TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 191, 1974

ON HOMEOMORPHISMS OF INFINITE DIMENSIONAL BUNDLES. II

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ABSTRACT. This paper presents some aspects of homeomorphism theory in the setting of (fibre) bundles modeled on separable Hilbert manifolds and generalizes results previously established. The main result gives a characterization of subsets of infinite deficiency in a bundle by means of their restriction to the fibres, from which we are able to prove theorems of the following types: (a) mapping replacement, (b) separation of sets, (c) negligibility of subsets, and (d) extending homeomorphisms.

1. Introduction. In [7] several aspects of homeomorphism theory are studied in the setting of (fibre) bundles with separable infinite dimensional spaces (manifolds) as fibres. Major results are established for bundles having fibre the Hilbert space l_2, or, equivalently, s, the countable infinite product of reals. In this paper we generalize such results to bundles with s-manifolds as fibres.

Our notation and definition follow that of [7]. In this paper we say a closed subset K of the total space E of a bundle (E, p, B) is a *fibre Z-set* provided $K \cap p^{-1}(b)$ is a Z-set in $p^{-1}(b)$ for each $b \in B$. Fibre bundle (E, p, B) will be denoted by its total space E. For any $K \subset E$, a map $f\colon K \to E$ is *B-preserving* (or *fibre-preserving*) if $pf(x) = p(x)$ for all $x \in K$.

Hypotheses. (1) *Throughout this paper let* (E, p, B) *denote a fibre bundle over base space* B *with fibre* M, *where* B, M *are given as follows.*

The base space B. We assume B is either (1) a polyhedron, or (2) a retract of a polyhedron. If B is in (2), let B_1 be a polyhedron for which there is a retraction $r\colon B_1 \to B$. Then any bundle (E, p, B) over B induces a pull-back bundle (E_1, p_1, B_1) which *contains* E. With this in mind it is not difficult to observe that all our results for B in (1) are also valid for B in (2) (see also the proof of Theorem 1.1 of [7]). Thus we will provide proof only for the case B = polyhedron.

Received by the editors November 18, 1970 and, in revised form, July 24, 1973.
AMS (MOS) subject classifications (1970). Primary 57A20, 58B05.
Key words and phrases. Bundle, polyhedron, homeomorphism, ambient isotopy.
(1) Research supported in part by NSF 14429.
(2) Research supported in part by NSF Grant GP20632.

The manifold M. We assume M is a paracompact manifold modeled on $s = (-1, 1)^\infty$, the countable infinite product of open intervals $(-1, 1)$. By the results of [6], M may be considered as an open set in s.

(2) *All spaces concerned are metrizable.*

Open question. For results of this paper, we do not know whether the hypotheses on B may be replaced by any ANR (for metric spaces). In fact it is not known even for $B = s$.

2. In the following we say $\mathfrak{U}$ is a *normal* open cover of $X\setminus K$, where K is closed in the metric space X, provided each map $f\colon X\setminus K \to X$ which is limited by $\mathfrak{U}$ has an extension $\tilde{f}\colon X \to X$ which is the identity on K.

Lemma 1. *Let $H, K, K_1, K_2, \cdots$ be a collection of closed fibre Z-sets in $E = (E, p, B)$ with fibre $M = s$. Then for any open cover $\mathfrak{U}$ of E, there is an isomorphism h of E limited by $\mathfrak{U}$ such that $h|_H = \mathrm{id}$ and $h(K\setminus H) \cap [\bigcup_{n>0}(K_n\setminus H)] = \varnothing$.*

Proof. By [7, Theorem 5.2] there is a B-preserving homeomorphism $f\colon E\setminus H \xrightarrow{\text{onto}} E$ such that f is limited by $\mathfrak{U}$. Thus $f(K\setminus H)$ is a closed fibre Z-set. Write $\bigcup_{n>0}(K_n\setminus H) = \bigcup_{n>0} T_n$ where each T_n is a closed subset of E such that $T_n \subset K_m$ for some m. Thus each T_n is a fibre Z-set and so is $f(T_n)$. Let $\mathfrak{U}_1$ be a normal open cover of $E\setminus H$ refining $\mathfrak{U}$ and let $f(\mathfrak{U}_1)$ denote the induced cover of E. Using Theorem 3.1 of [7] there is an isomorphism g of E such that $gf(K\setminus H) \cap (\bigcup_{n>0} f(T_n)) = \varnothing$ and g is limited by $f(\mathfrak{U}_1)$. Then $f^{-1}gf\colon E\setminus H \to E\setminus H$ is a B-preserving homeomorphism which is limited by $\mathfrak{U}_1$ and satisfies $f^{-1}gf(K\setminus H) \cap [\bigcup_{n>0}(K_n\setminus H)] = \varnothing$. Since $\mathfrak{U}_1$ is normal, we may assume $h = f^{-1}gf$ is an isomorphism of E such that $h|_H = \text{identity}$. h fulfills our requirements.

Theorem 1 (Strong separation). *Let $K, K_1, K_2, \cdots$ be closed fibre Z-sets of $E = (E, p, B)$. Then, for any open cover $\mathfrak{U}$ of E, there is an isomorphism h of E limited by $\mathfrak{U}$ such that $h(K) \cap (\bigcup_{n>0} K_n) = \varnothing$.*

Proof. First suppose E is the product bundle $(B \times M, p, B)$. Let $\{U_i\}$ be a countable star-finite open cover of M which refines $\mathfrak{U}$ and is ordered as in Anderson-Henderson-West [2, Theorem 2] for which there is a homeomorphism $\phi_i\colon \mathrm{cl}(U_i) \xrightarrow{\text{onto}} \mathrm{cl}(V_i)$, where V_i is a basic open set in s. Note that $\phi_i(\mathrm{cl}(U_i))$ is homeomorphic to s and $\phi_i(\mathrm{Bd}(U_i))$ is a Z-set in $\phi_i(\mathrm{cl}(U_i))$. Thus $B \times \mathrm{Bd}(U_i)$ is a Z-set in $B \times \mathrm{cl}(U_i)$. By Lemma 1 there is a B-preserving homeomorphism f_1 of $B \times \mathrm{cl}(U_1)$ satisfing $f_1|_{B\times \mathrm{Bd}(U_1)} = \mathrm{id}$ and $f_1(K \cap (B \times U_1)) \cap [(\bigcup_{n>0} K_n) \cap (B \times U_1)] = \varnothing$. Then extend f_1 to an isomorphism $\tilde{f}_1$ of $B \times M$ such that $f_1|_{(B\times M)\setminus(B\times \mathrm{cl}(U_1))} = \mathrm{id}$.

Applying the same procedure we have an isomorphism γ_2 of $B \times M$ such that $\gamma_2|_{(B\times M)\setminus(B\times \mathrm{cl}(U_2))} = \mathrm{id}$ and $\gamma_2(\gamma_1(K) \cap (B \times U_2)) \cap [(\bigcup_{n>0} K_n) \cap (B \times U_2)] = \emptyset$. Continue this process in the same manner to get isomorphisms $\gamma_1, \gamma_2, \cdots$ of $B \times M$ such that $\gamma_{i+1}(\gamma_i \cdots \gamma_1(K)) \cap [(\bigcup_{n>0} K_n) \cap (B \times U_{i+1})] = \emptyset$. The convergence procedure of [2, Theorem 2] implies that $h = \lim_{n\to\infty} \gamma_n \cdots \gamma_1$ gives an isomorphism of $B \times M$. We now claim that $h(K) \cap (\bigcup_{n>0} K_n) = \emptyset$. To this end choose $x \in K$ and let $i(0)$ be the smallest integer such that $x \in B \times U_{i(0)}$. Then $\gamma_{i(0)-1} \cdots \gamma_1(x) = x$. This implies $\gamma_{i(0)-1} \cdots f_1(x) \in B \times U_{i(0)}$. Thus $\gamma_{i(0)} \cdots \gamma_1(x) \notin (\bigcup_{n>0} K_n)$. If $\gamma_{i(0)} \cdots \gamma_1(x) \notin B \times U_{i(0)+1}$, then $\gamma_{i(0)+1} \cdots \gamma_1(x) \notin \bigcup_{n>0} K_n$. If $\gamma_i(0) \cdots \gamma_1(x) \in B \times U_{i(0)+1}$, then $\gamma_{i(0)+1}$ acts to move $\gamma_{i(0)} \cdots \gamma_1(x)$ away from $\bigcup_{n>0} K_n$. In any case we have $\gamma_{i(0)+k} \cdots \gamma_1(x) \notin \bigcup_{n>0} K_n$ for all k. But since each point is moved only finitely many times, which follows from the ordering of $\{U_i\}$ provided by [2, Theorem 2], we have $h(x) \notin \bigcup_{n>0} K_n$. Since by Lemma 1 each γ_n can be obtained limited by any open cover of E, h can be required to be limited by $\mathfrak{U}$.

For the general case we may take B to be equal to $|A|$ for some lfsc A. By giving A a finer triangulation we may suppose each $p^{-1}(|\sigma|)$, $\sigma \in A$, is trivial. Furthermore it suffices to assume $|A|$ is connected. Thus $|A|$ can be written as $\bigcup_{i>0} |A_i|$ such that each A_i is a subcomplex and $p^{-1}(A_i)$ is trivial. We can use the special case proven above to obtain an isomorphism h_1 of E limited by an open cover $\mathfrak{U}_1$ of E, where $\mathrm{St}^3(\mathfrak{U}_1)$ refines $\mathfrak{U}$, such that $h_1(K \cap p^{-1}(A_1)) \cap (\bigcup_{n>0}(K_n \cap p^{-1}(A_1))) = \emptyset$. Inductively we can obtain isomorphisms $h_1, \cdots, h_n$ so that h_n is limited by an open cover $\mathfrak{U}_n$, where $\mathrm{St}^3(\mathfrak{U}_n)$ refines $\mathfrak{U}_{n-1}$ and

$$h_n \cdots h_1(K \cap p^{-1}(\tilde{A}_n)) \cap \left(\bigcup_{n>0} (K_n \cap p^{-1}(\tilde{A}_n))\right) = \emptyset,$$

where $\tilde{A}_n = A_1 \cup \cdots \cup A_n$. By the convergence procedure of [1] we can obtain an isomorphism h for the theorem.

The following theorem characterizes all subsets of infinite deficiency of a bundle by means of their restriction to the fibres (compare with Theorem 1.1 of [7]). For any product space $X \times Y$ we denote the projection $X \times Y \to X$ by p_X.

Theorem 2 (characterization). *Let K be a closed set in $B \times M$ of product bundle $(B \times M, p, B)$. The following are equivalent statements:*

(A) *K is a fibre Z-set;*

(B) *$K \cap p^{-1}(b)$ is s-deficient in each $p^{-1}(b)$;*

(C) *there is an isomorphism ϕ of $B \times M$ such that $\phi(K)$ is an M-projective Z-set; and*

(D) *there is a B-preserving homeomorphism h of $B \times M$ onto $B \times M \times s$ which carries K into $B \times M \times \{0\}$.*

Moreover, if K satisfies any one of the above conditions and B is compact, then, for any cover $\mathfrak{U}$ of $B \times M$, we may choose h in (D) *so that the projection $p_{B\times M}h$: $B \times M \to B \times M$ is limited by $\mathfrak{U}$.*

Proof. (A) $\Leftrightarrow$ (B), (C) $\Leftrightarrow$ (D) are well known (see, for example, Chapman [5]). Obviously (C) or (D) implies (A). If B is a polyhedron, using Theorem 1 of this paper and Lemma 3.4 of [7], a proof for the implication (A) $\Rightarrow$ (C) may be given in exactly the same way as Theorem 1.1 of [7]. If B is not a polyhedron but satisfies (2) in the hypothesis for base space B, the proof follows easily from the discussion there.

To prove the last part of Theorem 2 we now suppose B is compact and the covering condition is imposed on $B \times M$. Using compactness of B it is evident that there is an open cover $\mathcal{V}$ of M such that any isomorphism of $B \times M$ is limited by $\mathfrak{U}$ provided it be limited by $B \times \mathcal{V} = \{B \times v\colon v \in \mathcal{V}\}$. It follows from [4] that there is a homeomorphism $f\colon M \to M \times s \times Q$ such that the projection $p_M f\colon M \to M$ is limited by $\mathcal{V}_1$, where $\mathcal{V}_1$ is any open cover of M such that $\mathrm{st}^2(\mathcal{V}_1)$ refines $\mathcal{V}$. Write $Q\backslash s$ as a countable union of compact sets $L_1, L_2, \cdots$. Let $K_i = B \times f^{-1}(M \times s \times L_i)$. Then each K_i is a fibre Z-set in $B \times M$. By Theorem 1 there is an isomorphism f_1 of $B \times M$ limited by $B \times \mathcal{V}_1$ such that $f_1(K) \cap (\bigcup_{n>0} K_n) = \emptyset$. Thus $K' = (\mathrm{id}_B \times f)f_1(K) \subset B \times M \times (s \times s)$ and K' is closed in $B \times M \times s \times Q$. Using Lemma 3.4 of [7] there is a $(B \times M)$-preserving homeomorphism f_2 of $(B \times M) \times s \times Q$ onto $(B \times M) \times s \times Q \times s$ such that $f_2(K') \subset (B \times M) \times s \times Q \times \{0\}$. Let $f_3 = \mathrm{id}_B \times f^{-1} \times \mathrm{id}_s\colon B \times (M \times s \times Q) \times s \to B \times M \times s$. Then $h = f_3 f_2(\mathrm{id}_B \times f)f_1$ is a B-preserving homeomorphism of $B \times M$ onto $B \times M \times s$ sending K into $B \times M \times \{0\}$ and the projection $p_{B\times M}h$ is limited by $\mathfrak{U}$.

Since we can always choose convex open covers for M (recall that M is being considered as an open subset in s), we have

Corollary 1. *If B is compact, then the projection $p_{B\times M}$: $B \times M \times s \to B \times M$ can be approximated by B-preserving homeomorphism h: $B \times M \times s \to B \times M$ (that is, for any open cover $\mathfrak{U}$ of $B \times M$, we can choose h so that h is $\mathfrak{U}$-close to $p_{B\times M}$).*

Corollary 2. *Let K be a closed fibre Z-set in $B \times M$ of bundle $(B \times M, p, B)$. Then there is an open imbedding h of $B \times M$ into $B \times s$ such that $h(K)$ is closed in $B \times s$.*

Proof. By Theorem 2 and the open imbedding theorem of Henderson [6], there is an open imbedding $f = \mathrm{id}_B \times f_1$ of $B \times M$ onto $B \times (U \times s_1 \times s_2)$ such that $f(K) \subset B \times U \times s_1 \times \{0\}$, where U is open in s_0 and $\{s_i\}_{i=0}^2$ are copies of s. Since $B \times M$ is topologically complete, so is K. Thus there is a closed imbedding g of

K into s_2. Then the map $g_1\colon K \to B \times s_0 \times s_1 \times s_2$ defined by $g_1(x) = (p_{B\times s_0\times s_1}f(x), g(x))$ is a closed imbedding. Since $f(K)$ can be regarded as a closed subset of $B \times U \times s_1$, the map $\phi\colon f(K) \to g(K)$ defined by $\phi(f(x)) = g(x)$ extends to a map ϕ_1 of $B \times U \times s_1$ into s_2. Now regard s_2 as a linear space with addition "+". Define f_1 of $(B \times U \times s_1) \times s_2$ onto itself by $f_1(x, y) = (x, \phi_1(x) + y)$. f_1 is a B-preserving homeomorphism such that $f_1 f(K) = g_1(K)$, which is closed in $B \times s$, where $s = s_0 \times s_1 \times s_2$. Then $h = f_1 f$ is a required imbedding.

Theorem 3 (extraction). *Let $K \subset E$ be locally closed such that $\mathrm{cl}(K)$ is a fibre Z-set in bundle (E, p, B). Then K can be strongly extracted from E.*

More generally if K is a countable union of locally closed sets $K_1, K_2, \cdots$ such that each $\mathrm{cl}(K_i)$ is a fibre Z-set in E, then for any open cover $\mathfrak{U}$ of E, there is an $(E, \mathfrak{U})$-extraction of K from E.

Proof. Let K be given by the first half of the theorem. If E is the product bundle $(B \times M, p, B)$ we can apply Theorem 2 of this paper and Lemma 5.1 of [7] to obtain a strong extraction of K from E. In general we can, by hypothesis of E, write E as $\bigcup_{i\geq 1} p^{-1}(B_i)$ where each B_i is a subpolyhedron of B and $p^{-1}(B_i)$ is trivial. Thus there is, for each i, a strong extraction of $K \cap p^{-1}(B_i)$ from $p^{-1}(B_i)$. Using the convergence procedure of [1] we can, in a straightforward manner, obtain a strong extraction of K from E. Once this is done, the convergence procedure also provides an $(E, \mathfrak{U})$-extraction of K from E, where K and $\mathfrak{U}$ are given by the second half of the theorem.

Theorem 4 (Mapping replacement). *Let A be a separable complete metric space and let $X \subset A$ be closed. Suppose $f\colon A \to E$ is a map such that $f|_X$ is an imbedding sending X onto a closed fibre Z-set. Then for any open cover $\mathfrak{U}$ of $\mathrm{cl}(f(A))$ there is an imbedding $g\colon A \to E$ $\mathfrak{U}$-close to f such that $g|_X = f|_X$, $pg(x) = pf(x)$ for all x and $g(A)$ is a closed fibre Z-set.*

The proof is based on the following lemma whose proof resembles Lemma 2.4 of [3].

Lemma 2. *Suppose E is trivial and A, X, f and $\mathfrak{U}$ are given as above. Then for any closed set $Y \subset A$ and any open set U for which $(X \cup Y) \subset U$, there is a map $g\colon A \to E$, $\mathfrak{U}$-close to f, satisfying (1) $g(x) = f(x)$ for $x \in X \cup (A\setminus U)$, (2) $pg(x) = pf(x)$ for all x, and (3) $g|_{X\cup Y}$ is an imbedding of $X \cup Y$ onto a closed fibre Z-set.*

Proof. By Anderson-Schori [4], Theorem 2 of this paper and Lemma 3.3 of [7] we may, without loss of generality, assume that M is $M \times s$ and $\mathrm{cl}(f(A)) \subset B \times M \times \{0\}$. By techniques of Anderson-McCharen [3] there is a sequence of maps $g_1, g_2, \cdots : B \times M \to [0, 1)$, such that (1) for all i, $g_i(x) > 0$ whenever $x \in \mathrm{cl}(f(A))$, and (2) for each $x \in \mathrm{cl}(f(A))$, there is a $U \in \mathfrak{U}$ for which $(x, y) \in \mathfrak{U}$ whenever $y = (y_i) \in s$ and $|y_i| \leq g_i(x)$ for all i. Let $\phi: A \to [0, 1]$ be a map such that $\phi^{-1}(0) = X \cup (A \backslash U)$. By hypothesis there is a closed imbedding h of A into s such that, for any $a \in A$, all coordinates of $h(a)$ are positive. Then $g: A \to E$ defined as follows fulfills all the requirements of the lemma:

$$g(a) = (f(a), f_1(a)),$$

where $f_1(a) = (\phi(a) \cdot g_i(f(a)) \cdot \pi_i h(a))_i \in s$ ($\pi_i h(a)$ = the ith-coordinate of $h(a)$).

Proof of Theorem 4. Let $\{T_i\}, \{V_i\}$ be locally finite open covers of B such that, for all i, $\mathrm{cl}(T_i) \subset V_i$ and $p^{-1}(\mathrm{cl}(V_i))$ is trivial. Let $Y_i = f^{-1}(p^{-1}(\mathrm{cl}(T_i)))$ and $U_i = f^{-1}(p^{-1}(V_i))$. Then $\{Y_i\}$ is a locally finite covering of A. Note that $\mathrm{cl}(U_i) \subset f^{-1}p^{-1}(\mathrm{cl}(V_i))$. By Lemma 2 there is a map g_1 of $A_1 = \mathrm{cl}(U_1)$ into E such that (1) $g_1(x) = f(x)$ for $x \in (X \cap A_1) \cup (A_1 \backslash U_1)$, (2) $pg(x) = pf(x)$ for all $x \in A_1$, (3) $g_1|_{(X \cap A_1) \cup Y_1}$ is a closed imbedding whose image is a fibre Z-set of E, and (4) g_1 is $\mathfrak{U}_1$-close to $f|_{A_1}$ where $\mathfrak{U}_1$ is an open cover of $\mathrm{cl}(f(A))$ such that $\mathrm{St}^3(\mathfrak{U}_1)$ refines $\mathfrak{U}$. By (1) we can extend the domain of g_1 to A satisfying $g_1(x) = f(x)$ for $x \in X$, $pg_1 = pf$, $g_1|_{X \cup Y_1}$ is a closed imbedding whose image is a fibre Z-set and g_1 is $\mathfrak{U}_1$-close to f. By the same manner we can construct mappings $g_2, g_3, \cdots$ of A into E satisfying, for each n, $g_n(x) = f(x)$ for $x \in X$, $pg_n = pf$, $g_n|_{X \cup Y_1 \cup \cdots \cup Y_n}$ is a closed imbedding whose image is a closed fibre Z-set and g_n is $\mathfrak{U}_n$-close to f, where $\mathfrak{U}_n$ is an open cover of $\mathrm{cl}(f(A))$ such that $\mathrm{St}^3(\mathfrak{U}_n)$ refines $\mathfrak{U}_{n-1}$. The mapping g for the theorem clearly follows.

Theorem 5 (Extending homeomorphism). *Let $G = \{g_t\}$ be a homotopy of a complete separable metric space A into E such that* (1) *g_0, g_1 are imbeddings of A onto closed fibre Z-sets in E, and* (2) *$pG(\{a\} \times I) = \{point\}$ for all $a \in A$. Then there is an isotopy $\{h_t\}$ on E such that $h_0 = \mathrm{id}$ and $h_1 g_0 = g_1$.*

Moreover, if E is trivial, then for any open neighborhood U of $\mathrm{cl}(G(A \times I))$ and any open cover $\mathfrak{U}$ of U for which G is limited, we may choose $\{h_t\}$ to be a $(U, \mathrm{St}^4(\mathfrak{U}))$-isotopy.

To give a proof we need the following lemma.

Lemma 3. *Let A be a space. Suppose there is a closed imbedding G of $A \times I$ onto a fibre Z-set in $B \times M$ of product bundle $(B \times M, p, B)$ such that for any $a \in A$, $pG(\{a\} \times I) = \{point\}$. Let $\phi: A \to I$ be a map.*

Then for any closed sets B_0, B_1 in B such that $B_0 \subset \mathrm{Int}(B_1)$, there is an isotopy on $B \times M$ such that (a) *$h_0 = \mathrm{id}$,* (b) *$h_1 G((a, \phi(a))) = G(a, 1)$ for all $(a, \phi(a)) \in A_1$, and* (c) *for each t, $h_t(x) = x$ for all $x \in A_2$ where*

$$A_1 = \{G(a, \phi(a)) : a \in A, pG(\{a\} \times I) \in B_0\}$$

and

$$A_2 = p^{-1}(pG(\phi^{-1}(1) \times I) \cup (B\setminus \text{Int } B_1)).$$

Moreover, we may choose $\{h_t\}$ *so that* $h_t G((a, \phi(a))) \in G(\{a\} \times I)$ *for all* a, t.

Proof. Using Theorem 2 of this paper, the open imbedding theorem of Henderson [6] together with the techniques of Anderson-McCharen, there is a B-preserving open imbedding f of $B \times M$ into $B \times s \times I$ such that $fG(A \times I) = A' \times [1/3, 2/3]$, where A' is a closed M-projective Z-set in $B \times M \times \{0\}$, and the imbedding fG takes each $\{a\} \times I$ order preservingly onto $\{a'\} \times [1/3, 2/3]$ for some $a' \in B \times M$.

By techniques of Lemma 4.1 of Anderson-McCharen, there is a bundle isotopy $\{f_t\}$ of product bundle $(B \times (s \times I), p, B)$ supported on $f(B \times M)$ such that $f_0 =$ id, $f_1 fG((a, \phi(a))) = fG(a, 1)$ for all $a \in A_1$ and, for all t, $f_t f(x) = f(x)$ for all $x \in A_2$. In fact $\{f_t\}$ is a motion such that, for each $a' \in B \times M$, $\{f_t|_{a' \times I}\}$ is an endpoints preserving isotopy such that $f_t fG((a, P(a))) \in fG(\{a\} \times I)$ for all a, t. $\{h_t = f^{-1} f_t f\}$ is an isotopy on $B \times M$ which clearly fulfills all the requirements of the lemma.

Proof of Theorem 5. The second half (where E is assumed trivial) of Theorem 5 follows immediately from Theorem 2 of this paper and Lemma 3.6 of [7].

To prove the first half, first suppose $g_0(A) \cap g_1(A) = \emptyset$. Thus $G|_{A \times \{0, 1\}}$ is a closed imbedding onto a fibre Z-set of E. By virtue of the mapping replacement theorem we may assume G is a closed imbedding. Let $\{T_i\}, \{V_i\}$ be star-finite open covers of B such that, for each i, $\text{cl}(T_i) \subset V_i$ and $p^{-1}(\text{cl}(V_i))$ is trivial. Let $Y_i = \{a \in A: pG(\{a\} \times I) \in T_i\}$. By virtue of Lemma 3 (in particular, Lemma 3(c)) there is an isotopy $\{h_{1t}\}$ on E which satisfies the following properties: (1) $h_{10} =$ id, (2) $h_{11} g_0(a) = g_1(a)$ for all $a \in Y_1$, and (3) for all a, $h_{11} g_0(a) \in G(\{a\} \times I)$.

h_{11} induces a map $\phi_1 : A \to I$ such that, for any a, $G(a, \phi_1(a)) = h_{11} g_0(a)$. By virtue of Lemma 3 again there is an isotopy $\{h_{2t}\}$ on E such that $h_{20} =$ id, $h_{21} g_0(a) = g_1(a)$ for all $a \in Y_2$, $h_{21} g_0(a) \in G(\{a\} \times I)$ for all a and $h_{2t}|_{p^{-1}(T_1)}$ $=$ id for all t.

Inductively we can construct isotopies $\{h_{nt}\}_{n \geq 1}$ on E such that for each n, $h_{n0} =$ id, $h_{n1} g_0(a) = g_1(a)$ for all $a \in Y_n$, $h_{n1} g_0(a) \in G(\{a\} \times I)$ for all a and $h_{nt}|_{p^{-1}(T_1 \cup \ldots \cup T_{n-1})} =$ id for all t. We now define an isotopy on E in the following. Fill in the levels $E \times [0, 1/2]$ (order preservingly) with $\{h_{1t}\}_t$; the levels $E \times [1/2, 2/3]$ with $\{h_{2t}h_{11}\}_t$; the levels $E \times [2/3, 3/4]$ with $\{h_{3t}h_{21}h_{11}\}_t$ and so on. Denote the homeomorphism on E at the tth level by h_t. Since each point $x \in E$ can be moved by at most finitely many h_t, the limit $\lim_{t\to 1} h_t = \lim_{n\to\infty} h_{n1} \cdots h_{11}$ exists. Denote the limit by h_1. $\{h_t\}$ is a desired isotopy.

Finally if $g_0(A) \cap g_1(A) \neq \emptyset$, by Theorem 1 there is an isotopy $\{\mu_t\}$ on E such that $\mu_0 = \mathrm{id}$ and $\mu_1(g_0(A)) \cap g_1(A) = \emptyset$. Thus we may use the special case above to construct an isotopy $\{\lambda_t\}$ on E such that $\lambda_0 = \mathrm{id}$ and $\lambda_1(\mu_1 g_0(a)) = g_1(a)$ for all $a \in A$. $\{\mu_t\}$ followed by $\{\lambda_t\}$ clearly gives an isotopy for the theorem.

Corollary 3. *Let K_1, K_2 be closed fibre Z-sets in $\Delta_n \times M$ of product bundle $(\Delta_n \times M, p, \Delta_n)$ over n-simplex Δ_n, $n \geq 1$, such that $K_1 \cap p^{-1}(\mathrm{Bd}\, \Delta_n) = K_2 \cap p^{-1}(\mathrm{Bd}\, \Delta_n)$. Suppose there is a bundle homotopy $\{g_t\}$ of K_1 onto K_2 such that $g_0 = \mathrm{id}$, $g_1: K_1 \to K_2$ is a homeomorphism and $g_t|_{K_1 \cap p^{-1}(\mathrm{Bd}\,\Delta_n)} = \mathrm{id}$ for all t; then there is a (bundle) isotopy $\{h_t\}$ on $B \times M$ such that $h_0 = \mathrm{id}$, $h_1|_{K_1} = g_1$ and $h_t|_{p^{-1}(\mathrm{Bd}\,\Delta_n)} = \mathrm{id}$ for all t.*

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