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CORRECTION TO "THE SEPARABLE CLOSURE OF SOME COMMUTATIVE RINGS"

BY

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In [3, 1.4, p. 112] we attempted to prove that every closed subgroupoid of a profinite groupoid is an intersection of open-closed subgroupoids. There is, however, an error in the proof: It is asserted that if G is a groupoid, H a subgroupoid and $f: G \to G'$ a groupoid homomorphism, then $f^{-1}(f(H))$ is a subgroupoid of G. This will not happen, however, if $f(H)$ is not a subgroupoid of G', and such things can occur, as pointed out in [1, p. 7]. The intersection theorem is, however, an interesting part of the Galois theory of commutative rings [2, 1.10, p. 96] and also plays a role in some subsequent developments. Thus we give a correction here.

We begin by recalling the relevant definitions. A groupoid is a category in which all morphisms are isomorphisms. A subgroupoid is a subcategory closed under inversion and containing all identities. A homomorphism of groupoids is a functor. A profinite groupoid is a (topological) inverse limit of finite groupoids, over a directed index set, where all transition maps are surjective (so two elements of a profinite groupoid are equal if and only if their homomorphic images in all finite groupoids are equal). A groupoid is connected if there is a map between any two of its objects, and every groupoid is a co-product of its maximal connected subgroupoids, which are called its (connected) components.

If G is a group and X a set, then $X \times X \times G$, with composition $(a, b, g) \cdot (b, c, h) = (a, c, gh)$ is a groupoid. As a first step towards the intersection theorem, we will show that every profinite groupoid is a subgroupoid of a groupoid of this type.

Let $\mathcal{G}$ be a profinite groupoid. We will find a profinite group $F(\mathcal{G})$ and a continuous functor $f: \mathcal{G} \to F(\mathcal{G})$ universal with respect to continuous functors from $\mathcal{G}$ to profinite groups: First, take the free group on the set $\mathcal{G}$. Let F_0 be the (topological) inverse limit of all the (discrete) quotients of the free group by normal subgroups of finite index each of whose cosets meets $\mathcal{G}$ in an open-closed set. Let h map $\mathcal{G}$ to F_0 and let $F(\mathcal{G})$ be the quotient of F_0 by the closed normal subgroup generated by all $h(s)h(t)h(st)^{-1}$ where s, t are composable maps in $\mathcal{G}$. (This construction is given in detail in [4, Definition 3].)

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To what extent are elements of $\mathcal{G}$ identified by the functor $\mathcal{G} \to F(\mathcal{G})$? The following lemma will lead to an answer to this question:

Lemma 1 *Let $\mathcal{G}$ be a connected finite groupoid, s, t distinct elements of $\mathcal{G}$ with s not an identity. Then there is a finite group G and a functor $h: \mathcal{G} \to G$ such that $h(s) \neq h(t)$.*

Proof. Since $\mathcal{G}$ is connected, the cardinality of all its hom-sets are equal. Suppose it is greater than 1. Let a, c and b, d be the domains and ranges of s, t respectively. Let G be the automorphism group of a. If a, b, c, d are all different, construct a functor $h: \mathcal{G} \to G$ as follows: for each object x of $\mathcal{G}$ choose a map f_x from a to x such that f_a is the identity, $f_b = s$, f_c is arbitrary and $f_d \neq tf_c$. If r in $\mathcal{G}$ has range and domain x and y, let $h(r) = f_y^{-1} r f_x$. Then $h(s) \neq h(t)$. (In case a, b, c, d are not all different, a suitable modification of the choices of the f_x's gives a similar functor.)

Now suppose there is a unique map $[x, y]$ between each pair of objects x, y of $\mathcal{G}$. Let G be the free $Z/Z3$-module on the objects of $\mathcal{G}$, and define $h: \mathcal{G} \to G$ by $h[x, y] = x - y$. Then h is a functor and if $(x, y) \neq (z, w)$ and $x \neq y$, $h[x, y] \neq h[z, w]$. So $h(s) \neq h(t)$.

The condition of Lemma 1 is certainly necessary, as all identities of a groupoid go to the identity of a group under any functor.

Lemma 2. *Let $\mathcal{G}$ be a profinite groupoid and s, t distinct elements of $\mathcal{G}$ with s not an identity. Then s and t have distinct images in $F(\mathcal{G})$.*

Proof. By the universal property of $F(\mathcal{G})$, it will be sufficient to find a finite group G and a continuous functor $\mathcal{G} \to G$ under which s and t have distinct images. Since $\mathcal{G}$ is profinite, there is a finite groupoid $\mathcal{G}'$ and a continuous functor $\mathcal{G} \to \mathcal{G}'$ under which the images of s and t are distinct and the former not an identity. So we may assume $\mathcal{G} = \mathcal{G}'$ is finite.

Suppose first that s and t are in different components. Use Lemma 1 to find a functor from the component of s to a finite group G such that the image of s is not the identity, and extend the functor to all of $\mathcal{G}$ by sending all other components to the identity. The functor then separates s and t.

If s and t are in the same component, apply Lemma 1 to that component and extend the functor trivially to all of $\mathcal{G}$; it still separates s and t.

Now let X be the (profinite) set of objects of the profinite groupoid $\mathcal{G}$. There is a continuous function $\mathcal{G} \to X \times X \times F(\mathcal{G})$ where the first two maps are the domain and range projections and the last is the functor defined above. Give $X \times X \times F(\mathcal{G})$ the groupoid structure we did above; then the function is seen to be a functor. It is immediate from Lemma 2 that the functor is injective, and so we have:

Proposition 3. *Every profinite groupoid is (isomorphic to) a closed subgroupoid of a profinite groupoid of the form* $X \times X \times G$*, where* X *is a profinite topological space and* G *a profinite group.*

This structural result will play a role in the proof of the intersection theorem, which we now state.

Theorem 4. *Let* $\mathcal{G}$ *be a profinite groupoid and* H *a closed subgroupoid. Then* H *is an intersection of open-closed subgroupoids of* $\mathcal{G}$.

Proof. By Proposition 3, we may assume $\mathcal{G} = X \times X \times G$, where X is a profinite space and G a profinite group. We next show that we can even take G to be finite: suppose s is in $\mathcal{G}$ but not in H. There is a profinite group G' and a continuous homomorphism $G \to G'$ such that the image of s is not in the image of H under the induced functor $X \times X \times G \to X \times X \times G'$. Since this map is injective on identities, the image of H is a subgroupoid [1, Proposition 1, p. 8]. Suppose there is an open-closed subgroupoid of $X \times X \times G'$ containing the image of H but not that of s. Then its inverse image is an open-closed subgroupoid of $X \times X \times G$ containing H but not s. Thus if we can prove Theorem 4 when $G = G'$ is finite, it will hold in general.

Now we assume G finite, and introduce some notation: If U, V are subsets of X, let $H(U, V) = \{g \in G \mid (a, b, g) \in H$ for some $a \in U$ and $b \in V\}$. Note that if $U' \subset U$ and $V' \subset V$ then $H(U', V') \subset H(U, V)$. We need some facts about these sets, beginning with:

For all a, b in X, $H(a, b) = \bigcap H(U, V)$, where the intersection ranges over all neighborhoods U of a and V of b: As we just noted, the left-hand side is contained in the right. If g is not in $H(a, b)$, then (a, b, g) is not in H, and since H is closed, some neighborhood of (a, b, g) misses H. This means there are neighborhoods U of a and V of b such that $U \times V \times g$ misses H, i.e., that g is not in $H(U, V)$. So g is not in the right-hand side.

Using the above plus the fact that G is finite, we can choose for each a and b in X neighborhoods $U(a)$ and $V(b)$ of a and b such that $H(a, b) = H(U(a), V(b))$. Let $W(a) = U(a) \cap V(a)$; we then have $H(a, b) = H(W(a), W(b))$. Since X is a profinite, hence zero-dimensional, space we can refine the open cover $\{W(a)\}$ by a partition $\{U_1, \cdots, U_n\}$ of X into open-closed sets, such that for each i there is an a_i in U_i with $U_i \subseteq W(a_i)$. If we are given a, b in X in advance, we can do the above so that a and b are among the a_i's. Thus we have shown:

Let a, b in X be given. Then there is a partition $\{U_1, \cdots, U_n\}$ of X into open-closed sets and elements a_i of U_i such that $a = a_i$ for some i and $b = a_j$ for some j, and $H(a_i, a_j) = H(U_i, U_j)$.

Now we can complete the proof of Theorem 4. Let (a, b, g) belong to $X \times X \times G$ but not to H. Choose a partition as above relative to this a and b. Let $H_{ij} = H(U_i, U_j)$, and let $K = \bigcup U_i \times U_j \times H_{ij}$ (the union being over all pairs i, j). We will show that K is a subgroupoid of G containing H but not (a, b, g). Since K is open-closed, this will finish the proof. Since $H(a_i, a_j) \cdot H(a_j, a_k) \subseteq H(a_i, a_k)$, $H_{ij} \cdot H_{jk} \subset H_{ik}$, and it follows that K is a groupoid. Since (a, b, g) is not in H, g is not in $H(a, b)$ and hence (a, b, g) is not in K. If (c, d, b) is in H, with c in U_i and d in U_j, then $b \in H(c, d) \subset H(U_i, U_j) = H_{ij}$, so $(c, d, b) \in U_i \times U_j \times H_{ij}$ which is contained in K, and K contains H.

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