfrak{a}(R)$, where $\Delta_f(x) = \overline{M}|f_x - f|$, say $\Delta_f(x) \sim \sum_{k=1}^{\infty} a_k e^{i\lambda_k x}$; and

(B) $\overline{M}_t \overline{M}_x |f(x+t) - f(t)| \cos \lambda x \le \overline{M}_x \overline{M}_t |f(x+t) - f(t)| \cos \lambda x$ for all λ in the additive subgroup of R generated by $\{\lambda_k\}_{k=1}^{\infty} \cup \{0\}$.

(A) may be replaced by any of the equivalent conditions (A0), (A1), or (A2).

References to other known characterizations of $\mathfrak{a}_{\overline{M}}(R)$ occur in the introduction to [5].(3)

4.13 **Theorem.** *Let G, $\mathfrak{a}(G)$, $\mathfrak{E}$, $\mathfrak{D}$, L be as in* 2.3(g) *or* 2.3(h). *Take $f \in \mathfrak{E}$. Then $f \in \mathfrak{a}_L(G)$ if and only if f satisfies* (A), (B) *and*

(M) *$t \longrightarrow L|f_t - f(t)|h$ is μ-measurable for all $h \in \mathfrak{a}(G)^r$.*

Proof. Take $f \in \mathfrak{a}_L(G)$, $h \in \mathfrak{a}(G)^r$ and let $\phi(t) = L|f_t - f(t)|h$. Assume first that $\|f\|_\infty < \infty$. To prove that ϕ is measurable we show that for every $\epsilon > 0$ there is a measurable function ψ such that $\|\phi - \psi\|_\infty < 2\epsilon$. Take $\epsilon > 0$. For each set $E \subset G$ let χ_E denote its indicator. Let $\{E_k\}_{k=1}^m$ be a measurable partition of G such that for appropriate $c_k \in C$

$$\left\| f - \sum_{k=1}^{m} c_k \chi_{E_k} \right\|_\infty < \epsilon/\|h\|_\infty. \tag{1}$$

Take $t_1, \cdots, t_n \in G$ such that for every $t \in G$ there is some $j \in \{1, \cdots, n\}$ satisfying

$$|h(xt^{-1}) - h(xt_j)| < \epsilon/2\|f\|_\infty \quad \text{for all } x \in G. \tag{2}$$

Let $\{B_j\}_1^n$ be a measurable partition of G such that when $t \in B_j$ (2) holds.

Let us write $F(t) \approx G(t)$ when $|F(t) - G(t)| \le \epsilon$. Fix arbitrary $t \in G$. Then

(3) Harold Donnelly has pointed out that the requirement that $f \in L_{1,\text{loc}}(R)$ in the main theorem of [5] may be weakened to measurability. Proofs remain the same.

$$\phi(t) = L_x|f(xt) - f(t)|h(x)$$

$$\approx L_x \left[|f(xt) - f(t)| \sum_{j=1}^{n} h(xtt_j)\chi_{B_j}(t) \right] \quad \text{by (2)}$$

$$= \sum_{j=1}^{n} \chi_{B_j}(t) L_x[|f(xt) - f(t)|h(xtt_j)] \quad \text{since only one } \chi_{B_j} \neq 0$$

$$\approx \sum_{j=1}^{n} \chi_{B_j}(t) L_x \left[\left| f(xt) - \sum_{k=1}^{m} c_k \chi_{E_k}(t) \right| h(xtt_j) \right] \quad \text{by (1)}$$

$$= \sum_{j,k} \chi_{B_j \cap E_k}(t) L_x[|f(xt) - c_k|h(xtt_j)]$$

$$= \sum_{j,k} [L|f - c_k|h_{t_j}]\chi_{B_j \cap E_k}(t) \quad \text{by (L1)}.$$

The right side is a measurable function within 2ϵ of ϕ. Thus ϕ is measurable when $\|f\|_\infty < \infty$.

If $\|f\|_\infty = \infty$, which might happen in 2.3(h), define

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n, \\ nf(x)/|f(x)| & \text{otherwise.} \end{cases}$$

It is not difficult to see that $f_n \in \mathfrak{a}_L(G)$, in the context of 2.3(h). Define $\phi_n(t) = L_x|f_n(xt) - f_n(t)|h(x)$. Fix $t \in G$ arbitrarily. For large n, say $n \ge n_0$, $f_n(t) = f(t)$. Now

$$|\phi(t) - \phi_n(t)| = |L|f_t - f(t)|h - L|f_{nt} - f_n(t)|h|$$

$$= |L[|f_t - f(t)|h - |f_{nt} - f_n(t)|h]|,$$

since $F \in \mathfrak{a}_L(G)$, $H \in \mathfrak{a}(G)$ implies that $|F_t - F(t)|H \in \mathfrak{a}_L(G)$ for fixed t and L is linear on $\mathfrak{a}_L(G)$. Thus

$$|\phi(t) - \phi_n(t)| \le \|h\|_\infty |L[|f_t - f(t)| - |f_{nt} - f(t)|]|, \qquad n \ge n_0,$$

$$\le \|h\|_\infty L|f_t - f_{nt}| = \|h\|_\infty \|f - f_n\| \xrightarrow[n]{} 0,$$

by the usual considerations [4, p. 100]. By the above ϕ_n is measurable, hence so is ϕ. Thus (M) is satisfied and (A), (B) are proven as before.

If $f \in \mathfrak{F}$ satisfies (A), (B) and (M), then the argument that $f \in \mathfrak{a}_L(G)$ is as before. This proves the theorem.

4.14 **Remarks.** (1) Let G, $\mathfrak{a}(G)$, $\mathfrak{F}$, $\mathfrak{D}$, L be as in 2.3(d) or (g). $\mathfrak{a}_L(G)$ is the set of *bounded* Besicovitch AP functions on G and has been characterized above. The set of Besicovitch AP functions on G (with respect to L), say $\mathfrak{b}_L(G)$, is defined to be the $\| \ \|$-closure of $\mathfrak{a}(G)$ in $L_{1,\mathrm{loc}}(G)$. Take $f \in L_{1,\mathrm{loc}}(G)$. Define

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n, \\ nf(x)/|f(x)| & \text{otherwise, } n = 1, 2, \cdots. \end{cases}$$

Then $f \in B_L(G)$ iff each f_n satisfies (A), (B), (M) and $\|f_n - f\| \xrightarrow[n]{} 0$. (M) may be omitted in the case of 2.3(d).

$(B_L(G), \| \|)$ may not be complete even though the sets $\{V_d\}_{d \in D}$ defining L satisfy

$$\lim_{d \in D} \frac{\mu(xV_d \Delta V_d)}{\mu(V_d)} = 0 \quad \text{for all } x \in G.$$

The example in §5 of [8] illustrates this.(4) Even when $B_L(G)$ is not complete its members have Fourier series which are unique and may be "summed" to the original function. When $B_L(G)$ is complete, it is a copy of $L_1(\overline{G})$, where $\overline{G}$ is the Bohr compactification of G (see [8,(5) §§3, 4]; completeness is not assumed in §3).

(2) The Weyl AP functions on an LC group (2.3(h)) are characterized in 4.13. Another characterization along more classical lines is in [7]. The classical characterization on R may be found in [4]. Different characterizations of Weyl AP functions on R occur in [9] and [12] (cf. remarks on pp. 23, 25 of [12]).

(3) There are some interesting Besicovitch spaces for which 4.9 does not appear applicable. For example, suppose $(G, \mathfrak{T})$ is a noncommutative σ-compact LC group satisfying at least one of the following:

(i) G is amenable,

(ii) G is first countable and nondiscrete,

(iii) G is free and discrete.

Then there is a sequence of open bounded subsets $\{V_n\}_1^\infty$, increasing to G such that

$$\lim_{n \to \infty} \frac{1}{\mu(V_n)} \int_{V_n} f \, d\mu = Mf$$

for all $f \in AP(G, \mathfrak{T})$ (cf., [11, 3.2]; [6]). If we define L on $L_{1,\mathrm{loc}}(G)^r$ (or even on its bounded members) by

$$Lf = \overline{\lim_{n \to \infty}} \frac{1}{\mu(V_n)} \int_{V_n} f \, d\mu,$$

(4) The sets V_d defined there are not open. They may be assumed open by taking the $W(n, \alpha)$'s to be open and using approximation considerations in case 1 of §5.

(5) In §3 of [8] $\sigma(G)$ should be defined as $[h]$, where h is any AP functions on G whose nonzero Fourier matrices are $\{D^\lambda : \lambda \in \Sigma\}$. This is not equivalent to the definition given. A similar modification of the definition of $\sigma'(G)$ in the proof of 3.1 should be made.

(L1) may not hold. It is not apparent that one can get around this difficulty in the fashion used in 2.3(g), (h). The closure of $AP(G, \mathfrak{T})$ in $L_{1,\mathrm{loc}}(G)$ by means of L is complete. The resulting Besicovitch space is a copy of $L_1(\overline{G})$, where $\overline{G}$ is the Bohr compactification of G [8, 2.4 and §4].

5. **Weakly AP functions are contained in the generalized AP functions.** Let $(G, \mathfrak{T})$ be an LC group. Let $CB(G, \mathfrak{T})$ be the continuous bounded complex-valued functions on G and let $WAP(G, \mathfrak{T})$ be the weakly AP functions on G: $f \in WAP(G, \mathfrak{T})$ iff $f \in CB(G, \mathfrak{T})$ and $O(f) = \{{}_xf : x \in G\}$ is relatively compact in the weak topology of $(CB(G, \mathfrak{T}), \| \|_\infty)$. For $f \in WAP(G, \mathfrak{T})$, Mf denotes its mean value. We shall use such basic facts about $WAP(G, \mathfrak{T})$ as may be found in [3].

If $(G, \mathfrak{T})$ is amenable, there is a directed set $(V_d, d \in D, \geq)$ of μ-positive bounded (= having compact closure) subsets of G such that

(a1)
$$\lim_{d \in D} \frac{\mu(xV_d \Delta V_d)}{\mu(V_d)} = 0 \quad \text{for all } x \in G.$$

In addition one may require that

(b) $d_1 \leq d_2$ implies $V_{d_1} \subset V_{d_2}$;

(c) $\bigcup_{d \in D} V_d = G$; and

(d) each V_d is open and bounded.

See [11] and [15, p. 43]. There the V_d's are compact but regularity of μ allows one to substitute slightly larger open sets in the conditions (A_{loc}) of [11].

5.1 **Theorem.** *Let $(G, \mathfrak{T})$ be an LC group.*

(i) *There is a net $(V_d, d \in D, \geq)$ of subsets of G satisfying* (b), (c), (d) *and*

(a2)
$$\lim_{d \in D} \frac{1}{\mu(V_d)} \int_{V_d} f \, d\mu = Mf \quad \text{for all } f \in WAP(G, \mathfrak{T}).$$

(ii) *Every net $(V_d, d \in D, \geq)$ of μ-positive bounded sets satisfying* (a1) *also satisfies* (a2).

(iii) *If $(G, \mathfrak{T})$ is abelian (hence amenable) and $(V_d, d \in D, \geq)$ is a net of μ-positive bounded sets of G satisfying* (a1), *then*

(a3) *for every $f \in WAP(G, \mathfrak{T})$*
$$\lim_{d \in D} \frac{1}{\mu(V_d)} \int_{V_d} f_x \, d\mu = MF \quad \text{uniformly in } x \in G.$$

5.2 **Remark.** (i) answers a question raised by Greenleaf in [15, p. 43]. There it is pointed out that (ii) follows from the argument of [17, 18.10]. I do not know if (a2) is true uniformly with respect to (one or two-sided) translations of f. It is if $f \in AP(G, \mathfrak{T})$.

Proof of 5.1. We need only prove (i), (iii). We use the notation of [3, see especially p. 146]. Let ΩG be the weak compactification of G and $\omega_G : G \to \Omega G$

canonical. $M(\Omega G)$ denotes the minimal ideal of ΩG. Recall that ω_G imbeds G isomorphically and homeomorphically onto an open dense subset of ΩG. We often identify G with $\omega_G(G)$. We may as well assume G is noncompact.

To prove (i) we establish several propositions [A], [B], [C]:

[A] If $K \subset G$ is compact, $G \sim K$ is dense in $\Omega G \sim K$.

For otherwise there exists $x \in \Omega G \sim K$ such that $x \notin (G \sim K)^{-\Omega G}$. But then $x \in \overline{K}^{\Omega G} = \overline{K}^G = K$, a contradiction.

[B] Let $f_1, \cdots, f_n \in WAP(G, \mathfrak{T})$ and let U be a bounded open neighborhood of e in G. Take $N, \epsilon > 0$. There exists *distinct* $a_1, \cdots, a_t \in G$ such that

$$\left|\frac{1}{t}\sum_{i=1}^{t} f_k(a_i) - Mf_k\right| < \epsilon, \quad 1 \leq k \leq n, \ t \geq N, \text{ and } a_iU \cap a_jU = \emptyset \text{ if } i \neq j.$$

To prove [B] notice that each $f_k/M(\Omega G)$ is AP since it is continuous on $M(\Omega, G)$, a compact group. We apply IV.1.1(f) of [3]. If $M(\Omega G)$ is finite, say with cardinality p, let $b_1, \cdots, b_t$ be a listing of the elements of $M(\Omega G)$ such that each element occurs q times in the list and $pq \geq N$. If $M(\Omega G)$ is infinite, apply 3.3 of [6] to obtain $b_1, \cdots, b_t \in M(\Omega G)$ such that $t \geq N$ and

$$\left|\frac{1}{t}\sum_{i=1}^{t} f_k(b_i) - Mf_k\right| < \epsilon/2, \quad 1 \leq k \leq n.$$

By continuity of f_k on ΩG and density of G in ΩG there exists $a_1 \in G$ such that $|f_k(b_1) - f_k(a_1)| < \epsilon/2$, $1 \leq k \leq n$. By [A] and continuity of each f_k, there exists $a_2 \in G$ such that $a_2U \cap a_1U = \emptyset$ (take $a_2 \in G \sim a_1(UU^{-1})^-$) and $|f_k(b_2) - f_k(a_2)| < \epsilon/2$, $1 \leq k \leq n$. (Recall that $M(\Omega G)$ and $\omega_G(G)$ are disjoint.) Continuing inductively one gets $a_1, \cdots, a_t \in G$ which evidently satisfy the required conditions.

[C] Let $f_1, \cdots, f_n \in WAP(G, \mathfrak{T})$ and take $M, \epsilon > 0$. There exists a bounded open set $B \subset G$ such that $\mu(B) \geq M$ and

$$\left|Mf_k - \frac{1}{\mu(B)}\int_B f_k\, d\mu\right| < 2\epsilon, \quad 1 \leq k \leq n.$$

To prove this use uniform continuity of the f_k to get a bounded open neighborhood U of e in G such that $|f_k(xu) - f_k(x)| < \epsilon$, $1 \leq k \leq n$, whenever $x \in G$, $u \in U$. Take N such that $N\mu(U) \geq M$ and let $a_1, \cdots, a_t \in G$ be as in [B]. It is not difficult to see that $B = \bigcup_{i=1}^{t} a_iU$ has the required properties.

To finish the proof of (i), one applies the arguments of 3.4 in [6], replacing $\alpha(G)$ and 3.3 in [6] by $WAP(G, \mathfrak{T})$ and [C] above.

To prove (iii) consider G as acting on $(WAP(G, \mathfrak{T}), \| \|_\infty)$ via left translation. The elements of $WAP(G, \mathfrak{T})$ are ergodic in the sense of Eberlein (Definition 3.1 of [10]). Thus it suffices to show that the transformations T_d defined on $WAP(G, \mathfrak{T})$ by

$$(T_d g)(t) = \frac{1}{\mu(V_d)} \int_{V_d t} g \, d\mu, \qquad d \in D, \ t \in G,$$

are a system of almost invariant integrals (in the sense of Eberlein's Definition 2.1 of [10]). It is clear that each T_d: $WAP(G, \mathfrak{T}) \to CB(G, \mathfrak{T})$ is linear and that $\{T_d\}_{d \in D}$ is equicontinuous. That $T_d g \in [\text{convex hull of } O(g)]^- \subset WAP(G, \mathfrak{T})$ for each $g \in WAP(G, \mathfrak{T})$ is not difficult to show from the uniform continuity of g (one approximates $(1/\mu(V))\int_{V} {}_x g \, d\mu$ by $\sum_{i=1}^n ((\mu(V_i)/\mu(V)) g(xt_i))$. Finally we must show that for each $g \in WAP(G, \mathfrak{T})$, $x \in G$,

$$\lim_{d \in D} \|{}_x(T_d g) - T_d g\|_\infty = \lim_{d \in D} \|T_d({}_x g) - T_d g\|_\infty = 0.$$

Now

$${}_x(T_d g)(t) = [T_d({}_x g)](t) = \frac{1}{\mu(V_d)} \int_{V_d} {}_{tx} g \, d\mu,$$

since G is abelian. Thus

$$|{}_x(T_d g)(t) - (T_d g)(t)| = |[T_d({}_x g)](t) - (T_d g)(t)| \le \frac{\|g\|_\infty \mu(xV_d \,\Delta\, V_d)}{\mu(V_d)} \xrightarrow[d \in D]{} 0$$

uniformly in $t \in G$. This proves the theorem.

5.3 **Remark.** We may write $WAP(G, \mathfrak{T}) = AP(G, \mathfrak{T}) \oplus WAP(G, \mathfrak{T})_0$ where $WAP(G, \mathfrak{T})_0 = \{f \in WAP(G, \mathfrak{T}): M|f|^2 = 0\}$. Since, for each $f \in WAP(G, \mathfrak{T})$, Mf may be uniformly approximated by finite convex sums of the form

$$(Sf)(x) = \sum_r \alpha_r f(xa_r) \qquad \left(a_r \in G;\ \alpha_r > 0,\ \sum \alpha_r = 1\right)$$

and since $\|f\|_\infty S|f| \ge S|f|^2 \ge (S|f|)^2$, we may write $WAP(G, \mathfrak{T})_0 = \{f \in WAP(G, \mathfrak{T}): M|f| = 0\}$. Let $\Phi = (V_d,\ d \in D, \ge)$ be a net μ-positive bounded set satisfying condition (a2) of 5.1. Define $W(G, \Phi)$, $B(G, \Phi)$ to be corresponding spaces of Weyl and Besicovitch AP functions on G, obtained by closing $AP(G, \mathfrak{T})$ in $L_{1,\text{loc}}(G)$ with the seminorm of 2.3(h), 2.3(g) respectively. By allowing (a2) rather than (a1), we allow $B(G, \Phi)$ to be more general than in 2.3(g). The characterization Theorems 4.9, 4.13 and 4.14(1) may not apply to $B(G, \Phi)$ unless G is abelian and (a1) holds. The elements of $W(G, \Phi)$ and $B(G, \Phi)$ have mean values given by

$$(\dagger) \qquad M_\Phi f = \lim_{d \in D} \frac{1}{\mu(V_d)} \int_{V_d} f \, d\mu.$$

Define $W(G, \Phi)_0 = \{f \in W(G, \Phi): M_\Phi |f| = 0\}$ and similarly for $B(G, \Phi)_0$. For $f \in W(G, \Phi)$, the limit in $(\dagger)$ is uniform with respect to translation of f (see, for example, the "proposition" of [9]). Hence we may write $W(G, \Phi)_0 = \{f \in L_{1,\text{loc}}(G): \|f\| = 0\}$. Here $\|\ \|$ is the seminorm defined in 2.2 from L as defined in 2.3(h). Similarly, $B(G, \Phi)_0 = \{f \in L_{1,\text{loc}}(G): \|f\| = 0\}$ for $\|\ \|$ as in 2.3(g).

The following extends Eberlein's theorem [10, #16] concerning the Weyl almost periodicity of weakly AP functions on R. It follows from 5.1 and the above discussion.

5.4 **Corollary to 5.1.** *Let* $(G, \mathfrak{T})$ *be an LC group and let* $\Phi = (V_d, d \in D, \geq)$ *be as above. Assume* (a2) *of* 5.1 *holds. Then* $WAP(G, \mathfrak{T})_0 \subset B(G, \Phi)_0$ *and hence* $WAP(G, \mathfrak{T}) \subset B(G, \Phi)$. *If* (a3) *holds, then* $WAP(G, \mathfrak{T})_0 \subset W(G, \Phi)_0$ *and hence* $WAP(G, \mathfrak{T}) \subset W(G, \Phi)$.

5.5 **Remarks.** (1) Suppose $(G, \mathfrak{T})$ is of the form $R^a \times Z^b \times F$, where F is compact and $Z = \{0, \pm 1, \cdots\}$ (see 2.3(c)). Let $V_n = (-n, n)^{a+b} \times F$, $n = 1, 2, \cdots$, in 5.1. In this case (a3) of 5.1 is satisfied even when G is not abelian. Thus in this case $WAP(G, \mathfrak{T}) \subset W(G, \Phi)$.

(2) One cannot expect $WAP(G, \mathfrak{T})_0 = W(G, \Phi)_0 \cap CB(G, \mathfrak{T})$ even when G is abelian. For example, suppose $G = \{0, \pm 1, \cdots\}$, $\mathfrak{T}$ is discrete and $\Phi = (\{1, \cdots, n\}, n = 1, 2, \cdots, \geq)$. Define $E = \{3^j - 2^i : j = 1, 2, \cdots; 1 \leq i \leq j\}$ and let χ_E be the indicator of E. Then $\chi_E \in W(G, \Phi)_0 \cap CB(G, \mathfrak{T})$. But

$$\lim_i \lim_j \chi_E(3^j - 2^i) = 1, \qquad \lim_i \lim_j \chi_E(3^j - 2^i) = 0$$

so $\chi_E \notin WAP(G, \mathfrak{T})$ [16, Proposition 7].

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