TRANSATIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 191, 1974

FREE S^1 ACTIONS AND THE GROUP OF DIFFEOMORPHISMS

BY

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ABSTRACT. Let S^1 act linearly on $S^{2p-1} \times D^{2q}$ and $D^{2p} \times S^{2q-1}$ and let $f: S^{2p-1} \times S^{2q-1} \to S^{2p-1} \times S^{2q-1}$ be an equivariant diffeomorphism. Then there is a well-defined S^1 action on $S^{2p-1} \times D^{2q} \cup_f D^{2p} \times S^{2q-1}$. An S^1 action on a homotopy sphere is decomposable if it can be obtained in this way. In this paper, we will apply surgery theory to study in detail the set of decomposable actions on homotopy spheres.

0. **Introduction.** Let S^1 act linearly on C^{m+1} by $(g, (u_0, \cdots, u_m)) = (gu_0, \cdots, gu_m)$ for $g \in S^1$ and $(u_0, \cdots, u_m) \in C^{m+1}$. Let $p = [(m+1)/2]$ and $q = m + 1 - p$. It is clear that $S^{2p-1} \times D^{2q}$, $D^{2p} \times S^{2q-1}$ and $S^{2p-1} \times S^{2q-1}$ are invariant subspaces. Let A denote the induced actions. Let f be an equivariant diffeomorphism of $(S^{2p-1} \times S^{2q-1}, A)$. We can define an action $A(f)$ on $\Sigma(f)$ where

$$\Sigma(f) = S^{2p-1} \times D^{2q} \cup_f D^{2p} \times S^{2q-1}$$

so that $A(f) | S^{2p-1} \times D^{2q} = A$ and $A(f) | D^{2p} \times S^{2q-1} = A$. A free S^1 action (Σ^{2m+1}, F) on a homotopy sphere Σ^{2m+1} is decomposable if there is an equivariant diffeomorphism f of $(S^{2p-1} \times S^{2q-1}, A)$ such that (Σ^{2m+1}, F) is equivalent to $(\Sigma(f), A(f))$. It is clear that if f is equivariantly pseudo-isotopic to g, then $(\Sigma(f), A(f))$ is equivalent to $(\Sigma(g), A(g))$. Hence the study of decomposable actions is reduced to the study of the group of equivariant pseudo-isotopy classes of equivariant diffeomorphisms of $(S^{2p-1} \times S^{2q-1}, A)$ or, equivalently, the group of diffeomorphisms of $S^{2p-1} \times S^{2q-1}/A$. Let $(\Sigma(f), A(f))$ and $(\Sigma(g), A(g))$ be two decomposable actions. We define $(\Sigma(f), A(f)) * (\Sigma(g), A(g)) = (\Sigma(f \cdot g), A(f \cdot g))$. We will show this is well defined and makes the set of decomposable free S^1 actions a group such that the splitting invariants are homomorphisms. Furthermore, we are able to calculate its rank and determine its torsion elements. As applications we have

Received by the editors March 5, 1973 and, in revised form, July 19, 1973.

AMS (MOS) subject classifications (1970). Primary 57E30.

Key words and phrases. Decomposable actions, splitting invariants, surgery, tangential homotopy complex projective spaces, characteristic spheres.

Theorem 5.1. *There are infinitely many topologically inequivalent free S^1 actions on homotopy $(2n+1)$-spheres with characteristic homotopy $(2k+1)$-spheres for $n \geq k \geq 3$ and* (a) *$n \geq 6$ if n is even,* (b) *$n \geq 5$ if both n and k are odd, and* (c) *$n \geq 7$ if n is odd but k is even.*

Theorem 5.6. *For $n \geq 1$, there are infinitely many topologically inequivalent free S^1 actions on homotopy $(4n+3)$-spheres which are not extendible to free S^3 actions.*

This paper is organized as follows: In §1, we outline the theory of surgery on simply-connected smooth manifolds. In §2, we study the group of pseudo-isotopy classes of diffeomorphisms of simply-connected manifolds which are homotopic to the identity. Let $s_{2k}(P)$ be the splitting invariants of a homotopy complex projective space P. Suppose f is a diffeomorphism of $S^{2p-1} \times S^{2q-1}/A$. Let $\theta_{2k}(f) = s_{2k}(\Sigma(f)/A(f))$. Then we study the splitting invariants of $\mathfrak{D}_0(S^{2p-1} \times S^{2q-1}/A)$ and will show θ_{2k} are homomorphisms. In §4, we will prove the main result that the set of decomposable actions is a group such that the splitting invariants are homomorphisms. In §5, we will prove Theorem 5.1 and Theorem 5.6.

1. Surgery on simply-connected smooth manifolds. In this section we outline the theory of surgery on simply-connected smooth manifolds as developed in [1].

Let X^m be a compact oriented smooth manifold with boundary (possibly empty) of dimension m, and let ξ^k be a linear k-plane bundle over X^m, k large. A normal map into X^m is a pair of maps $f: (M^m, \partial M^m) \to (X^m, \partial X^m)$, $b: \nu^k \to \xi^k$, where M^m is a compact oriented smooth manifold, ν^k is its normal bundle in a high dimensional euclidean space, f is of degree 1 (i.e., $f_*[M] = [X]$, where $[M]$, $[X]$ are the orientation classes, $[M] \in H_m(M^m)$, etc.) and b is a bundle map covering f.

A normal cobordism of a normal map (f, b) is a pair (F, B), where $F: W^{m+1} \to X \times [0, 1]$, $\partial W = M \cup \partial M \times I \cup M'$, $F \mid M = f: M \to X \times 0$ and $F \mid (x, t) = (f(x), t)$ for $x \in \partial M$. If ω^k is the normal bundle of $(W, \partial M \times I) \subseteq (D^{m+k} \times I, S^{m+k-1} \times I)$ (where $(M, \partial M) \subseteq (D^{m+k} \times 0, S^{m+k-1} \times 0)$, $(M', \partial M') \subseteq (D^{m+k} \times 1, S^{m+k-1} \times 1)$) B is a linear bundle map, $B: \omega \to \xi$ lying over F such that $B \mid (\omega \mid M) = B \mid \nu = b$ and $B \mid (u, t) = (b(u), t)$.

We note that any homotopy equivalence $g: M \to X$ can be made into a normal map by taking $\xi = g^{-1*}(\nu)$ and taking a bundle map covering g, but the bundle map is only well defined up to a bundle automorphism of ξ or ν.

Lemma 1.1. *The set of normal cobordism classes of normal maps into $(X, \partial X)$, ξ for all choices of ξ, modulo bundle automorphisms of ξ, is in one-to-one correspondence with the set of homotopy classes $[X/\partial X, G/O]$, where $G = \lim_{k\to\infty}((S^k)^{S^k}$, degree $\pm 1)$ and $O = \lim_{k\to\infty} O(k+1)$.*

We refer to [1] for a proof.

Let X^m be a compact oriented smooth manifold of dimension m with boundary (possibly empty) and consider pairs (M^m, h), where M is an oriented smooth manifold and $h: (M, \partial M) \to (X, \partial X)$ is a homotopy equivalence of pairs such that $h \mid \partial M$ is a diffeomorphism. Two pairs (M_1, f_1) and (M_2, f_2) are equivalent if there is a pair (W, H), where $\partial W = M_1 \cup \partial M_1 \times I \cup M_2$ and $H: W \to X \times I$ is a homotopy equivalence such that $H \mid \partial M_1 \times I: \partial M_1 \times I \to \partial M_1 \times I$ is a diffeomorphism and $H \mid M_i = f_i$, $i = 1, 2$. Let $[M, f]$ be the equivalence class of (M, f). The set of equivalence classes of pairs will be called the set of homotopy smoothings on X, $hS(X, \partial X)$.

A. *Exact sequence of surgery.* Let $m = \dim X \geq 5$ and $P_m = Z, 0, Z_2, 0$ if $m \equiv 0, 1, 2, 3 \pmod 4$ respectively. Then there are maps ω, η, σ such that

$$\cdots \xrightarrow{\omega} hS(X \times I, \partial(X \times I)) \xrightarrow{\eta} [\Sigma(X/\partial X), G/O]$$

$$\xrightarrow{\sigma} P_{m+1} \xrightarrow{\omega} hS(X, \partial X) \xrightarrow{\eta} [X/\partial X, G/O] \xrightarrow{\sigma} P_m$$

is an exact sequence of sets (of groups from P_{m+1} to the left).

This theorem includes much of the theory of surgery as developed by Milnor, Kervaire, Novikov, Browder and Sullivan.

B. *Properties of σ, the surgery obstruction.* (a) If $M = M_1 \cup M_2$, $X = X_1 \cup X_2$ union of submanifolds along a submanifold of codimension 0 of the boundary, M_i and X_i are simply connected, (f, b) is a normal map $f: M \to X$, $f \mid M_1 \cap M_2 \to X_1 \cap X_2$ is a homotopy equivalence, and if σ is defined for all maps involved, then

$$\sigma(f, b) = \sigma(f|M_1, b|M_1) + \sigma(f|M_2, b|M_2).$$

(b) Let K^{4n-1} be a closed simply-connected manifold of dimension $4n-1$. Then

$$\sigma: [K \times I/\partial(K \times I), G/O] \to Z$$

is equal to

$$\sigma(f) = (1/8)\langle L(K \times I)(L(\xi) - 1), [K \times I]\rangle$$

where $f: K \times I/\partial(K \times I) = \Sigma K \to G/O$ and ξ is the bundle induced by $K \times I \xrightarrow{f} G/O \to BO$.

Since $L(K \times I)$ and $L(\xi)$ are classes in $H^*(\Sigma K)$, ΣK being a suspension, products of positive dimensional classes are zero. Hence

$$\begin{aligned}\sigma(f) &= (1/8)\langle L(K \times I)L(\xi) - L(K \times I), [K \times I]\rangle \\ &= (1/8)\langle L(\xi) - 1, [K \times I]\rangle = (1/8)\langle L_n(\xi), [K \times I]\rangle \\ &= c\langle p_n(\xi), [K \times I]\rangle \quad \text{for some constant } c,\end{aligned}$$

where L_n is the nth Hirzebruch class and p_n is the nth Pontrjagin class and L is the total Hirzebruch class.

2. $hS(K \times I, \partial)$ and $\mathfrak{D}_0(K)$. Let K be a closed simply-connected smooth manifold. Let $[M, H] \in hS(K \times I, \partial)$ where $H: M \to K \times I$ is a homotopy equivalence such that $H \mid \partial M: \partial M \to \partial(K \times I)$ is a diffeomorphism. $(M; \partial_0 M, \partial_1 M)$ is an h-cobordism. Hence by the h-cobordism theorem, M is diffeomorphic to $K \times I$ by a diffeomorphism D^{-1}. Then $[M, H] = [K \times I, H \cdot D]$. Let $h_0 = H \cdot D \mid K \times 0$ and $h_1 = H \cdot D \mid K \times 1$. Then

$$[K \times I, H \cdot D] = [K \times I, H \cdot D \cdot (h_i^{-1} \times \mathrm{id})], \quad i = 0, 1.$$

Note that $H \cdot D \cdot (h_i^{-1} \times \mathrm{id}) \mid K \times i = \text{identity}$. Hence in each class $u \in hS(K \times I, \partial)$ we may choose representative $u = [K \times I, H]$ such that either $H \mid K \times 0 = \text{identity}$ or $H \quad K \times 1 = \text{identity}$.

We now define a composition in $hS(K \times I, \partial)$ as follows. Let $[K \times I, F]$ and $[K \times I, G] \in hS(K \times I, \partial)$ such that $F \mid K \times 1 = \text{identity}$ and $G \mid K \times 0 = \text{identity}$. Let $F(x, t) = (F_1(x, t), F_2(x, t)) \in K \times I$ and $G(x, t) = (G_1(x, t), G_2(x, t)) \in K \times I$. Then we define

$$[K \times I, F] * [K \times I, G] = [K \times I, F * G]$$

where

$$F * G(x, t) = \begin{cases} (F_1(x, t), \tfrac{1}{2} F_2(x, t)) & \text{for } 0 \le t \le \tfrac{1}{2}, \\ (G_1(x, t), \tfrac{1}{2}(1 + G_2(x, t)) & \text{for } \tfrac{1}{2} \le t \le 1. \end{cases}$$

Lemma 2.1. *The composition* $*$ *is well defined.*

Proof. Suppose $[(K \times I)_0, F_i] = [(K \times I)_1, G_i]$, $i = 0, 1$, such that $F_i I K \times (1 - i) = \text{identity}$ and $G_i \mid K \times (1 - i) = \text{identity}$. There are $(W_i, \tilde{H}_i)$, where $\partial W_i = (K \times I)_0 \cup \partial(K \times I) \times I \cup (K \times I)_1$ and $\tilde{H}_i: W \to (K \times I) \times I$ are homotopy equivalences such that $\tilde{H}_i \mid \partial(K \times I) \times I$ are diffeomorphisms and $\tilde{H}_i \mid (K \times I)_0 = F_i$, $\tilde{H}_i \mid (K \times I)_1 = G_i$. In fact, we may choose that $W_i = (K \times I) \times I$. Let $W = W_0 \cup W_1$ by identifying $(K \times I) \times 1 \subset W_0$ with $(K \times I) \times 0 \subset W_1$. Let $\tilde{H}_{0,1}(x, s, t) = \tilde{H}_0(x, 1, t)$, $\tilde{H}_{1,0}(x, s, t) = \tilde{H}_1(s, 0, t)$ for $(x, s, t) \in (K \times I) \times I$. Let $\tilde{H}_i \cdot \tilde{H}_{i,1-i}^{-1} = (H_i^{(1)}, H_i^{(2)}, H_i^{(3)})$. Then we define

$$H(x, s, t) = \begin{cases} (H_0^{(1)}(x, s, t), H_0^{(2)}(x, s, t), \tfrac{1}{2} H_0^{(3)}(x, s, t)) & \text{for } 0 \le t \le \tfrac{1}{2}, \\ (H_1^{(1)}(x, s, t), H_1^{(2)}(x, s, t), \tfrac{1}{2}(1 + H_1^{(3)}(x, s, t))) & \text{for } \tfrac{1}{2} \le t \le 1. \end{cases}$$

Then $\partial W = (K \times I)_0 \cup \partial(K \times I) \times I \cup (K \times I)_1$ and $H \mid (K \times I)_0 = F_0 * F_1$, $H \mid (K \times I)_1 = G_0 * G_1$. Hence (W, H) gives the required equivalence of $F_0 * F_1$ and $G_0 * G_1$.

Lemma 2.2. *Suppose* $\dim K = 2k-1$, *then* $(hS(K \times I, \partial), *)$ *is an abelian group.*

Proof. It is well known that G/O is an H-space and $[\Sigma K, G/O]$ is an abelian group. It is clear that $\eta(u * v) = \eta(u) + \eta(v)$. Now the lemma follows from the fact that η is a monomorphism when $\dim K$ is odd.

Let $\mathfrak{D}_0(K)$ be the group of pseudo-isotopy classes of diffeomorphisms of K which are homotopic to the identity. Define a map ψ: $hS(K \times I, \partial) \longrightarrow \mathfrak{D}_0(K)$ as follows. For $u \in hS(K \times I, \partial)$, let $u = [K \times I, F]$ such that $F \mid K \times 0 =$ identity. We then define $\psi(u) = \{(F \mid K \times 1)^{-1}\}$.

Lemma 2.3. ψ: $hS(K \times I, \partial) \longrightarrow \mathfrak{D}_0(K)$ *is well defined and is an epimorphism.*

Proof. Suppose $[K \times I, F] = [K \times I, G]$ and $F \mid G \times 0 =$ identity, $G \mid K \times 0 =$ identity. It is well known [6] that there is a diffeomorphism D: $K \times I \longrightarrow K \times I$ such that $D \cdot F$ is homotopic to G rel $\partial(K \times I)$. Hence $D \cdot F \mid \partial(K \times I) = G \mid \partial(K \times I)$. Let $f = F \mid K \times 1$ and $g = G \mid K \times 1$. It is clear that $D \mid K \times 0 =$ identity. Let $d = D \mid K \times 1$. Then d is pseudo-isotopic to the identity. Now it follows from $d \cdot f = g$ that f is pseudo-isotopic to g.

For $u, v \in hS(K \times I, \partial)$. Let $u = [K \times I, F]$ and $v = [K \times I, G]$ such that $F|K \times 0 =$ identity and $G|K \times 0 =$ identity. Let $f = F|K \times 1$ and $g = G|K \times 1$. $[K \times I, F] = [K \times I, F \cdot (f^{-1} \times \mathrm{id})]$. Then

$$[K \times I, F] * [K \times I, G] = [K \times I, \; F \cdot (f^{-1} \times \mathrm{id}) * G]$$

$$= [K \times I, \; ((F \cdot (f^{-1} \times \mathrm{id})) * G) \cdot (f \times \mathrm{id})].$$

Note that $((F \cdot (f^{-1} \times \mathrm{id})) * G) \cdot (f \times \mathrm{id})|K \times 0 =$ identity and $((F \cdot (f^{-1} \times id)) * G) \cdot (f \times \mathrm{id})|K \times 1 = g \cdot f$. Hence

$$\psi(u * v) = \{(g \cdot f)^{-1}\} = \{f^{-1}\}\{g^{-1}\} = \psi(u) \cdot \psi(v).$$

Let $\{f\} \in \mathfrak{D}_0(K)$. There is a map H: $K \times I \longrightarrow K \times I$ such that $H|K \times 0 = f$ and $H|K \times 1 =$ identity. It is clear that $H \cdot (f^{-1} \times \mathrm{id})$ is a homotopy equivalence and $H \cdot (f^{-1} \times \mathrm{id})|\partial(K \times I)$ is a diffeomorphism. Hence $[K \times I, H \cdot (f^{-1} \times \mathrm{id})] \in hS(K \times I, \partial)$ and $\psi([K \times I, H \cdot (f^{-1} \times \mathrm{id})]) = \{f\}$.

Corollary 2.4. *Let K be a closed simply-connected smooth manifold of dimension $2k-1$. Then $\mathfrak{D}_0(K)$ is an abelian group.*

Proof. Since $(hS(K \times I, \partial), *)$ is abelian and ψ is an epimorphism.

Proposition 2.5. *Let K be a closed simply-connected smooth manifold of dimension $2k-1$. Then* $\operatorname{Ker}\psi$ *is finite.*

Proof. Consider the following diagram

$$\begin{array}{ccc} \operatorname{Ker}\psi & & [\Sigma K_+, G] \\ \downarrow & & \downarrow i \\ 0 \to hS(K \times I, \partial) & \xrightarrow{\eta} & [\Sigma K_+, G/O] \\ \downarrow \psi & & \downarrow j \\ \mathfrak{D}_0(K) & & [\Sigma K_+, BO]. \end{array}$$

For $u \in \operatorname{Ker}\psi$, let $u = [K \times I, H]$ such that $H|K \times 0 =$ identity and $H|K \times 1 =$ identity. Hence H induces a homotopy equivalence h of $K \times S^1$ and a homotopy equivalence $\bar{h}$ of ΣK_+. Note that $h^*\tau(K \times S^1) + \tau(K \times S^1)^{-1}|K \times s$ is trivial. Hence it follows that

$$0 \to KO(\Sigma K_+) \xrightarrow{\phi} KO(K \times S^1) \to KO(K \times s) \to 0;$$

there is a unique $\gamma \in KO(\Sigma K_+)$ such that

$$\phi(\gamma) = h^*\tau(K \times S^1) + \tau(K \times S^1)^{-1}.$$

It is clear that $\gamma = j \cdot \eta([K \times I, H])$.

If $\operatorname{Ker}\psi$ were infinite, there would be infinitely many H such that h^*: $H^{4*}(K \times S^1, Z) \to H^{4*}(K \times S^1, Z)$ is the identity map. Let p be the total Pontrjagin class.

$$p(h^*\tau(K \times S^1)) = h^*p\tau(K \times S^1) = p(K \times S^1).$$

Then $p(h^*\tau(K \times S^1) + \tau(K \times S^1)^{-1}) = 0$. But the Pontrjagin class is a complete invariant modulo torsion [2]. Hence $h^*\tau(K \times S^1) + \tau(K \times S^1)^{-1}$ is a torsion in $KO(K \times S^1)$. So $j \cdot \eta([K \times I, H])$ is a torsion in $KO(\Sigma K_+)$. Since there are only finitely many torsions in $KO(\Sigma K_+)$, $\operatorname{Ker} j \cdot \eta$ is infinite. But $\operatorname{Im} i = \operatorname{Ker} j \supset \eta(\operatorname{Ker} j \cdot \eta)$ and η is a monomorphism, $\operatorname{Im} i$ is infinite. This contradicts the well-known fact that $[\Sigma K_+, G]$ is finite.

Theorem 2.6. *Let K be a closed simply-connected smooth manifold of dimension $2k-1$. Then* $\operatorname{rank}\mathfrak{D}_0(K) \otimes Q = \operatorname{rank} H^{4*}(\Sigma K_+, Q) - t$ *where $t = 0$ if k is odd and $t = 1$ if k is even.*

Proof. Consider the following exact sequence

$$0 \to hS(K \times I, \partial) \otimes Q \xrightarrow{\eta} [\Sigma K_+, G/O] \otimes Q \xrightarrow{\sigma} \begin{cases} 0 & \text{if } k \text{ is odd}, \\ Q & \text{if } k \text{ is even}, \end{cases}$$

and the sequence

$$[\Sigma K_+, G/O] \otimes Q \xrightarrow{i} [\Sigma K_+, BO] \otimes Q \xrightarrow{L} H^{4*}(\Sigma K_+; Q)$$

where i is an isomorphism, because the homotopy group of BG is finite, L is well known to be an isomorphism. Hence

(a) $Li\eta$ is an isomorphism if k is odd;

(b) a computation of σ in terms of the Hirzebruch polynomial L shows that $Li\eta$ is an isomorphism onto a corank 1 subspace of $H^{4*}(\Sigma K_+, Q)$. Now the theorem follows from Proposition 2.5.

Remark 2.7. $\mathfrak{D}_0(K)$ is finitely generated, because ψ is onto and $[\Sigma K_+, G/O]$ is finitely generated.

Remark 2.8. Let M^m be a simply connected closed smooth manifold of dimension m. If $m \not\equiv 3 \pmod 4$ and $\operatorname{rank} \Sigma H^{4i-1}(M, Q) \geq 1$ or $k \equiv 3 \pmod 4$ and $\operatorname{rank} \Sigma H^{4i-1}(M, Q) \geq 2$, then there exists a diffeomorphism f of M which is homotopic to the identity and f^q is not pseudo-isotopic to the identity for $q > 0$.

3. Splitting invariants. Let $S^1 = \{g \in C \mid |g| = 1\}$, $S^{2m-1} = \{u = (u_1, \cdots, u_m) \in C^m \mid \|u\| = 1\}$ and $D^{2m} = \{u = (u_1, \cdots, u_m) \in C^m \mid \|u\| \leq 1\}$. Let $gu = (gu_1, \cdots, gu_m)$ for $g \in S^1$ and $u \in C^m$. Define S^1 actions on $S^{2p-1} \times S^{2q-1}$, $S^{2p-1} \times D^{2q}$ and $D^{2p} \times S^{2q-1}$ by the equation $(g, (u, v)) = (gu, gv)$. It is clear that all the actions are free. We always assume that $q \geq p$. Let $n = p + q$. Let $K = K^{n,q} = S^{2p-1} \times S^{2q-1}/S^1$, $M = M^{n,q} = S^{2p-1} \times D^{2q}/S^1$ and $N = N^{n,q} = D^{2p} \times S^{2q-1}/S^1$. Note that $CP^{n-1} = M \cup K \times I \cup N$.

Define a map ω: $hS(K \times I, \partial) \to hS(CP^{n-1})$ as follows. For $u \in hS(K \times I, \partial)$, let $u = [K \times I, F]$ such that $F|K \times 0 =$ identity. Let $f = F|K \times 1$. Then id + F + id: $M + K \times I + N \to M + K \times I + N$ is compactible with the identifications, hence gives a map id $\cup F \cup$ id: $M \cup K \times I \cup_{f-1} N \to M \cup K \times I \cup N$. Let $\omega(u) = [M \cup K \times I \cup_{f-1} N, \text{id} \cup F \cup \text{id}]$.

Let $\mathfrak{D}^{n,q} = \mathfrak{D}_0(K^{n,q})$. We now observe that it is possible to define a map P: $\mathfrak{D}^{n,q} \to hS(CP^{n-1})$ as follows with the property that $\omega = p \cdot \psi$. Let f be a diffeomorphism of $K^{n,q}$ which is homotopic to the identity. There is F: $K \times I \to K \times I$ such that $F|K \times 0 =$ identity and $F|K \times 1 = f$. Then id $\cup F \cup$ id: $M \cup K \times I \cup_{f-1} N \to CP^{n-1}$ is a homotopy equivalence and $[M \cup K \times I \cup_{f-1} N, \text{id} \cup F \cup \text{id}]$ is independent of the choice of homotopy F because any two homotopy equivalences from homotopy complex projective space HCP^{n-1} to CP^{n-1} are homotopic.

Let s_{2k}: $hS(CP^{n-1}) \to P_{2k}$ be the splitting invariants [8]. Define $\theta_{2k} = s_{2k} \cdot P$: $\mathfrak{D}^{n,q} \to P_{2k}$. We shall show that θ_{2k} are homomorphisms.

Theorem 3.1. *The following diagram commutes.*

$$\begin{array}{ccccccccc}
 & \overset{\psi}{\swarrow} & hS(K^{n,q}\times I,\partial) & \xrightarrow{\eta} & [\Sigma K_+^{n,q}, G/O] & \xrightarrow{\iota} & [\Sigma K_+^{k,r}, G/O] & \overset{\sigma}{\searrow} & \\
\mathfrak{D}^{n,q} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} & & P_{2k-2} \\
 & \underset{p}{\searrow} & hS(CP^{n-1}) & \xrightarrow{\bar\eta} & [CP^{n-1}, G/O] & \xrightarrow{\bar\iota} & [CP^{k-1}, G/O] & \underset{\sigma}{\nearrow} &
\end{array}$$

We will denote the homomorphism $\sigma\cdot\iota\cdot\eta$ by $\bar S_{2k-2}$, $c\colon CP^{n-1}\to\Sigma K_+$ will denote the collapsing map and c also will denote the induced map $c\colon [\Sigma K_+, G/O]\to[CP^{n-1}, G/O]$.

Proof. By our earlier observation $\omega = p\cdot\psi$. We will break the remaining part of the proof into Lemmas 3.2, 3.3, and 3.4.

Lemma 3.2. *The following diagram commutes.*

$$\begin{array}{ccc}
hS(K\times I,\partial) & \xrightarrow{\eta} & [\Sigma K_+, G/O] \\
\downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle c} \\
hS(CP^{n-1}) & \xrightarrow{\eta} & [CP^{n-1}, G/O]
\end{array}$$

Proof. Let $u=[K\times I, F]\in hS(K\times I,\partial)$ such that $F|K\times 0=$ identity. Let $f=F|K\times 1$. Let $\nu(K\times I)$ be the stable normal bundle of $K\times I$. Let $\xi=(F^{-1})^*\nu(K\times I)$ and choose a bundle map B covering F such that $B\,|\,(\nu(K\times I)|K\times 0)=$ identity. Then $\eta(u)=(F,B)$. Let $b=B|(\nu(K\times I)|K\times 1)$. Let $\nu(M)$, $\nu(N)$ be the stable normal bundles of M and N respectively. Let

$$\tilde F=\mathrm{id}\cup F\cup\mathrm{id}\colon M\cup K\times I\underset{f^{-1}}{\cup}N\to M\cup K\times I\cup N$$

and $\tilde\xi=\nu(M)\cup\xi\cup\nu(N)$. $\nu(M\cup K\times I\cup_{f^{-1}}N)=\nu(M)\cup\nu(K\times I)\cup_{b^{-1}}\nu(N)$. Let $\tilde B$: $\mathrm{id}\cup B\cup\mathrm{id}\colon\nu(M\cup K\times I\cup_{f^{-1}}N)\to\tilde\xi$ which is a bundle map covering F. It is easy to check $c\cdot\eta(u)=(\tilde F,\tilde B)$.

But $\omega(u)=[M\cup K\times I\cup_{f^{-1}}N,\ \mathrm{id}\cup F\cup\mathrm{id}]$.

$$((\mathrm{id}\cup F\cup\mathrm{id})^{-1})^*\nu(M\cup K\times I\underset{f^{-1}}{\cup}N)$$

$$=(\mathrm{id}\cup F^{-1}\cup\mathrm{id})^*\nu(M)\cup\nu(K\times I)\underset{b^{-1}}{\cup}\nu(N)$$

$$=\nu(M)\cup(F^{-1})^*\nu(K\times I)\cup\nu(N)=\nu(M)\cup\xi\cup\nu(N)=\bar\xi.$$

Hence $\bar\eta\cdot\omega(u)=c\cdot\eta(u)$.

Lemma 3.3. *The following diagram commutes.*

$$\begin{array}{ccc} [\Sigma K_+, G/O] & \xrightarrow{\sigma} & \\ \downarrow c & & P_{\dim K+1} \\ [CP^{n-1}, G/O] & \xrightarrow{\bar{\sigma}} & \end{array}$$

Proof. Let $F: (W, \partial W) \to (K \times I, \partial(K \times I))$, $B: \nu^k \to \xi$, where W is a compact manifold, ν is its stable normal bundle, ξ is a linear bundle over $K \times I$, F is of degree 1, $F|\partial W$ is a diffeomorphism, B is a bundle map lying over F. We may assume that $F|\partial_0 W =$ identity and $B|(\nu|\partial_0 W) =$ identity. Let $f = F|\partial_1 W$ and $b = B|(\nu|\partial_1 W))$. Let $\widetilde{W} = M \cup W \cup_{f-1} N$ and $\widetilde{F} = \mathrm{id} \cup F \cup \mathrm{id}$. $\nu(\widetilde{W}) = \nu(M) \cup \nu(W) \cup_{b-1} \nu(N)$. Let $\widetilde{\xi} = \nu(M) \cup \xi \cup \nu(N)$ and $\widetilde{B} = \mathrm{id} \cup B \cup \mathrm{id}$. Then $c(F, B) = (\widetilde{F}, \widetilde{B})$.

$$\bar{\sigma}(\widetilde{F}, \widetilde{B}) = \sigma(\mathrm{id}, \mathrm{id}) + \sigma(F, B) + \sigma(\mathrm{id}, \mathrm{id}) = \sigma(F, B).$$

Let $r \leq q$ and $k - r \leq n - q$. Then we have inclusions

$$S^{2(k-r)-1} \times S^{2r-1} \subset S^{2(n-q)-1} \times S^{2q-1},$$

$$S^{2(k-r)-1} \times D^{2r} \subset S^{2(n-q)-1} \times D^{2q},$$

$$D^{2(k-r)} \times S^{2r-1} \subset D^{2(n-q)} \times S^{2q-1}.$$

Then we have inclusions $K^{k,r} \subset K^{n,q}$, $M^{k,r} \subset M^{n,q}$ and $N^{k,r} \subset N^{n,q}$. Let ι: $[\Sigma K_+^{n,q}, G/O] \to [\Sigma K_+^{k,r}, G/O]$ be the map induced by inclusion. Let $\bar{\iota}$: $[CP^{n-1}, G/O] \to [CP^{k-1}, G/O]$ be the map induced by the inclusion.

Lemma 3.4. *The following diagram commutes.*

$$\begin{array}{ccc} [\Sigma K_+^{n,q}, G/O] & \xrightarrow{\iota} & [\Sigma K_+^{k,r}, G/O] \\ \downarrow c & & \downarrow c \\ [CP^{n-1}, G/O] & \xrightarrow{\bar{\iota}} & [CP^{k-1}, G/O] \end{array}$$

Proof. It is obvious if one uses Lemma 1.1.

Lemma 3.5. *The following diagram commutes.*

$$\begin{array}{ccc} hS(K^{n,q} \times I, \partial) & \xrightarrow{\bar{s}_{2k-2}} & \\ \downarrow \psi & & P_{2k-2} \\ \mathfrak{D}^{n,q} & \xrightarrow{\theta_{2k-2}} & \end{array}$$

Proof. Consider the following diagram

$$
\begin{array}{ccccc}
hS(K^{n,q}\times I,\partial) & & \xrightarrow{\ \bar{s}_{2k-2}\ } & & P_{2k-2}\\
\downarrow \psi & \searrow \omega & & \nearrow s_{2k-2} & \\
 & & hS(CP^{n-1}) & & \\
 & \nearrow P & & & \\
\mathfrak{D}^{n,q} & & \xrightarrow{\ \theta_{2k-2}\ } & & P_{2k-2}
\end{array}
$$

where $\bar{s}_{2k-2} = s_{2k-2}\cdot\omega$ and $\theta_{2k-2} = s_{2k-2}\cdot P$. By chasing the diagram, it is easy to prove the lemma.

Theorem 3.6. $\theta_{2k}\colon \mathfrak{D}_0(K^{n,q}) \to P_{2k}$ *are homomorphisms.*

Proof. Since $\bar{s}_{2k}$ is a homomorphism and ψ is an epimorphism.

Proposition 3.7. $\theta_{4k}\colon \mathfrak{D}_0(K^{n,q}) \to Z$ *are trivial for* $k < q/2$, *but are nontrivial for* $q/2 \le k < n/2$.

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{Ker}\,\psi & & & & \\
\downarrow & & & & \\
hS(K^{n,q}\times I,\partial) & \xrightarrow{\ \eta\ } & [\Sigma K^{n,q}_+, G/O] & \xrightarrow{\ \iota\ } & [\Sigma K^{2k+1,r}_+, G/O]\\
\downarrow \psi & & \searrow \bar{s}_{4k} & & \downarrow \sigma\\
\mathfrak{D}_0(K^{n,q}) & & \xrightarrow{\ \theta_{4k}\ } & & Z\\
\downarrow & & & & \\
0 & & & &
\end{array}
$$

Let $f\colon K^{n,q}\times I \to G/O$.

$$\sigma\cdot\iota(f) = \sigma(f\cdot\iota) = c\langle p(i\cdot f\cdot\iota), [K^{2k+1,r}\times I]\rangle = c\langle \iota^* p(i\cdot f), [K^{2k+1,r}\times I]\rangle$$

where $\iota^*\colon H^{4*}(\Sigma K^{n,q}_+) \to H^{4*}(\Sigma K^{2k+1,r}_+)$. If $k < q/2$, $H^{4k}(\Sigma K^{n,q}_+) = 0$. Hence $\sigma\cdot\iota(f) = 0$. But ψ is onto, so $\theta_{4k} = 0$.

On the other hand, to show θ_{4k} are nontrivial it is sufficient to show that $\bar{s}_{4k}\otimes Q$ is nontrivial, since $\psi\otimes Q$ is an isomorphism. For $q/2 \le k < n/2$, $H^{4k}(\Sigma K^{n,q}_+, Q) \ne 0$. Consider the following diagram:

$$
\begin{array}{ccccc}
[\Sigma K^{n,q}_+, G/O]\otimes Q & \xrightarrow{\ i\ } & [\Sigma K^{n,q}_+, BO]\otimes Q & \xrightarrow{\ L\ } & H^{4*}(\Sigma K^{n,q}_+, Q)\\
\downarrow \iota & & \downarrow b & & \downarrow c\\
[\Sigma K^{2k+1,r}_+, G/O]\otimes Q & \xrightarrow{\ i\ } & [\Sigma K^{2k+1,r}_+, BO]\otimes Q & \xrightarrow{\ L\ } & H^{4*}(\Sigma K^{2k+1,r}_+, Q)
\end{array}
$$

where i and L are isomorphisms. $H^{4k}(\Sigma K^{2k+1,r}) \ne 0$. Hence there is $x \in$

$[\Sigma K_+^{2k+1,r}, G/O] \otimes Q$ such that $L(i(x)) = 1 + L_k(i(x))$ with $L_k(i(x)) \neq 0$. Let $y \in H^{4k}(\Sigma K_+^{n,q}, Q)$ such that $c(y) = L_k(i(x))$. Then there is $z \in [\Sigma K_+^{n,q}, G/O] \otimes Q$ such that $L(i(z)) = 1 + y$. Hence $\iota(z) = x$, because $c\colon H^{4k}(\Sigma K_+^{n,q}, Q) \to H^{4k}(\Sigma K_+^{2k+1,r}, Q)$ is an isomorphism.

(i) If $\dim(K^{n,q}) \equiv 1 \pmod 4$, then $\sigma(z) = 0$. There is $u \in bS(K^{n,q} \times I)$ such that $\eta(u) = z$.

(ii) If $\dim(K^{n,q}) \equiv 3 \pmod 4$, then

$$\sigma(z) = (1/8)\langle L(i(z)) - 1, [K \times I]\rangle = (1/8)\langle y, [K^{n,q} \times I]\rangle.$$

But $y \in H^{4k}(\Sigma K_+^{n,q}, Q)$, therefore $\sigma(z) = 0$. There is $u \in bS(K^{n,q} \times I, \partial)$ such that $\eta(u) = z$.

But $\bar{s}_{4k}(u) = (1/8)\langle L(i(x)) - 1, [K^{2k+1,r} \times I]\rangle = (1/8)\langle L_k(i(x)), [K^{2k+1,r} \times I]\rangle \neq 0$.

Theorem 3.8. $\operatorname{rank} \mathfrak{D}_0(K^{n,q}) = r_{n,q}$ *where*

$$r_{n,q} = \begin{cases} [n - q/2] - 1 & \textit{if both } n \textit{ and } q \textit{ are odd,} \\ [n - q/2] & \textit{otherwise.} \end{cases}$$

Proof. This is just a special case of Theorem 2.6.

Theorem 3.9. *For $u \in \mathfrak{D}_0(K^{n,q})$, u is of finite order if and only if $P(u)$ is tangential homotopy equivalent to CP^{n-1}.*

Proof. It is easy to see that a homotopy complex projective space HCP^{n-1} is tangential equivalent to CP^{n-1} if and only if $s_{4k}(HCP^{n-1}) = 0$ for all k.

Define a map

$$\bigoplus_{q/2 \le k < n/2} \bar{s}_{4k} : \mathfrak{D}_0(K^{n,q}) \to Z^{r_{n,q}}.$$

Since $\bar{s}_{4k}$ are homomorphisms and are nontrivial for $q/2 \le k < n/2$ and $\operatorname{rank} \mathfrak{D}_0(K^{n,q}) = r_{n,q}$. Hence

$$\bigoplus_{q/2 \le k < n/2} \bar{s}_{4k} \oplus Q : \mathfrak{D}_0(K^{n,q}) \oplus Q \to Q^{r_{n,q}}$$

is an isomorphism. Hence $\bar{s}_{4k}(u) = 0$ for all k if and only if u is of finite order.

4. **Decomposable actions.** Let f be a diffeomorphism of $K^{n,q}$ which is homotopic to the identity and let $\bar{f}$ be its covering which is an equivariant diffeomorphism of $S^{2p-1} \times S^{2q-1}$. The manifold $\Sigma(\bar{f}) = S^{2p-1} \times D^{2q} \cup_{\bar{f}} D^{2p} \times S^{2q-1}$ obtained by gluing along $S^{2p-1} \times S^{2q-1}$ via $\bar{f}$ is a homotopy sphere supporting a free S^1 action defined by $g(x, y) = (gx, gy)$ where $g \in S^1$ and $(x, y) \in$

$S^{2p-1} \times D^{2q}$ or $D^{2p} \times S^{2q-1}$. It is clear that this action depends only on the pseudo-isotopy class of f and will be denoted by $(\Sigma(u), S^1)$ where u is the pseudo-isotopy class of f. Then $P(u)$ is its orbit space.

Lemma 4.1. $(\Sigma(u), S^1)$ *is a free* S^1 *action on homotopy* $(2n-1)$*-sphere with characteristic* $(2q-1)$*-sphere such that the induced action is linear.*

Proof. Obvious.

Let $A^{n,q}$ be the set of all free S^1 actions on homotopy $(2n-1)$-spheres with characteristic $(2q-1)$-spheres such that the induced actions are linear, or, equivalently, the set of all homotopy complex $(n-1)$-projective spaces with characteristic standard complex $(q-1)$-projective spaces.

Lemma 4.2. *Let* $P\colon \mathfrak{D}_0(K^{n,q}) \to hS(CP^{n-1})$ *be the map defined in* §3. *Then* $\operatorname{Im} P = A^{n,q}$.

Proof. It is obvious that $\operatorname{Im} P \subset A^{n,q}$. Let $X \in A^{n,q}$. Then using the techniques of G. R. Livesay and C. B. Thomas [6] (see also [12]) it is easy to show that there is a diffeomorphism f of K such that $X \cong M \cup_f N$. It remains to show that we can choose a diffeomorphism which is homotopic to the identity. Since $X \cong M \cup_f N$. Let $h\colon M \cup N \to M \cup_f N$ be a homotopy equivalence.

$$CP^{q-1} \subset M \subset M \cup N \xrightarrow{h} M \cup_f \underbrace{N \supset M \supset CP^{q-1}}_{i} .$$

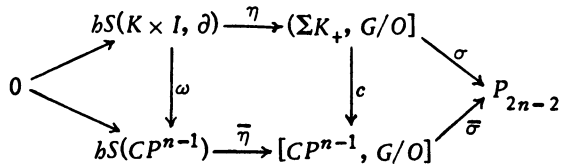
Note that $\Sigma^{2n-1} \to M \cup_f N$ is $(2n-1)$-universal and both $h|CP^{q-1}$ and $CP^{q-1} \hookrightarrow M \cup_f N$ are classifying maps for $S^{2q-1} \to CP^{q-1}$. Hence $h|CP^{q-1}$ is homotopic to the inclusion i. By the theorem of Haefliger, $h|CP^{q-1}$ is isotopic to i. By the isotopy extension theorem h is isotopic to a homotopy equivalence g such that $g|CP^{q-1} = i$.

Claim $\bar{\eta}(M \cup_f N, g')|CP^{q-1}$ is trivial, where g' is the homotopy inverse of g.

Following Sullivan, let k be large, then we can approximate $g \times 0\colon M \cup N \to (M \cup_f N) \times D^k$ by an embedding. Let E be the normal tubular neighborhood of $M \cup N$ in $(M \cup_f N) \times D^k$.

$$\begin{array}{ccccc} E & \xrightarrow{j} & (M \cup_f N) \times D^k & \xrightarrow{p_2} & D^k \\ \downarrow & \nearrow_{g \times 0} & & & \\ M \cup N & & & & \\ \uparrow & & & & \\ CP^{q-1} & & & & \end{array}$$

Then $\bar{\eta}(M \cup_f N, g') = (E, p_2 \cdot j)$. Since $g|CP^{q-1}$ = identity, $E|CP^{q-1}$ is trivial and $p_2 \cdot j|(E|CP^{q-1})$ is trivial. Hence $\bar{\eta}(M \cup_f N, g')|CP^{q-1} = 0$. CP^{q-1} is a deformation retract of M, hence $\bar{\eta}(M \cup_f N, g')|M = 0$. Similarly, $\bar{\eta}(M \cup_f N, g')|N = 0$. Now consider the following commutative diagram:



Now there is $u \in [\Sigma K_+, G/O]$ such that $c(u) = \bar{\eta}(M \cup_f N, g')\sigma(u) = \bar{\sigma} c(u) = \bar{\sigma}\bar{\eta}(M \cup_f N, g') = 0$. So there is $y \in hS(K \times I, \partial)$ such that $\eta(y) = u$.

$$\bar{\eta}\omega(y) = c\eta(y) = c(u) = \bar{\eta}([M \cup_f N, g']).$$

Since $\bar{\eta}$ is monic, $\omega(y) = [M \cup_f N, g']$. Let $y = [K \times I, H]$ where $H|K \times 0$ = identity and let $h = H|K \times 1$. Then

$$[M \cup_f N, g'] = [M \cup K \times I \cup_{h^{-1}} N, \text{id} \cup H \cup \text{id}].$$

Hence $X \cong M \cup K \times I \cup_{h^{-1}} N \cong M \cup_{h^{-1}} N$ and h^{-1} is homotopic to the identity.

Proposition 4.3. *For $u, v \in \mathfrak{D}_0(K^{n,q})$, then $P(u) \cong P(v)$ if and only if there are $\eta, \rho \in \mathfrak{D}_0(K^{n,q})$ with η extendible to a diffeomorphism of $M^{n,q}$ and ρ extendible to a diffeomorphism of $N^{n,q}$ such that $u = \rho \cdot v \cdot \eta$.*

Proof. Let $f \in u$, $g \in v$. Then $P(f) \cong P(g)$ if and only if there are diffeomorphisms h of $M^{n,q}$ and k of $N^{n,q}$ such that $f = h \cdot g \cdot k$ [12]. Since f and g are homotopic to the identity, then by exactly the same argument in [2] we can show $h|K^{n,q}$ and $k|K^{n,q}$ are homotopic to the identity. Let $\rho = \{h|K^{n,q}\}$ and $\eta = \{k|K^{n,q}\}$. Then $\rho, \eta \in \mathfrak{D}_0(K^{n,q})$ and $u = \rho \cdot v \cdot \eta$.

Theorem 4.4. *For $n/2 \le q \le 2n/3$, there is a well-defined group structure on $A^{n,q}$ defined by $P(u) * P(v) = P(u \cdot v)$, and $P: \mathfrak{D}_0(K^{n,q}) \to A^{n,q}$ is an epimorphism.*

Proof. Suppose $P(u) = P(u')$. Then there are $\eta, \rho \in \mathfrak{D}_0(K^{n,q})$ such that η extends to a diffeomorphism of M and ρ extends to a diffeomorphism of N such that $u' = \rho \cdot u \cdot \eta$. $u' \cdot v = \rho \cdot u \cdot \eta \cdot v = \rho \cdot u \cdot v \cdot \eta$. Hence $P(u' \cdot v) = P(u \cdot v)$.

Let $G^{n,q}$ be the subgroup of $\mathfrak{D}_0(K^{n,q})$ which is generated by those diffeomorphisms of $K^{n,q}$ which are extendible to either a diffeomorphism of M or a diffeomorphism of N. It is clear that $\operatorname{Ker} P = G^{n,q}$.

Proposition 4.5. *$G^{n,q}$ is a finite group.*

Proof. This is a special case of a more general theorem in [12]. Here we will give a simple proof. For $u \cdot v \in G^{n,q}$ where u is extendible to a diffeomorphism of M, and v is extendible to a diffeomorphism of N, $P(u \cdot v) \cong CP^{n-1}$. Hence, by Theorem 3.9, $u \cdot v$ is of finite order. But $\mathfrak{D}_0(K^{n,q})$ is finitely generated. Therefore, $G^{n,q}$ is finite.

Theorem 4.6. *For $n/2 \leq q \leq 2n/3$, $A^{n,q}$ is a finitely generated abelian group whose torsions are tangential homotopy complex projective spaces, and* $\operatorname{rank} A^{n,q} = \operatorname{rank} \mathfrak{D}_0(K^{n,q})$.

Definition 4.7. A free S^1 action (Σ^{2n-1}, S^1) is decomposable if there is a diffeomorphism f of $K^{n,q}$, for some q, such that (Σ^{2n-1}, S^1) is equivalent to $(\Sigma(f), S^1)$.

It is clear that $A^{n,[n+1/2]}$ is the set of all decomposable S^1 actions on homotopy $(2n-1)$-spheres.

Theorem 4.8. *There is a natural group structure on the set of all decomposable free S^1 actions on homotopy $(2n-1)$-spheres which is a finitely generated abelian group whose torsion subgroup consists of all tangential homotopy complex projective spaces and* $\operatorname{rank} A^{n,[n+1/2]} = [n/4] - e$ *where $e = 1$ if $n \equiv 1 \pmod 4$, or $e = 0$ otherwise. Furthermore $s_{2k}\colon A^{n,[n+1/2]} \to P_{2k}$ are homomorphisms.*

5. **Applications.** Let $s_{2k}\colon hS(CP^n) \to P_{2k}$ be the splitting invariants. Then $s_{2k}^{-1}(0)$ consists of all free S^1 actions on homotopy $(2n+1)$-spheres with characteristic $(2k+1)$-spheres. By Theorem 4.8 and Proposition 3.7 we have the following:

Theorem 5.1. *There are infinitely many topologically inequivalent free S^1 actions on homotopy $(2n+1)$-spheres with characteristic homotopy $(2k+1)$-spheres for $n \geq k \geq 3$ and* (a) *$n \geq 6$ if n is even,* (b) *$n \geq 5$ if both n and k are odd, and* (c) *$n \geq 7$ if n is odd but k is even.*

Remark 5.2. Similar results have been obtained by H. T. Ku [5] in the case k = even and $n \geq 10$ using the techniques of W. C. Hsiang [4]. See also [3], [9].

Remark 5.3. Theorem 5.1 was first proved by using Theorem 3.6, Theorem 3.8 and Proposition 4.5, which was announced in [10].

Remark 5.4. We have omitted the analogs of S^3 actions. Similarly, one can show the rank of the group of decomposable S^3 actions on homotopy $(4n+3)$-spheres is equal to $[n-1/2]$.

Recall the following.

Theorem 5.5 ([9]). *A free S^3 action on homotopy $(4n+3)$-spheres is decomposable if and only if its restriction to S^1 is decomposable.*

Theorem 5.6. *For $n \geq 1$, there are infinitely many topologically inequivalent free S^1 actions on homotopy $(4n+3)$-spheres which are not extendible to free S^3 actions.*

Proof. Note that the rank of the group of decomposable S^1 actions on homotopy $(4n+3)$-spheres is equal to $[n+1/2]$. Hence the theorem follows from Remark 5.4. and Theorem 5.5.

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