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RATIONAL APPROXIMATION ON PRODUCT SETS[1]

BY

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ABSTRACT. Our object here is to study pointwise bounded limits, decomposition of orthogonal measures and distance estimates for $R(K_1 \times K_2)$ where K_1 and K_2 are compact sets in the complex plane.

1. Introduction and main results. When X is a compact subset of $\mathbf{C}^N$, then $R(X)$ denotes the algebra of continuous functions which can be approximated uniformly on X by rational functions with singularities off X. A measure μ on X is called orthogonal to $R(X)$, we write $\mu \in R(X)^\perp$, if $\int f\, d\mu = 0$ for all $f \in R(X)$. A positive measure λ on X is a representing measure for $x \in X$ if $\int f\, d\lambda = f(x)$ for all $f \in R(X)$. The restriction of a measure μ to a subset E of X is denoted μ_E, and E is called a nullset for $R(X)^\perp$ if $\mu_E = 0$ whenever $\mu \in R(X)^\perp$. We refer to [6] for further details on terminology.

First we will obtain the following decomposition theorem.

Theorem 1. *Let K_i be compact subsets of $\mathbf{C}$ $(i = 1, 2)$, and let Q_i be the set of non peak points for $R(K_i)$. If μ is a measure on $K_1 \times K_2$ orthogonal to $R(K_1 \times K_2)$, then μ decomposes uniquely into $\mu = \mu_0 + \mu_1 + \mu_2$ where the μ_j's $(j = 0, 1, 2)$ are pairwise mutually singular and orthogonal to $R(K_1 \times K_2)$. The measure μ_0 belongs to the band of measures on $K_1 \times K_2$ generated by representing measures for points in $Q_1 \times Q_2$. μ_1 is supported on a set $E_1 \times K_2$, and μ_2 is supported on a set $K_1 \times E_2$ where E_i are nullsets for $R(K_i)^\perp$.*

Such a decomposition theorem has been obtained by B. Cole (unpublished) for the bidisc algebra. His proof carries over to the algebra $A(U \times V)$, but not to $R(K_1 \times K_2)$. Here we make the appropriate modifications, using Vitushkin's technique, to obtain this extension of Cole's decomposition.

When Q is a subset of X, we introduce the algebra $B(Q, R(X))$ of pointwise bounded limits as follows.

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$$B(Q, R(X)) = \{f: Q \to \mathbf{C};\ \textit{there is a bounded sequence } \{f_n\} \textit{ in } R(X) \textit{ such that } f_n(z) \to f(z) \textit{ for all } z \in Q\}.$$

Our second main result is now

Theorem 2. *Let K_i and Q_i $(i = 1, 2)$ be as in Theorem 1. Let $f: Q_1 \times Q_2 \to \mathbf{C}$ be bounded. The following are equivalent.*

(a) *$f \in B(Q_1 \times Q_2, R(K_1 \times K_2))$.*

(b) *$f(z, \ ) \in B(Q_2, R(K_2))$ for all $z \in Q_1$, and $f(\ , w) \in B(Q_1, R(K_1))$ for all $w \in Q_2$.*

(c) *There is a sequence $\{f_n\}$ in $R(K_1 \times K_2)$ with $\|f_n\| \le \|f\|$ and $f_n(z, w) \to f(z, w)$ for $(z, w) \in Q_1 \times Q_2$.*

Here $\| \ \|$ means the uniform norm over the respective sets of definition.

The proof of Theorem 2 employs Vitushkin techniques, especially the characterization of $B(Q, R(K))$ for compact $K \subset \mathbf{C}$, in terms of analytic capacity due to Gamelin and Garnett [8], and also Davie's theorem in [4] telling that $B(Q, R(K))$ is uniformly closed. We also here rely heavily on some unpublished ideas of B. Cole.

When σ is a positive measure on $K_1 \times K_2$, we define $H^\infty(\sigma)$ as the weak-star closure of $R(K_1 \times K_2)$ in $L^\infty(\sigma)$. Our third main result is

Theorem 3. *Let K_i and Q_i $(i = 1, 2)$ be as in Theorem 1. Let σ be the measure $\sigma = dx\,dy_{Q_1} \times dx\,dy_{Q_2}$. Assume Q_i is dense in K_i for $i = 1, 2$, and let $u: K_1 \times K_2 \to \mathbf{C}$ be continuous. Then*

$$\operatorname{dist}(u, H^\infty(\sigma)) = \operatorname{dist}(u, R(K_1 \times K_2)).$$

Distance equalities like those in Theorem 3 have been obtained first by Sarason [10] for the disc algebra, and more recently by Gamelin and Garnett [8] for $A(U)$ and $R(K)$. Our proof of Theorem 3 employs a general criterion in [8].

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2. A-measures and the B-norm. A-measures were introduced by Henkin [9], and were employed to pointwise bounded approximation for $A(D)$ when $D \subset \mathbf{C}^N$ is strictly pseudoconvex by Cole and Range [3].

Here let A be a uniform algebra on a compact metric space X, and Q be a fixed Borel set in X. A measure ν on X is called an A**-measure** for A on Q if $f_n \to 0$ weak-star in $L^\infty(|\nu|)$ whenever $\{f_n\}$ is bounded in A and $f_n \to 0$ pointwise on Q. To verify that certain measures are A-measures, the following property (cf. [9, Theorem 1.4]) turns out to be useful.

Definition 2.1. A measure ν on X has property (H) if there is a subfamily S of $C(X)$ with the algebra generated by A and S dense in $C(X)$, such that for each $g \in S$ and each bounded sequence $\{f_n\}$ in A with $f_n \to 0$ pointwise on Q, there exists a bounded sequence $\{F_n\}$ in A satisfying

(a) $F_n \to 0$ pointwise on Q.

(b) $F_n - gf_n \to 0$ weak-star in $L^\infty(|\nu|)$.

It is easy to prove the following.

Lemma 2.2. *If ν is orthogonal to A or is a representing measure for a point in Q, then ν is an A-measure if and only if ν has property* (H).

Let now $\mathfrak{M}_0$ denote the band of measures on X generated by representing measures for points in Q. We want to show that A-measures are in $\mathfrak{M}_0$. First however we prove the following Forelli-type lemma (cf. [6, II. 7.3]).

Lemma 2.3. *Let A be a uniform algebra on a compact X. Let ϕ_k be multiplicative linear functionals on A with M_k as the set of representing measures for ϕ_k, $k = 1, 2, \cdots$. Let F be an F_σ-set which is a nullset for M_k for $k = 1, 2, \cdots$. Then there is a sequence $\{f_n\}$ in A with $\|f_n\| \le 1$ such that $|f_n| \to 1$ on F, and $f_n \to 0$ weak-star in $L^\infty(\lambda)$ for each $\lambda \in M_k$ for $k = 1, 2, \cdots$.*

Proof. Let first $E \subset X$ be closed, $\phi \in M_A$ with $M_\phi(E) = 0$. As in [6, II. 7.3], we get $b_n \in A$ with $\operatorname{Re} b_n > 0$, $\operatorname{Re} b_n \ge n^2$ on E and with $\operatorname{Re} \phi(b_n) < 1/n$ and $\operatorname{Im} \phi(b_n) = 0$. Now $a_n = b_n^{-1} \in A$ and satisfies $\operatorname{Re} a_n \ge 0$, $\operatorname{Re} a_n \le n^{-2}$ on E and $\operatorname{Re} \phi(a_n) > n$. Next we return to our situation, and let $F = \bigcup_n F_n$ where each F_n is closed and $F_n \subset F_{n+1}$. By the arguments above there are $a_n^{(k)} \in A$ with $\operatorname{Re} a_n^{(k)} \ge 0$, $\operatorname{Re} a_n^{(k)} \le n^{-2}$ on F_n and $\operatorname{Re} \phi_k(a_n^{(k)}) \ge n$. For each n we put $a_n = a_n^{(1)} + \cdots + a_n^{(n)}$, and $f_n = \exp(-a_n)$.

This now applies to

Proposition 2.4. *Assume Q is the union of countably many parts for A. If ν is an A-measure, then $\nu \in \mathfrak{M}_0$.*

Proof. Decomposing ν relative to the band $\mathfrak{M}_0$, we may assume $\nu \in \mathfrak{M}_0'$. Choose one ϕ_k in each of the countably many parts whose union is Q. Let M_k be as in Lemma 2.3. By [6, II. 7. 4], there is an F_σ-set F with $|\nu|(X \setminus F) = 0$ and $\lambda(F) = 0$ for $\lambda \in M_k$, $k = 1, 2, \cdots$. Let now $\{f_n\}$ be a sequence as in Lemma

2.3. If $x \in Q$ with representing measure λ_x, then there is λ in some M_k with $\lambda_x \ll \lambda$ [6, pp. 143–144], so $\lambda_x = g\lambda$ with $g \in L^1(\lambda)$. Now $f_n(x) = \int g f_n \, d\lambda \to 0$, so $f_n \to 0$ pointwise on Q. Then $f_n \to 0$ weak-star in $L^\infty(|\nu|)$. Since $|f_n| \to 1$ on F, we have $|\nu|(F) = 0$, so $\nu = 0$.

When σ is a positive measure on Q, we define

$$B(\sigma, A) = \{f \in L^\infty(\sigma);\ \textit{there is bounded } \{f_n\} \textit{ in } A \textit{ with } f_n \to f \textit{ a.e. } \sigma\}.$$

On $B(\sigma, A)$ we introduce a norm, called the B-norm $\| \ \|_B$, by

$$\|f\|_B = \inf\{\sup \|f_n\|;\ f_n \in A,\ f_n \to f \text{ a.e. } \sigma\}.$$

Out of B. Cole's more general scheme of double duals and reducing bands (unpublished) one obtains the following answer to when the B-norm is a "sup-norm". A proof is included in [1].

Theorem 2.5. *Let σ be a positive measure on Q such that $f_n \to 0$ pointwise on Q whenever $\{f_n\}$ is bounded in A and $f_n \to 0$ a.e. σ. If every representing measure for points in Q are A-measures, then the B-norm on $B(\sigma, A)$ satisfies*

$$\|f^2\|_B = \|f\|_B^2, \qquad f \in B(\sigma, A).$$

3. A-measures for $R(K_1 \times K_2)$. When f is a bounded Borel function on $\mathbf{C}$, and ϕ is a smooth function with compact support in $\mathbf{C}$, we define

$$T_\phi f(z) = \phi(z) f(z) + \frac{1}{\pi} \int \frac{f(\xi)}{\xi - z} \frac{\partial \phi}{\partial \bar{\xi}} \, dx\, dy(\xi).$$

For properties of this T_ϕ-operator we refer to Chapter VIII of [6].

When $K \subset \mathbf{C}$ is compact, and Q is the set of non peak points for $R(K)$, we also define

$$H_\phi f(z) = \frac{1}{\pi} \int_{\mathbf{C} \setminus Q} \frac{f(\xi)}{\xi - z} \frac{\partial \phi}{\partial \bar{\xi}} \, dx\, dy(\xi),$$

and $H_\phi f \in R(K)$ (cf. [8]). We finally define

$$R_\phi f(z) = \frac{1}{\pi} \int_Q \frac{f(\xi)}{\xi - z} \frac{\partial \phi}{\partial \bar{\xi}} \, dx\, dy(\xi)$$

and obtain $\phi f + R_\phi f = T_\phi f - H_\phi f \in R(K)$ if $f \in R(K)$.

We apply this to

Lemma 3.1. *Let $K \subset \mathbf{C}$ be compact and let Q be the set of non peak points for $R(K)$. If λ is a representing measure for a point in Q, then λ is an A-measure for $R(K)$ on Q.*

Proof. We verify property (H). Let S be the family of restrictions to K of smooth functions with compact support in $\mathbf{C}$. S is dense in $C(K)$. If $\phi \in S$ and $\{f_n\}$ is bounded in $R(K)$ with $f_n(z) \to 0$ for $z \in Q$, we define $F_n = \phi f_n + R_\phi f_n$, and obtain a sequence as in 2.1.

In the rest of this section let K_i and Q_i be as in Theorem 1. Proceeding as in Cole's band decomposition for the bidisc algebra (cf. Theorem 1), we introduce three bands of measures on $K_1 \times K_2$ as follows.

$\mathfrak{M}_0$ = band generated by representing measures for points in $Q_1 \times Q_2$.

$\mathfrak{M}_1$ = measures supported on sets of the form $E_1 \times K_2$ where E_1 is a nullset for $R(K_1)^\perp$.

$\mathfrak{M}_2$ = measures supported on sets of the form $K_1 \times E_2$ where E_2 is a nullset for $R(K_2)^\perp$.

It is well known (cf. [7, p. 200]) that a continuous function $f: K_1 \times K_2 \to \mathbf{C}$ belongs to $R(K_1 \times K_2)$ if $f(z, \ ) \in R(K_2)$ for each $z \in K_1$ and $f(\ , w) \in R(K_1)$ for each $w \in K_2$.

Proposition 3.2. *Let ν be a measure on $K_1 \times K_2$. If ν belongs to $\mathfrak{M}_0$, or ν is orthogonal to $R(K_1 \times K_2)$ and singular to both $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then ν is an A-measure for $R(K_1 \times K_2)$ on $Q = Q_1 \times Q_2$.*

Proof. Again we verify property (H). Here let S be the family of restrictions to $K_1 \times K_2$ of functions g of the form $g(z, w) = \phi(z)$ or $g(z, w) = \phi(w)$, where ϕ is smooth with compact support in $\mathbf{C}$. The algebra generated by S is dense in $C(K_1 \times K_2)$. Let $g \in S$, say $g(z, w) = \phi(z)$. Let $\{f_n\}$ be bounded in $R(K_1 \times K_2)$ converging pointwise to zero on $Q_1 \times Q_2$. Define

$$R_g f_n(z, w) = \frac{1}{\pi} \int_{Q_1} \frac{f_n(\xi, w)}{\xi - z} \frac{\partial \phi}{\partial \bar{\xi}} \, dx\, dy(\xi)$$

and $F_n = g f_n + R_g f_n$. We will show that $\{F_n\}$ satisfies the conditions of 2.1.

The comment before Lemma 3.1 implies $F_n(\ ,w) \in R(K_1)$ for each $w \in K_2$. When τ is orthogonal to $R(K_2)$, then

$$\int R_g f_n(z,w)\,d\tau(w) = \frac{1}{\pi}\iint \frac{f_n(\xi,w)}{\xi - z}\frac{\partial\phi}{\partial\bar{\xi}}\,d\tau(w)\,dx\,dy(\xi) = 0$$

by the Fubini theorem. Thus $F_n(z,\ ) \in R(K_2)$ for each $z \in K_1$. Then $\{F_n\}$ is a bounded sequence in $R(K_1 \times K_2)$. Easily $F_n(z,w) \to 0$ for $(z,w) \in Q_1 \times Q_2$, and it remains to show that

$$F_n - gf_n = R_g f_n \to 0 \quad \text{weak-star in } L^\infty(|\nu|).$$

It is enough to prove $\int hR_g f_n\,d\nu \to 0$ for bounded h. By the Fubini theorem we have

$$\int hR_g f_n\,d\nu = \frac{1}{\pi}\int \frac{\partial\phi}{\partial\bar{\xi}}\int \frac{f_n(\xi,w)}{\xi - z}h(z,w)\,d\nu(z,w)\,dx\,dy(\xi)$$

and since $\int(d|\nu|(z,w)/|\xi - z|) \in L^1(dx\,dy)$, it is enough to prove that

$$\int \frac{f_n(\xi,w)}{\xi - z}h(z,w)\,d\nu(z,w) \to 0 \quad \text{for a.a. } \xi.$$

Since furthermore $h(z,w)/(\xi - z) \in L^1(\nu)$ for a.a. ξ, it is enough to show that for each $\xi \in Q_1$ $f_n(\xi,\ ) \to 0$ weak-star in $L^\infty(|\nu|)$. In fact we show that $f_n(\xi,\ ) \to 0$ a.e. ν. Put $L = \{(z,w);\ f_n(\xi,w) \not\to 0\}$. Define $h_n \in R(K_2)$ by $h_n(w) = f_n(\xi,w)$. Then $\{h_n\}$ is a bounded sequence in $R(K_2)$ and $h_n \to 0$ pointwise on Q_2. Put $E_2 = \{w;\ h_n(w) \not\to 0\}$. Lemma 3.1 tells that E_2 is a nullset for $R(K_2)^\perp$, so if ν is singular to $\mathfrak{M}_2$, then $|\nu|(L) = |\nu|(K_1 \times E_2) = 0$. If ν is a representing measure for $(z_0, w_0) \in Q_1 \times Q_2$, then the projection $\pi: K_1 \times K_2 \to K_2$ induces a representing measure $\pi^*\nu$ on K_2 for w_0 w.r.t. $R(K_2)$. Again Lemma 3.1 gives $\nu(L) = \pi^*\nu(E_2) = 0$, and this now completes the proof.

Since we know that each Q_i is the union of countably many parts for $R(K_i)$ (cf. [6, p. 146]), we can combine the Propositions 2.4 and 3.2 to get

Corollary 3.3. *A measure ν on $K_1 \times K_2$ is an A-measure for $R(K_1 \times K_2)$ on $Q_1 \times Q_2$ if and only if ν belongs to the band $\mathfrak{M}_0$.*

4. Proof of Theorem 1. Let the bands $\mathfrak{M}_0$, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be as in §3. The following two lemmas are essentially due to B. Cole (unpublished).

Lemma 4.1. *$\mathfrak{M}_1$ and $\mathfrak{M}_2$ are reducing bands (i.e. if $\mu \in R(K_1 \times K_2)^\perp$ decomposes $\mu = \mu_a + \mu_s$ relative to $\mathfrak{M}_i$, then $\mu_a \in R(K_1 \times K_2)^\perp$). $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both singular to $\mathfrak{M}_0$.*

Proof. Let $\mu \in R(K_1 \times K_2)^\perp$ decompose $\mu = \mu_a + \mu_s$ relative to the band $\mathfrak{M}_1$. There is an F_σ-set E_1 in K_1, E_1 being a nullset for $R(K_1)^\perp$ with μ_a supported on $E_1 \times K_2$ and $|\mu_s|(E_1 \times K_2) = 0$. When $E_1 = \bigcup_k F_k$ with each F_k closed, then $F_k \times K_2$ is a peak set for $R(K_1 \times K_2)$ (cf. [6, p. 56]), so $\mu_a|_{F_k \times K_2} = \mu|_{F_k \times K_2} \in R(K_1 \times K_2)^\perp$. Then $\mu_a \in R(K_1 \times K_2)^\perp$. Next let ν be representing measure for some $(z, w) \in Q_1 \times Q_2$. Each F_k consists of peak points for $R(K_1)$ so $z \notin F_k$. Let $f \in R(K_1 \times K_2)$ peak at $F_k \times K_2$. Then $\nu(F_k \times K_2) = \lim_n \int f^n d\nu = \lim_n f(z, w)^n = 0$. Thus $\nu(E_1 \times K_2) = 0$, and each measure in $\mathfrak{M}_1$ is singular to $\mathfrak{M}_0$. The proofs are similar for $\mathfrak{M}_2$.

Lemma 4.2. *$\mathfrak{M}_1 \cap R(K_1 \times K_2)^\perp$ is singular to $\mathfrak{M}_2 \cap R(K_1 \times K_2)^\perp$.*

Proof. Let $\mu \in \mathfrak{M}_1 \cap R(K_1 \times K_2)^\perp$ decompose $\mu = \mu_a + \mu_s$ relative to $\mathfrak{M}_2$. Then μ_a is supported on $E_1 \times E_2$ where E_i is a nullset for $R(K_i)^\perp$. Let $F_i \subset E_i$ $(i = 1, 2)$ be closed. Each F_i is a peak interpolation set for $R(K_i)$, and $F_1 \times F_2$ is a peak set for $R(K_1 \times K_2)$. Then $R(K_1 \times K_2)|_{F_1 \times F_2}$ is closed, and $C(F_1 \times F_2) = C(F_1) \otimes C(F_2) = R(K_1 \times K_2)|_{F_1 \times F_2}$. Since $\mathfrak{M}_2$ is reducing, $\mu_a \in R(K_1 \times K_2)^\perp$, and now $\mu_a|_{F_1 \times F_2} = 0$. Thus $\mu_a = 0$ and $\mu = \mu_s$.

Finally we can conclude with

Proof of Theorem 1. Decompose $\mu = \mu_1 + \mu_s$ relative to the band $\mathfrak{M}_1$, and decompose $\mu_s = \mu_2 + \mu_0$ relative to $\mathfrak{M}_2$. Then $\mu = \mu_0 + \mu_1 + \mu_2$, $\mu_1 \in \mathfrak{M}_1$ and $\mu_2 \in \mathfrak{M}_2$, μ_0, μ_1 and μ_2 are pairwise mutually singular and orthogonal to $R(K_1 \times K_2)$. Since μ_0 is singular to $\mathfrak{M}_1$ and $\mathfrak{M}_2$, Proposition 3.2 says μ_0 is an A-measure. Then $\mu_0 \in \mathfrak{M}_0$ by Proposition 2.4. The decomposition is unique because of the two lemmas above. This completes the proof.

A measure is called completely singular if it is singular to all representing measures for our algebra. It is well known (cf. [6, p. 47]) that $R(K)^\perp$ has no nonzero completely singular measures. This is no longer true for $R(K_1 \times K_2)^\perp$. However, looking for extreme points in the unit ball of $R(K_1 \times K_2)^\perp$, we have the following result, which also has been obtained for the bidisc algebra by B. Cole.

Corollary 4.3. ball $R(K_1 \times K_2)^\perp$ *has no completely singular extreme points.*

Proof. Let $\mu \in$ ball $R(K_1 \times K_2)^\perp$ be a completely singular extreme point. Being an extreme point, μ must belong to one of the bands $\mathfrak{M}_0$, $\mathfrak{M}_1$ or $\mathfrak{M}_2$, and being completely singular, $\mu \notin \mathfrak{M}_0$. Say $\mu \in \mathfrak{M}_1$, so μ is supported on a set $E \times K_2$ where E is a nullset for $R(K_1)^\perp$. If E_1 and E_2 are disjoint with $E = E_1 \cup E_2$, we define μ_1 and μ_2 as the restrictions of μ to $E_1 \times K_2$ and $E_2 \times K_2$ respectively. Then $\mu = \mu_1 + \mu_2$, μ_1 and μ_2 are mutually singular and belong to $\operatorname{ball} R(K_1 \times K_2)^\perp$. Since μ is extreme, $\mu_1 = 0$ or $\mu_2 = 0$. Thus the support of μ must be a set of the form $\{z\} \times K_2$. When δ_z is the point mass at z, we may view μ as $\mu = \delta_z \times \mu_0$ where $\mu_0 \in R(K_2)^\perp$. By the Glicksberg-Wermer decomposition μ_0 is absolutely continuous to some representing measure for $R(K_2)$, and thus μ cannot be completely singular. This contradiction proves the result.

Yet another consequence of the decomposition theorem is

Corollary 4.4. *If Q_i is dense in K_i $(i = 1, 2)$, then* $\operatorname{ball}\mathfrak{M}_0 \cap R(K_1 \times K_2)^\perp$ *is weak-star dense in* $\operatorname{ball} R(K_1 \times K_2)^\perp$.

Proof. By the Kreĭn-Milman theorem it is enough to show that every extreme point in $\operatorname{ball} R(K_1 \times K_2)^\perp$ is in the weak-star closure of $\operatorname{ball} \mathfrak{M}_0 \cap R(K_1 \times K_2)^\perp$, so let μ be such an extreme point. By the Glicksberg-Wermer decomposition and Corollary 4.3, then μ is absolutely continuous to some representing measure λ for a non peak point (z, w). If $(z, w) \in Q_1 \times Q_2$, then $\mu \in \mathfrak{M}_0$. Assume z is a peak point for $R(K_1)$. Then λ and μ are supported on $\{z\} \times K_2$, and we may write

$$\lambda = \delta_z \times \lambda_0 \quad \text{and} \quad \mu = \delta_z \times \mu_0$$

where λ_0 is representing measure for w w.r.t. $R(K_2)$ and $\mu_0 \in R(K_2)^\perp$. Let now $z_n \in Q_1$ with $z_n \to z$, and let λ_n be representing measure for z_n w.r.t. $R(K_1)$. Since z is a peak point, and any weak-star cluster point of $\{\lambda_n\}$ must be a representing measure for z, we have $\lambda_n \to \delta_z$ weak-star. Defining $\mu_n = \lambda_n \times \mu_0$, we obtain a sequence $\{\mu_n\}$ in $\operatorname{ball} \mathfrak{M}_0 \cap R(K_1 \times K_2)^\perp$ converging weak-star to μ.

5. **Proof of Theorem 2.** Let σ_i $(i = 1, 2)$ denote the area measure on the set Q_i of non peak points for $R(K_i)$, and put $\sigma = \sigma_1 \times \sigma_2 = dx\,dy_{Q_1} \times dx\,dy_{Q_2}$. Then σ is a positive measure on $Q = Q_1 \times Q_2$, and to simplify notation we put

$$B(Q_1 \times Q_2) = B(Q_1 \times Q_2, R(K_1 \times K_2)), \qquad B(\sigma) = B(\sigma, R(K_1 \times K_2)).$$

We easily obtain (cf. [4])

Lemma 5.1. *If $f \in B(\sigma)$ and $\{f_n\}$ is bounded in $R(K_1 \times K_2)$ and $f_n \to f$ a.e. σ, then in fact $\{f_n\}$ converges everywhere on $Q_1 \times Q_2$ to a unique $\tilde{f} \in B(Q_1 \times Q_2)$. Moreover $\|\tilde{f}\| \le \|f\|_B$.*

Proof. Let $\epsilon > 0$ and $(z, w) \in Q_1 \times Q_2$. Define $P_\epsilon(z) = \{z'; |f(z) - f(z')| < \epsilon, f \in \text{ball}\, R(K_1)\}$, and similarly $P_\epsilon(w)$ and $P_\epsilon(z, w)$ in terms of $R(K_2)$ and $R(K_1 \times K_2)$. Easily we have $P_{\epsilon/2}(z) \times P_{\epsilon/2}(w) \subseteq P_\epsilon(z, w)$. By Browder's metric density theorem (cf. [2]) we have $\sigma(P_\epsilon(z, w)) > 0$. Then there is $(z', w') \in P_\epsilon(z, w)$ with $f_n(z', w') \to f(z', w')$. Then $|f_n(z, w) - f_m(z, w)| \leq (2M + 1)\epsilon$ for n and m big enough, when now $M = \sup \|f_n\|$. Thus $\{f_n(z, w)\}$ converges for each $(z, w) \in Q_1 \times Q_2$, and we define

$$\tilde{f}(z, w) = \lim_n f_n(z, w).$$

That $\tilde{f}$ only depends on f and not on the particular sequence $\{f_n\}$, is proved similarly. Finally, the proof also gives $\|\tilde{f}\| \leq M$, which implies $\|\tilde{f}\| \leq \|f\|_B$.

Again to simplify notation we let $B(Q_1) \# B(Q_2)$ be the bounded functions on $Q_1 \times Q_2$ satisfying (b) of Theorem 2. The following lemma is a trivial consequence of Davie's theorem [4].

Lemma 5.2. *$B(Q_1) \# B(Q_2)$ is uniformly closed and contains $B(Q_1 \times Q_2)$.*

To show that the two spaces in this lemma in fact are equal, we use the Vitushkin approximation scheme (cf. [6, p. 210]). First however let us state the following version of [6, VIII. 6.3].

Lemma 5.3. *Let $E \subset \mathbf{C}$ be bounded with analytic capacity $\gamma(E) > 0$, analytic center w_0 and analytic diameter $\beta(E)$. Then there are functions g and h both analytic off some compact subset of E and satisfying*

(i) $g(\infty) = h(\infty) = 0$,
(ii) $g'(\infty) = 1$, $h'(\infty) = 0$,
(iii) $\beta(g, w_0) = 0$, $\beta(h, w_0) = 1$,
(iv) $\|g\| \leq 14/\gamma(E)$,
(v) $\|h\| \leq 6/\gamma(E)\beta(E)$.

Proof. Let f_1 and f_2 be as in the proof of [6, VIII. 6.3], and put $g = f_2/\gamma(E)$ and $h = f_1/\gamma(E)\beta(E)$.

We recall the approximation scheme from [6, VIII.7]. For each $\delta > 0$ cover the complex plane with discs $\Delta_k = \Delta(z_k, \delta)$ and choose smooth functions ϕ_k such that

(i) ϕ_k has support in Δ_k,
(ii) $\Sigma_k \phi_k = 1$,
(iii) $\|\partial \phi_k / \partial \bar{\xi}\| \leq 4/\delta$,
(iv) no point $z \in \mathbf{C}$ is contained in more than M of the discs Δ_k, where M is a universal constant.

If now f is a bounded measurable function with compact support, and we define

$$f_k(z) = T_{\phi_k} f(z) = \frac{1}{\pi} \int \frac{f(z) - f(\xi)}{z - \xi} \frac{\partial \phi_k}{\partial \bar{\xi}} dx\, dy(\xi),$$

then $f = \Sigma_k f_k$.

Employing this approximation scheme we prove the equivalence of (a) and (b) in Theorem 2, which we here formulate as follows.

Proposition 5.4. $B(Q_1 \times Q_2) = B(Q_1) \# B(Q_2)$.

Proof. Let $f \in B(Q_1) \# B(Q_2)$. Extend f to be zero off $Q_1 \times Q_2$. For each $\delta > 0$ let Δ_k and ϕ_k be as above. Define

$$f_k(z, w) = \frac{1}{\pi} \int \frac{f(\xi, w) - f(z, w)}{\xi - z} \frac{\partial \phi_k}{\partial \bar{\xi}} dx\, dy(\xi),$$

$$f_k'(\infty)(w) = -\frac{1}{\pi} \int f(\xi, w) \frac{\partial \phi_k}{\partial \bar{\xi}} dx\, dy(\xi),$$

$$\beta(f_k, t_k)(w) = -\frac{1}{\pi} \int f(\xi, w)(\xi - t_k) \frac{\partial \phi_k}{\partial \bar{\xi}} dx\, dy(\xi),$$

where t_k is an analytic center of $E_k = \Delta(z_k, 3\delta) \setminus K_1$. As usual we get $f = \Sigma_k f_k$, $\|f_k\| \leq 32\, \|f\|$. Since $f(\ ,w) \in B(Q_1)$, the characterization of $B(\sigma_1)$ in [8] gives us the estimate $|f_k'(\infty)(w)| \leq A_1\, \|f\|\, \gamma(E_k)$. Arguing as in [6, p. 216] we obtain

$$|\beta(f_k, t_k)(w)| \leq A_2 \|f\| \gamma(E_k) \beta(E_k)$$

where A_1 and A_2 are universal constants. By Lemma 5.3 applied to $E = E_k$, we have functions g_k and h_k with the first three coefficients in their Laurent expansions at ∞ as in (i)–(iii) and with norm estimates as in (iv)–(v) of that lemma. Now we define

$$H_k = f_k'(\infty) g_k + \beta(f_k, t_k) h_k \quad \text{and} \quad F_\delta = \sum_k H_k.$$

Since g_k and h_k are in $R(K_1)$ for each k, we get $F_\delta(\ , w) \in R(K_1)$ for each $w \in K_2$. We use Fubini's theorem to verify the condition of [8] characterizing $B(\sigma_2)$ thus showing $f_k'(\infty) \in B(\sigma_2)$ and $\beta(f_k, t_k) \in B(\sigma_2)$ for each k. Then $H_k(z,\ )$

and in turn $F_\delta(z, \ ) \in B(\sigma_2)$ for each $z \in K_1$. We have $\|F_\delta\| \le A_4\|f\|$, and as in [8, §3], one shows that for each $w \in Q_2$, then $F_\delta(z, w) \to f(z, w)$ for σ_1-a.a. z. As in [4] (cf. our Lemma 5.1), we get $F_\delta(z, w) \to f(z, w)$ for $(z, w) \in Q_1 \times Q_2$. Next we do this argument all over again for each F_δ, but now in the second variable. For $\rho > 0$ we get $g_{\delta,\rho} \in R(K_1 \times K_2)$ with $\|g_{\delta,\rho}\| \le A_5\|f\|$ and $g_{\delta,\rho} \xrightarrow[\rho]{} F_\delta$ pointwise on $Q_1 \times Q_2$. The bounded net $\{g_{\delta,\rho}\}$ has a subsequence $\{g_n\}$ converging weak-star in $L^\infty(\sigma)$. We may assume $\{g_n\}$ converges a.e. σ, and then $g_n \to f$ pointwise on $Q_1 \times Q_2$ by Lemma 5.1. Thus $f \in B(Q_1 \times Q_2)$, and the proof is completed.

As a consequence of this result and Lemma 5.2 (cf. [4]) we note

Corollary 5.5. *$B(Q_1 \times Q_2)$ is uniformly closed.*

Trivially (c) $\Rightarrow$ (a) in Theorem 2, and now we turn to (a) $\Rightarrow$ (c). Then let T: $B(Q_1 \times Q_2) \to B(\sigma)$ be the inclusion map. Lemma 5.1 tells us that T is one-to-one and onto, and T^{-1} is continuous from $B(\sigma)$ with the B-norm to $B(Q_1 \times Q_2)$ with uniform norm. $B(\sigma)$ with the B-norm is complete, and we just established completeness of $B(Q_1 \times Q_2)$. Then T is an isomorphism by the closed-graph theorem.

Our Lemma 5.1 also tells that the measure σ satisfies the condition in Theorem 2.5. We proved in §3 that each representing measure for a point in $Q_1 \times Q_2$ is an A-measure, so by Theorem 2.5 we can conclude $\|f^2\|_B = \|f\|_B^2$, $f \in B(\sigma)$.

Thus T is an isometry, which exactly means that (c) in Theorem 2 is satisfied for all $f \in B(Q_1 \times Q_2)$.

This now completes the proof of Theorem 2.

The Kreĭn-Šmulian theorem and Theorem 2 give that $B(\sigma)$ is weak-star closed, and then must coincide with the weak-star closure $H^\infty(\sigma)$ of $R(K_1 \times K_2)$. Thus we can note (cf. [8]).

Corollary 5.6. *If $f \in H^\infty(\sigma)$, then there is a sequence $\{f_n\}$ in $R(K_1 \times K_2)$ such that $\|f_n\| \le \|f\|$ and $f_n \to f$ a.e. σ.*

The equivalence of (a) and (b) in the following corollary was noted already in 3.3, but we include it here for completeness.

Corollary 5.7. *Let μ be a measure on $K_1 \times K_2$ orthogonal to $R(K_1 \times K_2)$. The following are equivalent:*

(a) *μ belongs to the band $\mathfrak{M}_0$.*

(b) *μ is an A-measure for $R(K_1 \times K_2)$ on $Q_1 \times Q_2$.*

(c) *The inclusion map $H^\infty(\sigma + |\mu|) \to H^\infty(\sigma)$ is an isometric isomorphism.*

Proof. Let $f \in H^\infty(\sigma)$, and let $\{f_n\}$ be a sequence in $R(K_1 \times K_2)$ with

$\|f_n\| \le \|f\|$ and $f_n \to f$ a.e. σ. Let $\hat{f}$ be a weak-star cluster-point of $\{f_n\}$ in $H^\infty(\sigma + |\mu|)$. Then $\hat{f} = f$ a.e. σ, and $\|\hat{f}\| = \|f\|$. Thus (c) holds if and only if $\hat{f}$ is uniquely determined by f. Let $B(\sigma + |\mu|)$ be the bounded weak-star closure of $R(K_1 \times K_2)$ in $L^\infty(\sigma + |\mu|)$. We have $\hat{f} \in B(\sigma + |\mu|)$, and $\hat{f}$ is unique in $B(\sigma + |\mu|)$ for all $f \in H^\infty(\sigma)$ if and only if μ is an A-measure. In this case $B(\sigma + |\mu|)$ becomes weak-star closed, and so coincides with $H^\infty(\sigma + |\mu|)$.

As a direct consequence of the Corollaries 4.4 and 5.7 we note

Corollary 5.8. *If Q_i is dense in K_i $(i = 1, 2)$, then the inclusion map $H^\infty(\sigma + |\mu|) \to H^\infty(\sigma)$ is an isometric isomorphism for a weak-star dense set of measures μ in* ball $R(K_1 \times K_2)^\perp$.

6. Proof of Theorem 3. The localization property. When A is a uniform algebra on a compact metric space X, σ is a positive measure on X, and $H^\infty(\sigma)$ is the weak-star closure of A in $L^\infty(\sigma)$, then the fiber over $p \in X$ in the maximal ideal space $M_{H^\infty(\sigma)}$ of $H^\infty(\sigma)$ is the set

$$\mathfrak{M}_p = \{\Psi \in M_{H^\infty(\sigma)};\ \Psi(f) = f(p),\ f \in A\}.$$

Theorem 3 will follow by verifying the assumptions in the following general criterion of [8] for such distance equalities.

Theorem 6.1 (Gamelin, Garnett). *Let A be a uniform algebra on a compact X. Let σ be a positive measure on X whose closed support coincides with X. Suppose*

(L) *If $p \in$ supp σ, $u \in C(X)$, $f \in H^\infty(\sigma)$ and $|f| \le u$ a.e. σ, then $|f| \le u(p)$ on $\mathfrak{M}_p$.*

(D) *The inclusion map $H^\infty(\sigma + |\mu|) \to H^\infty(\sigma)$ is an isometric isomorphism for a weak-star dense set of measures μ in ball $A^\perp$.*

For any continuous $u: X \to \mathbf{C}$ we then have $\operatorname{dist}(u, A) = \operatorname{dist}(u, H^\infty(\sigma))$.

In Theorem 3 we apply this to $X = K_1 \times K_2$, $A = R(K_1 \times K_2)$ and $\sigma = dx\,dy_{Q_1} \times dx\,dy_{Q_2}$, and assuming Q_i is dense in K_i $(i = 1, 2)$. Property (D) was established in the proceding section, so the only thing left to verify is now the localization property (L). We have identified $H^\infty(\sigma)$ with $B(Q_1 \times Q_2)$, so (L) now follows from

Proposition 6.2. *Let $f \in B(Q_1 \times Q_2)$, and let Ψ be in the fiber over $(z_0, w_0) \in K_1 \times K_2$ in the maximal ideal space of $B(Q_1 \times Q_2)$. For $\delta > 0$ define $W_\delta = \Delta(z_0, \delta) \times \Delta(w_0, \delta)$. For any $\delta > 0$ we have*

$$|\Psi(f)| \leq \|f\|_{(Q_1 \times Q_2) \cap W_\delta}.$$

Proof. Put $f \equiv 0$ outside $Q_1 \times Q_2$. Choose a smooth function ϕ with compact support in $\Delta(z_0, \delta)$ satisfying $\phi = 1$ on $\Delta(z_0, \delta/2)$, $\|\partial\phi/\partial\bar{\xi}\| \leq 4/\delta$ and $\|\phi\| \leq 1$. Define

$$R_\phi f(z, w) = \frac{1}{\pi} \int \frac{f(\xi, w)}{\xi - z} \frac{\partial \phi}{\partial \bar{\xi}} d\sigma_1(\xi) \quad \text{and} \quad F = \phi f + R_\phi f.$$

When $\{f_n\}$ is bounded in $R(K_1 \times K_2)$ and $f_n \to f$ pointwise on $Q_1 \times Q_2$, then we put $F_n = \phi f_n + R_\phi f_n$. We saw in the proof of Proposition 3.2 $F_n \in R(K_1 \times K_2)$, and $\{F_n\}$ is bounded and $F_n \to F$ pointwise on $Q_1 \times Q_2$. Thus $F \in B(Q_1 \times Q_2)$. When $|z - z_0| < \delta/2$, then

$$\text{(i)} \qquad (F - f)(z, w) = R_\phi f(z, w) = \frac{1}{\pi} \int \frac{f(\xi, w)}{\xi - z} \frac{\partial \phi}{\partial \bar{\xi}} d\sigma_1(\xi)$$

which is analytic in z for each $w \in Q_2$.

Next we define $G_1 \colon Q_1 \times Q_2 \to \mathbf{C}$ by

$$G_1(z, w) = \frac{(F - f)(z, w) - (F - f)(z_0, w)}{z - z_0}$$

and want to show that $G_1 \in B(Q_1 \times Q_2)$. When $\{f_n\}$ is as above and $r \in R(K_2)^\perp$, then $\int R_\phi f_n(z_0, w)\, dr(w) = 0$ by the Fubini theorem. Thus $R_\phi f_n(z_0, \ ) \in R(K_2)$, and we get a bounded sequence satisfying $R_\phi f_n(z_0, w) \to R_\phi f(z_0, w) = (F - f)(z_0, w)$ for $w \in Q_2$. Then $(F - f)(z_0, \ ) \in B(Q_2)$. For each $z \in Q_1$, $z \neq z_0$, then $G_1(z, \ ) \in B(Q_2)$. Furthermore

$$G_1(z_0, w) = \frac{1}{\pi} \int \frac{f(\xi, w)}{(\xi - z_0)^2} \frac{\partial \phi}{\partial \bar{\xi}} d\sigma_1(\xi),$$

and similar use of Fubini gives $G_1(z_0, \ ) \in B(Q_2)$. Totally $G_1(z, \ ) \in B(Q_2)$ for all $z \in Q_1$. Next we fix $w \in Q_2$. Now $(F - f)(\ , w) - (F - f)(z_0, w) \in B(Q_1)$, is analytic and zero at z_0, and then $G_1(\ , w) \in B(Q_1)$ (see [8, 6.1]).

Define next $b\colon Q_1 \times Q_2 \to \mathbf{C}$ by $b(z, w) = (F - f)(z_0, w)$ and put $b = 0$ outside $Q_1 \times Q_2$. Choose a new smooth function ϕ with compact support in $\Delta(w_0, \delta)$ satisfying $\phi = 1$ on $\Delta(w_0, \delta/2)$, $\|\partial\phi/\partial\bar{\xi}\| \le 4/\delta$ and $\|\phi\| \le 1$. Define $H = \phi b + R_\phi b$. As above $H \in B(Q_1 \times Q_2)$, and $H - b$ extends analytically in w to w_0 for each $z \in Q_1$.

We may define

$$G_2(z, w) = \frac{(H - b)(z, w) - (H - b)(z, w_0)}{w - w_0}$$

and we get $G_2 \in B(Q_1 \times Q_2)$. Now we may write

$$(F - f)(z, w) = (z - z_0)G_1(z, w) + (w_0 - w)G_2(z, w) + H(z, w) + (b - H)(z, w_0).$$

The last term here is in fact a constant, and when $\Psi \in \mathfrak{M}_{(z_0, w_0)}$, then

(ii) $$|\Psi(F - f)| \le |\Psi(H)| + |(b - H)(z, w_0)|.$$

Here

(iii) $$(b - H)(z, w_0) = -\frac{1}{\pi}\int \frac{(F - f)(z_0, \xi)}{\xi - w_0} \frac{\partial\phi}{\partial\bar{\xi}}\, d\sigma_2(\xi).$$

From the integral formulas (i) and (iii) we get the estimates

$$\|F - f\|_{\{z_0\}\times(Q_2 \cap \Delta(w_0, \delta))} \le A_1 \|f\|_{(Q_1 \times Q_2)\cap W_\delta}$$

$$|(b - H)(z, w_0)| \le A_1 \|F - f\|_{\{z_0\}\times(Q_2 \cap \Delta(w_0, \delta))} \le A_2 \|f\|_{(Q_1 \times Q_2)\cap W_\delta}.$$

We get

$$\|H\|_{Q_1 \times Q_2} \le A_3 \|b\|_{Q_1 \times (Q_2 \cap \Delta(w_0, \delta))}$$

and by (i) we get

$$\|b\|_{Q_1 \times (Q_2 \cap \Delta(w_0, \delta))} \le A_4 \|f\|_{(Q_1 \times Q_2)\cap W_\delta}.$$

Adding up all these estimates in (ii), we conclude

(iv) $$|\Psi(F - f)| \leq A_5 \|f\|_{(Q_1 \times Q_2) \cap W_\delta}.$$

Next we go through the whole argument all over again with F instead of f, but reversing the order of the R_ϕ-operations, i.e. first we choose smooth ϕ with compact support in $\Delta(w_0, \delta)$, $\phi = 1$ on $\Delta(w_0, \delta/2)$, $\|\partial\phi/\partial\bar{\xi}\| \leq 4/\delta$ and $\|\phi\| \leq 1$, and we define $G = \phi F + R_\phi F \in B(Q_1 \times Q_2)$. Next we proceed as above, and finally we reach the conclusion

(v) $$|\Psi(G - F)| \leq A_5 \|F\|_{(Q_1 \times Q_2) \cap W_\delta}.$$

Looking at the definition of F we have

(vi) $$\|F\|_{Q_1 \times Q_2 \cap W_\delta} \leq \|F\|_{Q_1 \times (Q_2 \cap \Delta(w_0, \delta))} \leq A_6 \|f\|_{(Q_1 \times Q_2) \cap W_\delta}.$$

Furthermore

(vii) $$\|G\|_{Q_1 \times Q_2} \leq A_7 \|F\|_{Q_1 \times (Q_2 \cap \Delta(w_0, \delta))}.$$

When we put together the estimates (iv), (v), (vi), and (vii), and by breaking up $\Psi(f) = \Psi(f - F) + \Psi(F - G) + \Psi(G)$, we conclude

(viii) $$|\Psi(f)| \leq A_8 \|f\|_{(Q_1 \times Q_2) \cap W_\delta}.$$

A_1 to A_8 are universal constants, and since (viii) holds for all $f \in B(Q_1 \times Q_2)$, we get

$$|\Psi(f)| = |\Psi(f^n)|^{1/n} \leq A_8^{1/n} \|f\|_{(Q_1 \times Q_2) \cap W_\delta}$$

for each $n = 1, 2, \cdots$ which finally proves

$$|\Psi(f)| \leq \|f\|_{(Q_1 \times Q_2) \cap W_\delta},$$

and the proof is complete.

This completes the proofs of our theorems. However, let us note the following simple example to the effect that the conditions on Q_i being dense in K_i ($i = 1, 2$) are really necessary.

Let $K_1 = \{z;\ |z| \leq 1\}$. Let $z_0 \notin K_1$, and let $K_2 = K_1 \cup \{z_0\}$. Let u be continuous on $K_1 \times K_2$ such that

$$u|_{K_1 \times K_1} \in R(K_1 \times K_1), \quad u|_{K_1 \times \{z_0\}} \notin R(K_1).$$

Then $\operatorname{dist}(u, H^\infty(\sigma)) = 0$, but $\operatorname{dist}(u, R(K_1 \times K_2) > 0$.

Finally, we should also note that similar results to those presented here hold for $R(K_1 \times K_2 \times \cdots \times K_N)$ for compacts $K_1, K_2, \cdots, K_N$ in $\mathbf{C}$. More details on this are in [1].

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