LIMIT THEOREMS FOR VARIATIONAL SUMS

BY

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ABSTRACT. Limit theorems in the sense of a.s. convergence, convergence in L_1-norm and convergence in distribution are proved for variational series. In the first two cases, if g is a bounded, nonnegative continuous function satisfying an additional assumption at zero, and if $\{X(t), 0 \le t \le T\}$ is a stochastically continuous stochastic process with independent increments, with no Gaussian component and whose trend term is of bounded variation, then the sequence of variational sums of the form $\sum_{k=1}^{n} g(X(t_{nk}) - X(t_{n,k-1}))$ is shown to converge with probability one and in L_1-norm. Also, under the basic assumption that the distribution of the centered sum of independent random variables from an infinitesimal system converges to a (necessarily) infinitely divisible limit distribution, necessary and sufficient conditions are obtained for the joint distribution of the appropriately centered sums of the positive parts and of the negative parts of these random variables to converge to a bivariate infinitely divisible distribution. A characterization of all such limit distributions is obtained. An application is made of this result, using the first theorem, to stochastic processes with (not necessarily stationary) independent increments and with a Gaussian component.

1. **Introduction and summary.** The expression "variational sum" refers to the sum of a function of independent random variables from an infinitesimal system of random variables. More precisely, let $\{\{X_{n1}, \cdots, X_{nk_n}\}\}$ be an infinitesimal system of random variables, i.e., for $n = 1, 2, \cdots$, the k_n random variables $X_{n1}, \cdots, X_{nk_n}$ are independent, and, in addition, $k_n \to \infty$ as $n \to \infty$ and for every sequence of integers $\{j_n\}$ such that $1 \le j_n \le k_n$, $X_{nj_n} \to 0$ in probability. Then, for a real-valued or vector-valued function g, the expression $\sum_{j=1}^{k_n} g(X_{nj})$ is a *variational* or *g-variational* sum. Three interrelated limit theorems on variational sums are proved in this paper.

The first of these theorems, Theorem 1 in §2, yields a conclusion of almost sure convergence and L_1-convergence for variational sums of the form $\sum_{j=1}^{n} g(X(t_{nj}) - X(t_{n,j-1}))$. Here $\{X(t), 0 \le t \le T\}$ is a stochastically continuous stochastic process with independent increments, no Gaussian component, and a trend function of bounded variation over $[0, T]$. The function g is a bounded,

nonnegative, continuous function over R^1 which satisfies $g(x) \leq M|x|^\gamma$, where $\max\{1, \beta\} < \gamma \leq 2$ or $\beta = \gamma = 2$ and $\beta = \inf\{\delta > 0: \int_{|x| \leq 1} |x|^\delta M_T(dx) < \infty\}$, $M_t(x)$ denoting the Lévy spectral measure corresponding to $X(t)$. This result extends theorems of S. M. Berman (Theorem 5.1 in [1]), R. Cogburn and H. G. Tucker (Theorem 2 in [3]) and P. W. Millar (Theorem 4.5 in [7]).

The second such theorem deals with the joint limiting distribution of the sum of the positive parts and the sum of the negative parts of random variables from an infinitesimal system. In particular, let $\{\{X_{n1}, \cdots, X_{nk_n}\}\}$ be an infinitesimal system of random variables as defined above. The *general hypothesis* is that there exists a sequence of real numbers $\{a_n\}$ such that the distribution function of $\sum_{j=1}^{k_n} X_{nj} + a_n$ converges to some limit distribution at all its points of continuity. This limit distribution is necessarily infinitely divisible and has a characteristic function of the form

$$f(u) = \exp\left\{i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) dM(x)\right\},$$

where γ and $\sigma^2 \geq 0$ are constants and $M(x)$ is a nondecreasing function over $(-\infty, 0)$ and over $(0, \infty)$ such that $M(-\infty) = M(+\infty) = 0$ and $\int_{-1}^{1} x^2 \, dM(x) < \infty$. For X any random variable let us denote $X^+ = XI_{[X \geq 0]}$ and $X^- = -XI_{[x \leq 0]}$, so that $X = X^+ - X^-$. The problem considered is: given the above general hypothesis, under what conditions do there exist sequences of constants $\{u_n\}$ and $\{v_n\}$ such that the joint distribution of the pair of centered sums

$$\sum_{j=1}^{k_n} X_{nj}^+ + u_n, \qquad \sum_{j=1}^{k_n} X_{nj}^- + v_n$$

converges as $n \to \infty$ to that of a bivariate infinitely divisible distribution function? Here $g: R^1 \to R^2$ is a function defined by $g(x) = (x^+, x^-)$. In §3 necessary and sufficient conditions are obtained for such a limit law to exist. This theorem is an improvement of Theorem 1 in [8] and a result of Loève [5]. A characterization of all such bivariate limit distributions is obtained.

In §4 there is a third such theorem. It is an application of the first two theorems to separable stochastic processes with independent increments which are centered and have no fixed discontinuities (i.e., $X(t - 0) = X(t + 0)$ a.s. for each t). In particular, if $\{X(t), 0 \leq t \leq T\}$ is such a process, then letting $0 = t_{n0} < t_{n1} < \cdots < t_{nn} = T$ be a sequence of partitions under refinement such that $\max\{t_{nj} - t_{n,j-1}; 1 \leq j \leq n\} \to 0$ as $n \to \infty$, one considers for each such partition the sum of the positive parts and the sum of the negative parts of the increments. Under suitable centering the bivariate limit distribution is shown always to exist. The results of §4 might overlap with those of P. Greenwood and B. Fristedt [4] who treat the more special case of processes with *stationary* independent increments.

2. **Extension of a theorem of Berman.** In this section a limit theorem is proved for g-variational sums for stochastic processes with independent increments. The limit here will exist *both* in the almost sure sense and in L_1-norm. The stochastic process is one with (not necessarily stationary) independent increments, and the function g is from a rather general class. Its interest is that for the most part it extends a theorem of S. M. Berman, namely, Theorem 5.1 in [1]. Also extended in a sense are Theorem 2 in [3] and a theorem of P. W. Millar, Theorem 4.5 in [7]. An application of this theorem will be given in §4.

In this section and in §4, $\{X(t), \ 0 \le t \le T\}$ will be a stochastic process with independent increments which is centered, has no fixed points of discontinuity and is separable. The characteristic function of $X(t)$ can be written as

$$f_{X(t)}(u) = \exp\left\{ i a(t)u - \frac{\sigma^2(t)u^2}{2} + \int_{-\infty}^{-0} + \int_{+0}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) M_t(dx) \right\},$$

where $a(t)$ is a continuous function over $[0, T]$, $\sigma^2(t)$ is a continuous nonnegative, nondecreasing function, and $M_t(\ )$ is the Lévy spectral measure whose properties are given in a number of places, e.g., Loève [6]. *In addition*, it is assumed that $a(t)$ is of *bounded variation* over $[0, T]$. Such a process $X(t)$ can be written as a sum of two *independent* separable stochastic processes $U(t)$ and $V(t)$. The process $\{U(t), \ 0 \le t \le T\}$, called the Gaussian component, has characteristic function $f_{U(T)}(u) = \exp\{-\sigma^2(t)u^2/2\}$, and it is known that almost all of its sample functions are continuous over $[0, T]$. The non-Gaussian component has characteristic function

$$f_{V(t)}(u) = \exp\left\{ i a(t)u + \int_{-\infty}^{-0} + \int_{+0}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) M_t(dx) \right\},$$

and almost all of its sample functions are right continuous and have left limits at each point. Each such sample function necessarily has discontinuities of the first kind only, and a countable number of them at that. Now, for every positive integer n, let $0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = T$ be such that

$$\{t_{n0}, \cdots, t_{nn}\} \subset \{t_{n+1,0}, \cdots, t_{n+1,n+1}\}$$

and such that $\max\{t_{n,k} - t_{n,k-1} : 1 \le k \le n\} \to 0$ as $n \to \infty$. We shall use the following notation in these two sections: $X_{nk} = X(t_{nk}) - X(t_{n,k-1})$, F_{nk} denotes the distribution function of X_{nk},

$$a_{nk} = a(t_{nk}) - a(t_{n,k-1}), \quad U_{nk} = U(t_{nk}) - U(t_{n,\ k-1}), \quad V_{nk} = V(t_{nk}) - V(t_{n,k-1}),$$

and

$$\beta = \inf\left\{ \delta > 0 : \int_{-1}^{1} |x|^{\delta} M_T(dx) < \infty \right\}.$$

The number β is an index originally due to Blumenthal and Getoor [2] and generalized by Berman [1]. For every $s \in (0, t]$, we define $J(s) = X(s) - X(s - 0) = V(s) - V(s - 0)$ as the jump at s. An arbitrary point in the probability space is denoted by ω. The theorem to be proved in this section follows.

Theorem 1. *Let g be a bounded, continuous, nonnegative function defined over R^1. Suppose there are constants $M > 0$ and γ such that $\gamma \in (\max(1, \beta), 2]$ or $\beta = \gamma = 2$ and $g(x) \leq M|x|^\gamma$ for all x. Let $\{X(t), 0 \leq t \leq T\}$ be the stochastic process defined above but without a Gaussian component. Then*

(A)
$$\sum_{k=1}^{n} \int g(x) dF_{nk}(x) \rightarrow \int g(x) M_T(dx)$$

as $n \rightarrow \infty$, and

(B)
$$\sum_{k=1}^{n} g(X_{nk}) \rightarrow \sum \{g(J(s)): 0 < s \leq T\}$$

almost surely and in L_1-mean.

It should be noted that if we required $g(x) \leq K|x|^\gamma$ to hold in a neighborhood of 0, then by the boundedness of g there exists an $M > 0$ such that $g(x) \leq M|x|^\gamma$ for all x.

By the size of a jump of the process at s we mean $|J(s)|$. Let us denote $I_{nk}(\epsilon)$ as the random variable which is 1 if no jumps of size greater than ϵ occur in $(t_{n,k-1}, t_{nk}]$ and which is zero otherwise. We also need the notation

$$Y^{(\epsilon)}(t) = X(t) - \alpha(t) - \sum \{J(s): |J(s)| > \epsilon, 0 < s \leq t\}$$

and

$$Y^{(\epsilon)}_{nk} = X_{nk} - \alpha_{nk} - \sum \{J(s): |J(s)| > \epsilon, t_{n,k-1} < s \leq t_{nk}\}.$$

Lemma 1. *If $\max\{1, \beta\} < \gamma \leq 2$, then*

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} E\left\{\sum_{j=1}^{n} g(X_{nk}) I_{nj}(\epsilon_m)\right\} = 0.$$

Proof. Since g is nonnegative, and since $X_{nj} = Y^{(\epsilon_m)}_{nj} + \alpha_{nj}$ whenever $I_{nj}(\epsilon_m) = 1$, we have, for every n and j,

$$g(X_{nj}) I_{nj}(\epsilon_m) \leq g(Y^{(\epsilon_m)}_{nj} + \alpha_{nj}).$$

Hence

(1)
$$E\left\{\sum_{j=1}^{n} g(X_{nj}) I_{nj}(\epsilon_m)\right\} \leq E\left\{\sum_{j=1}^{n} g(Y^{(\epsilon_m)}_{nj} + \alpha_{nj})\right\}.$$

Now by hypothesis $g(x) \leq M|x|^\gamma$, so

$$(2) \qquad E\left\{\sum_{j=1}^{n} g(Y_{nj}^{(\epsilon_m)} + \alpha_{nj})\right\} \leq ME\left\{\sum_{j=1}^{n} |Y_{nj}^{(\epsilon_m)} + \alpha_{nj}|^{\gamma}\right\}.$$

Since $\gamma > 1$, we may apply the inequality $|a + b|^{\gamma} \leq 2^{\gamma-1}(|a|^{\gamma} + |b|^{\gamma})$ (see Loève [6, p. 155] to obtain

$$(3) \qquad E\left\{\sum_{j=1}^{n} |Y_{nj}^{(\epsilon_m)} + \alpha_{nj}|^{\gamma}\right\} \leq 2^{\gamma-1}E\left\{\sum_{j=1}^{n} |Y_{nj}^{(\epsilon_m)}|^{\gamma}\right\} + 2^{\gamma-1}\sum_{j=1}^{n} |\alpha_{nj}|^{\gamma}.$$

Now

$$(4) \qquad \limsup_{n \to \infty} \sum_{j=1}^{n} |\alpha_{nj}|^{\gamma} = 0$$

since $\alpha(t)$ is continuous and of bounded variation over $[0, T]$ and $\gamma > 1$. We obtain

$$(5) \qquad \lim_{m \to \infty} \limsup_{n \to \infty} E\left\{\sum_{j=1}^{n} |Y_{nj}^{(\epsilon_m)}|^{\gamma}\right\} = 0$$

by applying two lemmas due to S. M. Berman, namely, first applying Lemma 5.3 and then applying Lemma 5.1 in his paper [1]. Combining (1)–(5), we obtain the lemma.   Q.E.D.

Lemma 2. *For every* m,

$$\lim_{n \to \infty} E\left\{\sum_{k=1}^{n} g(X_{nk})(1 - I_{nk}(\epsilon_m))\right\} = E\left(\sum \{g(J(s)) : |J(s)| > \epsilon_m, 0 < s < T\}\right)$$

$$= \int_{|x| > \epsilon_m} g(x)M_T(dx).$$

Proof. Let $K = \sup\{g(x) : x \in R^1\} < \infty$. Let N denote the number of jumps during $[0, T]$ of size greater than ϵ_m. Then N is a random variable; it has the Poisson distribution with expectation $\int_{|x| > \epsilon_m} M_T(dx) < \infty$. Thus, if we denote $Z_n = \sum_{k=1}^{n} g(X_{nk})(1 - I_{nk}(\epsilon_m))$, we have $0 \leq Z_n \leq KN$ a.s. In addition, $Z_n \to \sum\{g(J(s)) : |J(s)| > \epsilon_m, 0 < s \leq T\}$ a.s. because almost all sample functions are right continuous and have left limits at every point and a finite number of jumps of size greater than ϵ_m, and because g is everywhere continuous. Thus, the first equality is obtained by the dominated convergence theorem. It is well known that the last two terms in the statement of the lemma are equal.   Q.E.D.

Lemma 3. *The following holds:*

$$\lim_{n \to \infty} \sum_{k=1}^{n} g(X_{nk}) = \sum \{g(J(s)): 0 < s \leq T\} \ a.s.$$

Proof. By hypothesis on M_T and since $\beta < \gamma \leq 2$ or $\beta = \gamma = 2$, $\sum\{|J(s)|^{\gamma}: 0 < s \leq T, |J(s)| < 1\}$ has finite expectation $\int_{-1}^{1} |x|^{\gamma} M_T(dx)$, and hence it is a finite random variable. Thus, for arbitrary $\epsilon > 0$ and $\eta > 0$ there exists a $\kappa > 0$ $(\kappa < 1)$ small enough so that

(1) $$P\left[\sum\{|J(s)|^{\gamma}: 0 < s \leq T, |J(s)| < \kappa\} > \epsilon/(M + 1)\right] < \eta,$$

where the constant M is from the hypothesis on g, and such that $\pm\kappa$ are continuity points of $M_T(x)$. Let us denote $A_n = \sum_{k=1}^{n} g(X_{nk}) I_{nk}(\kappa)$ and $B_n = \sum_{k=1}^{n} g(X_{nk})(1 - I_{nk}(\kappa))$. Let $\{Y(t), 0 \leq t \leq T\}$ be a stochastic process defined by

$$Y(t) = X(t) - \sum\{J(s): 0 < s \leq t, |J(s)| \geq \kappa\},$$

and denote $Y_{nk} = Y(t_{nk}) - Y(t_{n,k-1})$. We now show that

(2) $$A_n - \sum_{k=1}^{n} g(Y_{nk}) \to 0 \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Indeed, since $X(\cdot)$ is continuous a.s. at each fixed $t \in [0, T]$, there exists an event Ω_0 of probability one such that, for each fixed $\omega \in \Omega_0$, $X(t, \omega)$ is continuous at all the t_{nk}'s. Take $\omega \in \Omega_0$. For each n there are no jumps at any of the endpoints of any of the subintervals. For each n there are at most $N(\omega)$ subintervals which are not included in the sum defining A_n but which are included in the sum $\sum_{k=1}^{n} g(Y_{nk})$, where N is a Poisson random variable with expectation $\int_{|x| \geq \kappa} M_T(dx) < \infty$. In each of these $N(\omega)$ subintervals the point at which a jump of size $\geq \kappa$ for $X(t)$ was located is a point of continuity of $Y(t)$. Hence, continuity of g at zero implies each of these $N(\omega)$ summands tends to zero as $n \to \infty$, which proves (2). In the proof of Lemma 2, it was established that

(3) $$B_n \to \sum\{g(J(s)): 0 < s \leq T, |J(s)| \geq \kappa\} \quad \text{a.s.} \quad \text{as } n \to \infty.$$

Thus, by Theorem 5.1 in Berman [1] in the case $\max\{1, \beta\} < \gamma \leq 2$ or by Theorem 2 in [3] in the case $\beta = \gamma = 2$, we have by hypothesis on g that

(4) $$\limsup_{n \to \infty} \sum_{k=1}^{n} g(Y_{nk}) \leq M \limsup_{n \to \infty} \sum_{k=1}^{n} |Y_{nk}|^{\gamma}$$

$$= M \sum\{|J(s)|^{\gamma}: 0 < s \leq T, |J(s)| < \kappa\}.$$

Hence, applying in order (3), (2), (4) and (1) we obtain

$$P\left[\limsup_{n\to\infty}\left|\sum_{k=1}^{n}g(X_{nk})-\sum\{g(J(s))\colon 0<s\le T\}\right|\ge\epsilon\right]$$

$$\le P\left[\limsup_{n\to\infty}\left|A_n-\sum\{g(J(s))\colon 0<s\le T,\ |J(s)|<\kappa\}\right|\right.$$

$$\left.+\limsup_{n\to\infty}\left|B_n-\sum\{g(J(s))\colon 0<s\le T,\ |J(s)|\ge\kappa\}\right|\ge\epsilon\right]$$

$$=P\left[\limsup_{n\to\infty}\left|A_n-\sum\{g(J(s))\colon 0<s\le T,\ |J(s)|<\kappa\}\right|\ge\epsilon\right]$$

$$=P\left[\limsup_{n\to\infty}\left|\sum_{k=1}^{n}g(Y_{nk})-\sum\{g(J(s))\colon 0<s\le T,\ |J(s)|<\kappa\}\right|\ge\epsilon\right]$$

$$\le P\left[(1+M)\sum\{|J(s)|^{\gamma}\colon 0<s\le T,\ |J(s)|<\kappa\}\ge\epsilon\right]<\eta,$$

which proves Lemma 3 by the arbitrariness of ϵ and η.   Q.E.D.

Lemma 4. *If* $\beta=\gamma=2$, *then the sequence* $\{\sum_{j=1}^{n}g(X_{nj})\}$ *is uniformly integrable.*

Proof. Let $Y(t)$ and Y_{nj} be as defined in the proof of Lemma 3, and let $Y_n=\sum_{j=1}^{n}Y_{nj}^2$. It is known (see, e.g., S. J. Wolfe [11, Theorem 2]) that each Y_{nj} has moments of all orders. We first compute EY_n. Let us define

$$M_{nj}=M_{t_{nj}}-M_{t_{n,j-1}}\quad\text{and}\quad\alpha_{nj}=\alpha(t_{nj})-\alpha(t_{n,j-1}).$$

The characteristic function of Y_{nj} is

$$f_{Y_{nj}}(u)=\exp\left\{iu\left(\alpha_{nj}-\int_{|x|\ge\kappa}\frac{x}{1+x^2}M_{nj}(dx)\right)+\int_{|x|<\kappa}\left(e^{iux}-1-\frac{iux}{1+x^2}\right)M_{nj}(dx)\right\}.$$

Differentiating twice with respect to u, and recalling that $EY_{nj}^2=-f''_{Y_{nj}}(0)$ and $M_T=\sum_{j=1}^{n}M_{nj}$, we obtain

$$EY_n=\sum_{j=1}^{n}E(Y_{nj}^2)$$

(2)
$$=\sum_{j=1}^{n}\left\{\alpha_{nj}-\int_{|x|\ge\kappa}\frac{x}{1+x^2}M_{nj}(dx)+\int_{|x|\le\kappa}\frac{x^3}{1+x^2}M_{nj}(dx)\right\}^2$$

$$+\int_{|x|<\kappa}x^2M_T(dx).$$

Let $Y=\sum\{(J(s))^2\colon 0<s\le T,\ |J(s)|<\kappa\}$. Then it is known (see, e.g., Lemma 5 in §3 of [2]) that $EY=\int_{|x|<\kappa}x^2M_T(dx)$. We now wish to prove that $EY_n\to EY$.

To do this, it is sufficient to prove that $\Gamma_n \to 0$ as $n \to \infty$, where Γ_n is defined by

$$\Gamma_n = \sum_{j=1}^{n} (b(t_{nj}) - b(t_{n,j-1}))^2$$

and

$$b(t) = \alpha(t) - \int_{|x| \geq \kappa} \frac{x}{1 + x^2} M_t(dx) + \int_{|x| \leq \kappa} \frac{x^3}{1 + x^2} M_t(dx).$$

We first make two observations:

(i) Since $M_t(x)$ is continuous in t for each x at which $M_T(\cdot)$ is continuous, we have by the Helly-Bray theorem that $\int_{|x| \geq \kappa} (x/(1 + x^2)) M_t(dx)$ is a continuous function in t. Since M_t as a measure is nondecreasing with increasing t, $\int_{x \geq \kappa} (x/(1 + x^2)) M_t(dx)$ and $-\int_{x \leq -\kappa} (x/(1 + x^2)) M_t(dx)$ are nondecreasing and continuous as functions of t.

(ii) Consider the finite measure $\mu_n(A) = \int_A x^2 M_{nj}(dx)$ defined over all Borel subsets A of $[-\kappa, \kappa]$. For $1 \leq j_n \leq n, X_{nj_n} \to 0$ in probability, and hence by a known theorem on convergence of sequences of infinitely divisible distributions (see, e.g., Theorem 2 on p.163 of [4]), $M_{nj_n}(x) \to 0$ as $n \to \infty$ at all $x \neq 0$. Thus, $\{\mu_n\}$ converges to the zero-measure over $[-\kappa, \kappa]$, and hence by the Helly-Bray theorem,

$$\max\left\{ \int_{|x| < \kappa} \left| \frac{x^3}{1 + x^2} \right| M_{nj}(dx) : 1 \leq j \leq n \right\} \to 0$$

as $n \to \infty$. This implies that, as a function of t, $\int_{|x| < \kappa} (x^3/(1 + x^2)) M_t(dx)$ is continuous, and, in addition, $\int_{0 < x < \kappa} (x^3/(1 + x^2)) M_t(dx)$ and $-\int_{-\kappa < x < 0} (x^3/(1 + x^2)) M_t(dx)$ are nondecreasing and continuous functions of t. Observations (i) and (ii) and the hypothesis on $\alpha(\cdot)$ imply that $b(t)$ is continuous and $b(t) - \alpha(t)$ is equal to a difference of two nondecreasing continuous functions, and hence b is both continuous and of bounded variation over $[0, T]$. We may write

$$\Gamma_n \leq \max_{1 \leq j \leq n} \{|b(t_{nj}) - b(t_{n,j-1})|\} \sum_{j=1}^{n} |b(t_{nj}) - b(t_{n,j-1})|.$$

The conclusion drawn from the two observations implies $\Gamma_n \to 0$ as $n \to \infty$. Thus we have shown $EY_n \to EY$ as $n \to \infty$. Now, by Theorem 1 in [3], $Y_n \to Y$ a.s. as $n \to \infty$. Since $\{Y_n\}$ are nonnegative, these two results are easily seen to imply $E|Y_n - Y| \to 0$ as $n \to \infty$, which in turn implies that $\{Y_n\}$ is uniformly integrable. Let us denote $S_n = \sum_{j=1}^{n} g(X_{nj}) I_{nj}(\kappa)$. Since by hypothesis $g(x) \leq Mx^2$ for all x we have

$$S_n \leq \sum_{j=1}^{n} g(Y_{nj}) \leq M \sum_{j=1}^{n} Y_{nj}^2,$$

and by what we have just proved, we obtain that $\{S_n\}$ is uniformly integrable. We

obtain the conclusion of the lemma from the inequality

$$\sum_{j=1}^{n} g(X_{nj}) \le S_n + \|g\|_{\infty} N. \quad \text{Q.E.D.}$$

We now prove (A) and (B). By Lemma 3 we immediately obtain the almost sure part of (B). This much of the conclusion and Lemma 4 imply in the case $\gamma = \beta = 2$ conclusion (A) and the L_1-mean convergence part of (B). We now prove conclusion (A) of Theorem 1 in the case $\max\{1, \beta\} < \gamma \le 2$. Indeed, one easily sees that, for every m,

$$\limsup_{n \to \infty} E\left(\sum_{k=1}^{n} g(X_{nk})(1 - I_{nk}(\epsilon_m))\right) \le \limsup_{n \to \infty} E\left(\sum_{k=1}^{n} g(X_{nk})\right)$$

$$\le \limsup_{n \to \infty} E\left(\sum_{k=1}^{n} g(X_{nk})(1 - I_{nk}(\epsilon_m))\right) + \limsup_{n \to \infty} E\left(\sum_{k=1}^{n} g(X_{nk})I_{nk}(\epsilon_m)\right).$$

These inequalities plus Lemmas 1 and 2 imply

$$\limsup_{n \to \infty} E\left(\sum_{k=1}^{n} g(X_{nk})\right) = \lim_{m \to \infty} \limsup_{n \to \infty} E\left(\sum_{k=1}^{n} g(X_{nk})(1 - I_{nk}(\epsilon_m))\right)$$

$$= \lim_{m \to \infty} \int_{|x| > \epsilon_m} g(x) M_T(dx) = \int g(x) M_T(dx),$$

which proves (A) in the case $\max\{1, \beta\} < \gamma \le 2$.

Now we use a well-known result which states that if Z_n and Z are nonnegative random variables with finite expectations, if $Z_n \to Z$ a.s. and if $EZ_n \to EZ$ as $n \to \infty$, then $E|Z_n - Z| \to 0$. Thus, by conclusion (A) and Lemma 3 we obtain conclusion (B) in the case $\max\{1, \beta\} < \gamma \le 2$. Q.E.D.

3. Existence and extent of bivariate limit laws. The univariate and bivariate forms of the general central limit theorem are used repeatedly for the rest of this paper. The general multivariate form is given here for easy reference. (See Theorem 2.3 in [8, pp. 190–191]).

Central limit theorem. *Let* $\{\{X_{n1}, \cdots, X_{nk_n}\}\}$ *be an infinitesimal system of p-dimensional random vectors, and let* H_{nj} *denote the distribution function of* X_{nj}. *In order that there exist a sequence of constant p-dimensional vectors* $\{c_n\}$ *such that* $\sum_{j=1}^{k_n} X_{nj} + c_n$ *converges in law it is necessary and sufficient that there exist a completely additive, nonnegative set function* $N(S)$ *defined on Borel sets not containing* 0 *and a nonnegative quadratic form* $Q(t)$ *defined over* $t \in R^p$ *such that*

(i) *For Borel sets* S *which are continuity sets of* N *and which are bounded away from zero,*

$$\sum_{j=1}^{k_n} \int_S dH_{nj}(x) \to N(S) \quad as \ n \to \infty,$$

(ii) $$\lim_{\epsilon \downarrow 0} \overline{\lim_{n \to \infty}} \sum_{j=1}^{k_n} \left\{ \int_{|x|<\epsilon} (t'x)^2 dH_{nj}(x) - \left(\int_{|x|<\epsilon} t'x \, dH_{nj}(x) \right)^2 \right\} = Q(t).$$

The p-dimensional vectors $\{c_n\}$ may be chosen by the formula

$$c_n = \sum_{j=1}^{k_n} \int_{|x|<\tau} x \, dF_{nj}(x) - \gamma,$$

where $\tau > 0$ is arbitrary and γ is a constant vector. The logarithm of the characteristic function of the limit distribution is given by

$$\log f(t) = i\gamma' t - Q(t)/2 + \int_{|x|>0} \left[\exp(it'x) - 1 - (t'x)(1 + |x|^2)^{-1} \right] N(dx).$$

Now let $\{\{X_{nj}\}\}$ be an infinitesimal system of random variables (as defined in §1), let F_{nj} be the distribution function of X_{nj}, and recall that *the general hypothesis* means that there exists a sequence of real numbers $\{a_n\}$ such that the sequence of centered sums $\sum_{j=1}^{k_n} X_{nj} + a_n$ converges in law. For every positive integer n and every $\epsilon > 0$, let us define

$$\sigma^2(\epsilon, n, +) = \sum_{j=1}^{k_n} \left\{ \int_0^\epsilon x^2 \, dF_{nj}(x) - \left(\int_0^\epsilon x \, dF_{nj}(x) \right)^2 \right\},$$

$$\sigma^2(\epsilon, n, -) = \sum_{j=1}^{k_n} \left\{ \int_{-\epsilon}^0 x^2 \, dF_{nj}(x) - \left(\int_{-\epsilon}^0 x \, dF_{nj}(x) \right)^2 \right\},$$

and

$$Cr(\epsilon, n) = \sum_{j=1}^{k_n} \left\{ \int_0^\epsilon x \, dF_{nj}(x) \int_{-\epsilon}^0 x \, dF_{nj}(x) \right\}.$$

Theorem 2. *Under the general hypothesis given above, necessary and sufficient conditions that there exist sequences of real numbers $\{u_n\}$ and $\{v_n\}$ such that the joint distribution function of $\sum_{j=1}^{k_n} X_{nj}^+ + u_n$ and $\sum_{j=1}^{k_n} X_{nj}^- + v_n$ converges to a limit distribution are that there exist constants $\sigma_{11} \geq 0$ and $\sigma_{22} \geq 0$ such that*

(1) $$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \binom{\sup}{\inf} \sigma^2(\epsilon, n, +) = \sigma_{11}$$

and

(2) $$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \binom{\sup}{\inf} \sigma^2(\epsilon, n, -) = \sigma_{22}.$$

Proof. Under the general hypothesis, $\{\{X_{n1}^+, \cdots, X_{nk_n}^+\}\}$ and $\{\{X_{n1}^-, \cdots, X_{nk_n}^-\}\}$ are infinitesimal systems. If there exist sequences of real numbers $\{u_n\}$ and $\{v_n\}$ such that $\Sigma_{j=1}^{k_n} X_{nj}^+ + u_n$ and $\Sigma_{j=1}^{k_n} X_{nj}^- + v_n$ converge jointly in law, then the centered sums each converge in law, and the central limit theorem shows that (1) and (2) are necessary. The proof of the sufficiency of (1) and (2) is similar to the proof of Theorem 1 in [8] and is given here for the sake of completeness. Define

$$S = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = 0, x_2 \geq 0 \text{ or } x_1 \geq 0, x_2 = 0\}.$$

Let

$$G_{nj}(x_1, x_2) = P[X_{nj}^+ \leq x_1, X_{nj}^- \leq x_2].$$

Then for any Borel set $B \subset \mathbf{R}^2$,

$$\int_B dG_{nj}(x_1, x_2) = 0 \qquad \text{if } B \cap S = \varnothing,$$
$$= \int_{C_B} dF_{nj}(x) \quad \text{otherwise,}$$

where $C_B = \{x \geq 0 : (x, 0) \in B\} \cup \{x < 0 : (0, -x) \in B\}$. By hypothesis and the central limit theorem, we have

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} (F_{nj}(x) - 1) = M(x) \quad \text{if } x > 0,$$

(3)

$$\lim_{n \to \infty} \sum_{j=1}^{k_n} F_{nj}(x) = M(x) \qquad \text{if } x < 0,$$

where in both cases x is a continuity point of M. Hence there is a Lévy spectral measure N over $\mathbf{R}^2 \backslash \{(0, 0)\}$ such that

(4) $$\lim_{n \to \infty} \sum_{j=1}^{k_n} \int_B dG_{nj}(x_1, x_2) = N(B) = \int_{C_B} dM(x)$$

for every Borel set $B \subset \mathbf{R}^2$ whose closure does not contain $(0, 0)$ and such that $N(\partial B) = 0$. Let us define

$$\sigma^2(\epsilon, n) = \sum_{j=1}^{k_n} \left\{ \int_{|x| < \epsilon} x^2 \, dF_{nj}(x) - \left(\int_{|x| < \epsilon} x \, dF_{nj}(x) \right)^2 \right\}.$$

One easily verifies that

$$\sigma^2(\epsilon, n) = \sigma^2(\epsilon, n, +) + \sigma^2(\epsilon, n, -) - 2 \, Cr(\epsilon, n).$$

By the general hypothesis and the central limit theorem we know that

(5) $$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \sigma^2(\epsilon, n) = \sigma^2.$$

Now, (1), (2) and (5) imply

(6)
$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} Cr(\epsilon, n) = \sigma_{12},$$

for some nonpositive constant σ_{12} which satisfies $\sigma^2 = \sigma_{11} + \sigma_{22} - 2\sigma_{12}$. Let $t = (t_1, t_2)$, $x = (x_1, x_2)$ and

$$Q_n(t) = \sum_{j=1}^{k_n} \left\{ \int_{|x|<\epsilon} (t, x)^2 dG_{nj}(x) - \left(\int_{|x|<\epsilon} (t, x) dG_{nj}(x) \right)^2 \right\},$$

where $(t, x) = t_1 x_1 + t_2 x_2$. One observes that

$$Q_n(t) = \sigma^2(\epsilon, n, +)t_1^2 + \sigma^2(\epsilon, n, -)t_2^2 + 2t_1 t_2 \, Cr(\epsilon, n).$$

Applying (1), (2) and (6) to this expression, and taking (4) into account, we obtain the sufficiency of (1) and (2) by the multivariate form of the central limit theorem given above.   Q.E.D.

It should be pointed out here that it is possible that the general hypothesis above holds and yet conditions (1) and (2) are not both fulfilled. An example of such can be easily deduced. Indeed, in [9] a distribution function F_X of a random variable X was constructed so that neither F_{X+} nor F_{X-} is in the domain of attraction of the normal distribution and yet F_X is. Thus, if $\{X_n\}$ is a sequence of independent identically distributed random variables with common distribution function F_X and with normalizing coefficients $\{B_n\}$, and if we denote $X_{nk} = X_k/B_n$, $1 \le k \le n = k_n$, then it is easy to see that the general hypothesis is satisfied but that neither of the conditions of Theorem 2 is satisfied.

We now present the set of all bivariate limit distributions obtained in Theorem 2.

Theorem 3. *Under the conditions of Theorem 2, the joint limiting bivariate distribution obtained has a characteristic function of the form*

(1) $$f(u_1, u_2) = \exp\left\{ i\gamma^t u - u^t \Sigma u/2 + \int_0^\infty A(u_1, x)M_1(dx) + \int_0^\infty A(u_2, x)M_2(dx) \right\},$$

where $\Sigma = (\sigma_{ij})$ with $\sigma_{12} = \sigma_{21} \le 0$, $A(u, x) = e^{iux} - 1 - iux/(1 + x^2)$, $M_1(x) = M(x)$, and $M_2(x) = -M(-x)$, where M is the Lévy spectral measure determined under the general hypothesis. Conversely, given any Lévy spectral measure M and a bivariate distribution of the above form (1) with this M, there always exists an infinitesimal system $\{\{X_{nj}, 1 \le j \le k_n\}\}$ which satisfies the general hypothesis and the conditions of Theorem 1 so that the limiting distribution of appropriately centered sums of the positive and negative parts is given by (1).

Proof. We need only prove the converse, the direct statement following immediately as a corollary to Theorem 1, using the multivariate form of the central limit theorem quoted earlier and the fact that $Cr(\epsilon, n) \le 0$.

In order to prove the converse, we show that if $\gamma = (\gamma_1, \gamma_2) \in R^2$, if $\Sigma = (\sigma_{ij})$ is any two-by-two nonnegative definite symmetric matrix for which $\sigma_{12} = \sigma_{21} \leq 0$, and if M is any Lévy spectral function, then there exists an infinitesimal system $\{\{X_{nj}, 1 \leq j \leq k_n\}\}$ which satisfies the general hypothesis and the conditions of Theorem 1 so that the limiting distribution of properly centered sums of the positive and negative components is of the form given in (1).

First we shall show that, given any matrix Σ just described, there is a sequence of independent, identically distributed random variables $\{X_n\}$ such that, for some sequences of normalizing constants $B_n \rightarrow +\infty$ and centering constants $\{\alpha_n\}$ and $\{\beta_n\}$, then the joint limiting distribution of $B_n^{-1} \sum_{j=1}^n X_n^+ + \alpha_n$, $B_n^{-1} \sum_{j=1}^n X_j^- + \beta_n$ is bivariate normal with zero mean vector and covariance matrix Σ. We must consider four cases.

Case (i). $\sigma_{12} = \sigma_{21} < 0$, $\det \Sigma > 0$.

Consider a random variable X which takes values $-a$, 0, b with probabilities $(1 - \theta)/2$, θ, $(1 - \theta)/2$ respectively, where

$$\theta = ((\sigma_{11}\sigma_{22})^{1/2} + \sigma_{12})^2 / (\sigma_{11}\sigma_{22} - \sigma_{12}^2),$$

$$b = 2\sigma_{11}^{1/2}/(1 - \theta^2)^{1/2}, \quad \text{and} \quad a = 2\sigma_{22}^{1/2}/(1 - \theta^2)^{1/2}.$$

One can easily verify in this case that $0 < \theta < 1$, $\text{var}(X^+) = \sigma_{11}$, $\text{var}(X^-) = \sigma_{22}$ and $EX^+EX^- = -\sigma_{12}$. Now let $\{X_n\}$ be independent identically distributed random variables with the same distribution as X. By Theorem 4 in [9] we may take $B_n = n^{1/2}$, $\alpha_n = -n^{1/2}EX^+$ and $\beta_n = -n^{1/2}EX^-$ and obtain the conclusion sought in this case.

Case (ii). $\sigma_{12} = \sigma_{21} < 0$, $\det \Sigma = 0$.

Consider a random variable X which takes values $2\sigma_{11}^{1/2}$ and $-\sigma_{22}^{1/2}$ with equal probability $\frac{1}{2}$. Apply Theorem 4 from [9] as above.

Case (iii). $\sigma_{12} = \sigma_{21} = 0$, $\sigma_{11} > 0$, $\sigma_{22} > 0$.

Let F be a symmetric distribution function, and assume that F is in the domain of attraction of the normal distribution but not in its domain of normal attraction. Then it is known that

$$\int_{-x}^{x} t^2 \, dF(t) = 2 \int_0^x t^2 \, dF(t)$$

is a slowly varying function at ∞. By Theorem 5 in [9] there exist normalizing coefficients $B_n \rightarrow \infty$ and centering constants $\{c_n\}$ so that the joint distribution function of

$$B_n^{-1}\left(\sum_{j=1}^n X_j^+ + c_n\right), \quad B_n^{-1}\left(\sum_{j=1}^n X_j^- + c_n\right)$$

converges to that of two independent normal random variables with mean zero and variance one. Now replace F by

$$G(x) = F(x\sigma_{11}^{-\frac{1}{2}}) \quad \text{if } x > 0,$$

$$= F(x\sigma_{22}^{-\frac{1}{2}}) \quad \text{if } x < 0.$$

Let $\{U_n\}$ be independent, identically distributed random variables with common distribution G, let $u_n = c_n\sigma_{11}^{\frac{1}{2}}/B_n$ and $v_n = c_n\sigma_{22}^{\frac{1}{2}}/B_n$. Then the limit distribution function of

$$B_n^{-1}\sum_{j=1}^{n} U_j^+ + u_n, \qquad B_n^{-1}\sum_{j=1}^{n} U_n^- + v_n$$

is bivariate normal with mean vector zero and covariance matrix Σ.

Case (iv). $\sigma_{12} = \sigma_{21} = 0$; $\sigma_{11} > 0$, $\sigma_{22} = 0$ or $\sigma_{22} > 0$ and $\sigma_{11} = 0$.

In this case, replace G in Case (iii) by

$$G_0(x) = G(x) \quad \text{if } x \geq 0, \qquad G_1(x) = 1 \quad \text{if } x \geq 0,$$
$$\text{or}$$
$$= 0 \quad \text{if } x < 0, \qquad = G(x) \quad \text{if } x < 0,$$

to obtain an analogous result.

In whichever case Σ is above, let F denote the distribution function for that case, let $\{B_n\}$ denote the normalizing constants for F, and denote $F_{nj}(x) = F(B_nx)$, $1 \leq j \leq n$. Now let $\epsilon_n > 0$ be such that $\pm\epsilon_n$ are points of continuity of M and $\epsilon_n \downarrow 0$. For each n let $K_n = M(-\epsilon_n) - M(\epsilon_n)$. Let $\{r_n\}$ be a sequence of positive integers satisfying $K_n/r_n \leq 1/n$. If $M \equiv 0$, take $r_n = 0$. If $r_n > 0$, define

$$G_n(x) = M(x)/r_n, \qquad \text{if } x \leq -\epsilon_n,$$
$$= M(-\epsilon_n)/r_n, \qquad \text{if } -\epsilon_n < x \leq 0,$$
$$= 1 + M(\epsilon_n)/r_n, \qquad \text{if } 0 < x \leq \epsilon_n,$$
$$= 1 + M(x)/r_n, \qquad \text{if } x > \epsilon_n.$$

Next, define $k_n = n + r_n$ and $F_{nj}(x) = G_n(x)$ for $n + 1 \leq j \leq k_n$. Finally, for every n let $\{X_{n,1}, \cdots, X_{n,k_n}\}$ be independent random variables such that the distribution function of X_{nj} is F_{nj}. It is easy to verify that $\{\{X_{nj}\}\}$ is an infinitesimal system which satisfies the general hypothesis and Theorem 1. Q.E.D.

4. **An application of Theorems 1 and 2.** Let $\{X(t),\ 0 \leq t \leq T\}$ be a stochastic process which is centered and has no fixed points of discontinuity. The same assumptions and notation from the beginning of §3 carry over here except that in this section $\{X(t),\ 0 \leq t \leq T\}$ is allowed to have a Gaussian component. The following theorem is an application of Theorem 1 to $X(t)$, and this section is devoted to its proof.

Theorem 4. *Under the above hypotheses on* $\{X(t), \ 0 \le t \le T\}$, *there is a sequence* $\{c_n\}$ *of constants such that the joint distribution function of* $\sum_{j=1}^{k_n} X_{nj}^+ + c_n$ *and* $\sum_{j=1}^{k_n} X_{nj}^- + c_n$ *converges to a bivariate infinitely divisible distribution function whose characteristic function is*

$$f(u_1, u_2) = \exp\left\{i(\gamma, \mathbf{u}) - \mathbf{u}^t \Sigma \mathbf{u}/2 + \int_0^\infty A(u_1, x)\, dM_1(x) + \int_0^\infty A(u_2, x)\, dM_2(x)\right\},$$

where

$$\Sigma = \begin{pmatrix} (\pi - 1)\sigma^2(T)/2\pi & -\sigma^2(T)/2\pi \\ -\sigma^2(T)/2\pi & (\pi - 1)\sigma^2(T)/2\pi \end{pmatrix}, \qquad A(u, x) = e^{iux} - 1 - iux/(1 + x^2),$$

and $M_1(x) = M_T(x)$ *and* $M_2(x) = -M_T(-x)$, *where* $x > 0$ *and* $\pm x$ *are continuity points of* M_T.

If X is a random variable with finite expectation, and if A is an event, we denote $E[A, X] = \int_A X\, dP$. According to Theorem 2 we need to prove that

(1) $\displaystyle \lim_{\epsilon \downarrow 0} \ \overline{\lim_{n \to \infty}} \ \sum_{k=1}^{n} \{E[0 \le X_{nk} < \epsilon, X_{nk}^2] - E^2[0 \le X_{nk} < \epsilon, X_{nk}]\} = \frac{(\pi - 1)\sigma^2(T)}{2\pi},$

and

(2) $\displaystyle \lim_{\epsilon \downarrow 0} \ \overline{\lim_{n \to \infty}} \ \sum_{k=1}^{n} \{E[-\epsilon < X_{nk} \le 0, X_{nk}^2] - E^2[-\epsilon < X_{nk} \le 0, X_{nk}]\} = \frac{(\pi - 1)\sigma^2(T)}{2\pi}.$

Equations (1) and (2) ensure the existence of two sequences, $\{c_n\}$ and $\{c'_n\}$, such that $\sum_{k=1}^{n} X_{nk}^+ + c_n$ and $\sum_{k=1}^{n} X_{nk}^- + c'_n$ converge jointly in law. However, since $X(T) + (c_n - c'_n) = \sum_{k=1}^{n} X_{nk}^+ + c_n - (\sum_{k=1}^{n} X_{nk}^- + c'_n)$, then $\{c_n - c'_n\}$ is a convergent sequence, and hence we may take $c_n = c'_n$.

Equation (1) will be proved for *any* process satisfying the above conditions. Now if $X(t)$ is such a process, so is $-X(t)$. Equation (2) follows from equation (1) by applying (1) to $-X(t)$. Thus it suffices to prove (1) in order to prove Theorem 4. The first part of the proof is devoted to separating the contributions of the Gaussian and non-Gaussian components to (1) and showing that the quantity in (1) is due to the Gaussian part alone. An easy computation will finish the proof.

Lemma 1. *The following holds:*

$$\lim_{\epsilon \downarrow 0} \ \limsup_{n \to \infty} \ \sum_{k=1}^{n} E[0 \le X_{nk} < \epsilon, V_{nk}^2] = 0.$$

Proof. It is sufficient to prove for every $\epsilon > 0$ that

(*) $$\lim_{\epsilon \downarrow 0} \ \limsup_{n \to \infty} \ \sum_{k=1}^{n} E[0 \le X_{nk} < \epsilon, |V_{nk}| < 2\epsilon, V_{nk}^2] = 0$$

and

$(**)$
$$\lim_{n\to\infty} \sum_{k=1}^{n} E[0 \le X_{nk} < \epsilon, \; |V_{nk}| \ge 2\epsilon, V_{nk}^2] = 0.$$

We first observe that for every n and k, $1 \le k \le n$,

$$E[0 \le X_{nk} < \epsilon, |V_{nk}| < 2\epsilon, V_{nk}^2] \le E[|V_{nk}| < 2\epsilon, V_{nk}^2].$$

An easy consequence of Theorem 1 (where we take $g(x) = \min\{x^2, 4\epsilon^2\}$) and the central limit theorem is that if $\pm 2\epsilon$ are continuity points of M_T, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} E[|V_{nk}| < 2\epsilon, V_{nk}^2] = \int_{-2\epsilon}^{2\epsilon} x^2 M_T(dx).$$

We obtain (*) by taking the limit as $\epsilon \downarrow 0$. We now prove (**). First we recall the well-known inequality: for $\lambda > 0$,

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{\lambda}^{\infty} e^{-x^2/2\sigma^2} dx \le \frac{\sigma}{\lambda\sqrt{2\pi}} e^{-\lambda^2/2\sigma^2}.$$

Since U_{nk} is normally distributed with mean zero and variance σ_{nk}^2, we have

$$P[|U_{nk}| > j\epsilon] \le (2\sigma_{nk}/j\epsilon\sqrt{2\pi})e^{-j^2\epsilon^2/2\sigma_{nk}^2}.$$

The monotone convergence theorem implies

$$\sum_{k=1}^{n} E[0 \le X_{nk} < \epsilon, |V_{nk}| > 2\epsilon, V_{nk}^2]$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{\infty} E[0 \le X_{nk} < \epsilon, (j+1)\epsilon < |V_{nk}| \le (j+2)\epsilon, V_{nk}^2].$$

But $X_{nk} = U_{nk} + V_{nk}$; hence, if $0 \le X_{nk} < \epsilon$ and $(j+1)\epsilon < |V_{nk}| \le (j+2)\epsilon$, then $|U_{nk}| \ge j\epsilon$. Hence the right side above is not greater than

$$\sum_{k=1}^{n} \sum_{j=1}^{\infty} E[|U_{nk}| \ge j\epsilon, (j+1)\epsilon \le |V_{nk}| < (j+2)\epsilon, V_{nk}^2].$$

Because U_{nk} and V_{nk} are independent, this is dominated by

$$\sum_{k=1}^{n} \sum_{j=1}^{\infty} (j+2)^2\epsilon^2 P[|U_{nk}| > j\epsilon]P[|V_{nk}| \ge 2\epsilon].$$

The upper bound for $P[|U_{nk}| > j\epsilon]$ determined above implies

$$\sum_{j=1}^{\infty} (j+2)^2\epsilon^2 P[|U_{nk}| > j\epsilon] \le 3\sigma_{nk}^2 \sum_{j=1}^{\infty} \frac{(j+2)\epsilon}{\sqrt{2\pi\sigma_{nk}^2}} e^{-(j\epsilon)^2/2\sigma_{nk}^2}$$

$$\le 3(\sigma_{nk}^2)^2/2\epsilon.$$

These inequalities imply

$$\sum_{k=1}^{n} E[0 \le X_{nk} < \epsilon, \, |V_{nk}| \ge 2\epsilon, \, V_{nk}^2]$$

$$\le (3\sigma^2(T)/2) \max\{\sigma_{nk}^2 P[|V_{nk}| \ge 2\epsilon]: \, 1 \le k \le n\}.$$

Taking the limit of both sides as $n \to \infty$, we obtain (**). Q.E.D.

Lemma 2. *Let $A \, \Delta \, B$ denote the symmetric difference between sets A and B. Then*

(1) $$\lim_{\epsilon \downarrow 0} \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} E[[0 \le X_{nk} < \epsilon] \, \Delta \, [0 \le U_{nk} < \epsilon], \, U_{nk}^2] = 0.$$

Proof. We observe that

$$[0 \le X_{nk} < \epsilon] \, \Delta \, [0 \le U_{nk} < \epsilon] = A_1^{(k)} \cup A_2^{(k)} \cup A_3^{(k)},$$

where

$$A_1^{(k)} = [0 \le X_{nk} < \epsilon, \, U_{nk} < 0] \cup [0 \le U_{nk} < \epsilon, \, X_{nk} < 0],$$

$$A_2^{(k)} = [0 \le X_{nk} < \epsilon, \, U_{nk} \ge \epsilon] \quad \text{and} \quad A_3^{(k)} = [0 \le U_{nk} < \epsilon, \, X_{nk} \ge \epsilon].$$

Equation (1) above will be proved by showing that

(2) $$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \sum_{k=1}^{n} E[A_i^{(k)}, \, U_{nk}^2] = 0, \quad \text{for } i = 1, 2, 3.$$

When $A_1^{(k)}$ occurs, X_{nk} and U_{nk} have opposite signs. Since $X_{nk} = U_{nk} + V_{nk}$, we have $A_1^{(k)} \subset [|U_{nk}| < |V_{nk}|]$. But $[|U_{nk}| < |V_{nk}|] = A_{11}^{(k)} \cup A_{12}^{(k)}$, where $A_{11}^{(k)} = [|U_{nk}| < |V_{nk}| < 2\epsilon]$, and $A_{12}^{(k)} = [|U_{nk}| < |V_{nk}|] \cap [|V_{nk}| \ge 2\epsilon]$. Now, using the fact that U_{nk} and V_{nk} are independent, applying Theorem 1 as in the proof of Lemma 1, and by the Central Limit Theorem, we have

$$\overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} E[A_1^{(k)}, \, U_{nk}^2] \le \sum_{j=1}^{2} \overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} E[A_{1j}^{(k)}, \, U_{nk}^2]$$

$$\le \overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} E[|V_{nk}| < 2\epsilon, \, V_{nk}^2] + \overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{n \to \infty}} \sum_{k=1}^{n} E[|V_{nk}| \ge 2\epsilon, \, U_{nk}^2]$$

$$\le \overline{\lim_{\epsilon \downarrow 0}} \int_{-\epsilon}^{\epsilon} x^2 M_T(dx) + \sigma^2(T) \overline{\lim_{\epsilon \downarrow 0}} \, \overline{\lim_{n \to \infty}} \max_{1 \le k \le n} \{P[|V_{nk}| \ge 2\epsilon]\} = 0,$$

which proves (2) for $i = 1$.

In order to prove (2) when $i = 2$, we write $A_2^{(k)} = A_{21}^{(k)} \cup A_{22}^{(k)}$ where

$A_{21}^{(k)} = [0 \leq X_{nk} < \epsilon, \epsilon \leq U_{nk} < 2\epsilon]$ and $A_{22}^{(k)} = [0 \leq X_{nk} < \epsilon, U_{nk} \geq 2\epsilon]$. Then

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[A_2^{(k)}, U_{nk}^2] \leq \sum_{j=1}^{2} \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[A_{2j}^{(k)}, U_{nk}^2]$$

$$\leq \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} 4\epsilon^2 \sum_{k=1}^{n} P(A_{21}^{(k)}) + \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[|V_{nk}| > \epsilon, U_{nk}^2].$$

By independence of U_{nk} and V_{nk}, the second term to the right of the last inequality is bounded by

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \max\{P[|V_{nk}| > \epsilon]: 1 \leq k \leq n\}\sigma^2(T)$$

which is zero. As for the first term on the right of the last inequality, it is clearly bounded above by

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} 4\epsilon^2 \sum_{k=1}^{n} P[U_{nk} \geq \epsilon],$$

which is zero by applying the central limit theorem to the infinitesimal system $\{\{U_{nj}\}\}$, keeping in mind that $U(T) = \sum_{j=1}^{n} U_{nj}$ for all n, and that $U(t)$ is Gaussian. This proves (2) for $i = 2$.

The argument for $i = 3$ is similar to that for $A_2^{(k)}$. We decompose $A_3^{(k)}$ as follows:

$$A_3^{(k)} = A_{31}^{(k)} \cup A_{32}^{(k)},$$

where

$$A_{31}^{(k)} = [0 \leq U_{nk} < \epsilon/2, X_{nk} \geq \epsilon] \quad \text{and} \quad A_{32}^{(k)} = [\epsilon/2 \leq U_{nk} < \epsilon, X_{nk} \geq \epsilon].$$

Since $A_{31}^{(k)} \subset [V_{nk} > \epsilon/2]$, we have for reasons given before in similar situations,

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[A_{31}^{(k)}, U_{nk}^2] \leq \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[V_{nk} > \epsilon/2, U_{nk}^2]$$

$$\leq \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \max\{P[V_{nk} > \epsilon/2]: 1 \leq k \leq n\}\sigma^2(T) = 0.$$

Also, since $A_{32}^{(k)} \subset [\epsilon/2 \leq U_{nk} < \epsilon]$, we have for by now obvious reasons

$$\varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[A_{32}^{(k)}, U_{nk}^2] \le \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \sum_{k=1}^{n} E[\epsilon/2 \le U_{nk} < \epsilon, U_{nk}^2]$$

$$\le \varlimsup_{\epsilon \downarrow 0} \varlimsup_{n \to \infty} \epsilon^2 \sum_{k=1}^{n} P[U_{nk} \ge \epsilon/2] = 0.$$

These last two strings of inequalities prove (2) for $i = 3$ and thus the lemma. Q.E.D.

Lemma 3. *If* $\chi_{[a,b]}(x) = 1$ *if* $x \in [a, b)$ *and* $= 0$ *if* $x \notin [a, b)$, *then*

(1) $$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \sup E \sum_{k=1}^{n} |U_{nk}^2 \chi_{[0,\epsilon)}(U_{nk}) - X_{nk}^2 \chi_{[0,\epsilon)}(X_{nk})| = 0,$$

(2) $$\lim_{n \to \infty} E \sum_{k=1}^{n} U_{nk}^2 \chi_{[0,\epsilon)}(U_{nk}) = \sigma^2(T)/2,$$

and

(3) $$\lim_{n \to \infty} \sum_{k=1}^{n} E^2\{U_{nk} \chi_{[0,\epsilon)}(U_{nk})\} = \frac{\sigma^2(T)}{2\pi}.$$

Proof. In order to prove (1) we observe that

$$E \sum_{k=1}^{n} |U_{nk}^2 \chi_{[0,\epsilon)}(U_{nk}) - X_{nk}^2 \chi_{[0,\epsilon)}(X_{nk})|$$

$$\le \sum_{k=1}^{n} E\{[0 \le U_{nk} < \epsilon] \Delta [0 \le X_{nk} < \epsilon], U_{nk}^2\}$$

$$+ 2 \sum_{k=1}^{n} E\{0 \le X_{nk} < \epsilon, |U_{nk}V_{nk}|\} + \sum_{k=1}^{n} E\{0 \le X_{nk} < \epsilon, V_{nk}^2\}.$$

Now let $n \to \infty$ and then $\epsilon \downarrow 0$. The first term on the right-hand side becomes 0 by Lemma 2. The third term on the right-hand side becomes zero by Lemma 1. Applications of the Cauchy-Schwarz inequality bound the middle term of the right-hand side by

$$2 \sum_{k=1}^{n} E^{1/2}\{0 \le X_{nk} < \epsilon, U_{nk}^2\} E^{1/2}\{0 \le X_{nk} < \epsilon, V_{nk}^2\}$$

$$\le 2 \left\{ \sum_{k=1}^{n} E\{0 \le X_{nk} < \epsilon, U_{nk}^2\} \right\}^{1/2} \left\{ \sum_{k=1}^{n} E\{0 \le X_{nk} < \epsilon, V_{nk}^2\} \right\}^{1/2}.$$

The first of these two multiplied terms is bounded above by $\sigma(T) < \infty$, and the second tends to zero by Lemma 1 if we first let $n \to \infty$ and then $\epsilon \downarrow 0$. This proves (1). We next prove (2). Since U_{nk} is Gaussian with mean zero and variance σ_{nk}^2, we obtain

$$E \sum_{k=1}^{n} U_{nk}^2 \chi_{[0,\epsilon]}(U_{nk}) = - \sum_{k=1}^{n} \frac{\sigma_{nk}\epsilon}{\sqrt{2\pi}} \exp\{-\epsilon^2/2\sigma_{nk}^2\} + \sum_{k=1}^{n} \frac{\sigma_{nk}^2}{\sqrt{2\pi}} \int_0^{\epsilon/\sigma_{nk}} e^{-t^2/2} dt.$$

Now for n sufficiently large,

$$\sigma_{nk} e^{-\epsilon^2/2\sigma_{nk}^2} < \sigma_{nk}^3,$$

from which we obtain that the limit of the first term on the right, as $n \to \infty$, is zero. The limit of the second term, as $n \to \infty$, is seen to be $\sigma^2(T)/2$ by an easy application of the dominated convergence theorem, and thus (2) is proved. In order to establish (3), we observe that

$$\lim_{n\to\infty} \sum_{k=1}^{n} E^2\{0 \le U_{nk} < \epsilon, U_{nk}\} = \lim_{n\to\infty} \frac{1}{2\pi} \sum_{k=1}^{n} \sigma_{nk}^2 \left(\int_0^{\epsilon/\sigma_{nk}} t e^{-t^2/2} dt \right)^2,$$

and this is seen to equal $\sigma^2(T)/2\pi$ by an application of the dominated convergence theorem.   Q.E.D.

For the convenience of the reader we state here a lemma due to P. Greenwood and B. Fristedt (Lemma 3.4 in [4]) which we shall use.

Lemma 4. *For $n = 1, 2, \cdots$, let $S_n = \sum_{k=1}^n S_{nk}$ and $T_n = \sum_{k=1}^n T_{nk}$ be sums of nonnegative independent random variables. Suppose that as $n \to \infty$,*

$$\sum_{k=1}^{n} (ES_{nk})^2 \to m_1, \qquad \sum_{k=1}^{n} E(S_{nk}^2) \to m_2 < \infty,$$

and

$$\limsup_{n\to\infty} E \sum_{k=1}^{n} |S_{nk}^2 - T_{nk}^2| \le \delta.$$

Then $\operatorname{Var} S_n \to m_2 - m_1$,

$$\limsup \left| \sum_{k=1}^{n} (ET_{nk})^2 - m_1 \right| \le [12\delta(m_2 + \delta)]^{1/2}, \qquad \limsup \left| \sum_{k=1}^{n} E(T_{nk}^2) - m_2 \right| \le \delta,$$

and

$$\limsup |\operatorname{Var} T_n - (m_2 - m_1)| \le \delta + 12\delta(m_2 + \delta).$$

We now prove Theorem 4. In the statement of Lemma 4 above, replace S_{nk} by $U_{nk}\chi_{[0,\epsilon]}(U_{nk})$ and T_{nk} by $X_{nk}\chi_{[0,\epsilon]}(X_{nk})$. Now denote

$$\delta(\epsilon) = \lim_{n \to \infty} \sup E \sum_{k=1}^{n} |U_{nk}^2 \chi_{[0,\epsilon)}(U_{nk}) - \ddot{X}_{nk}^2 \chi_{[0,\epsilon)}(X_{nk})|.$$

By (1) in Lemma 3, $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$. By Lemma 3, m_2 and m_1 in Lemma 4 become $m_2 = \sigma^2(T)/2$ and $m_1 = \sigma^2(T)/2\pi$. Hence by Lemma 4, we have

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \sup \left| \text{Var}\left(\sum_{k=1}^{n} X_{nk} \chi_{[0,\,\epsilon)}(X_{nk}) \right) - \frac{\pi - 1}{2\pi} \sigma^2(T) \right| = 0,$$

which proves Theorem 4.   Q.E.D.

An immediate consequence of this result is that there always exists a sequence of centering constants $\{c_n\}$ such that $\Sigma_{j=1}^{n} |X_{nj}| + c_n$ converges in law.

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