

ON HOMEOMORPHISMS OF INFINITE DIMENSIONAL BUNDLES. III

BY

T. A. CHAPMAN⁽¹⁾ AND R. Y. T. WONG⁽²⁾

ABSTRACT. In this paper we continue the study of homeomorphisms and prove an analogue of the homeomorphism extension theorem for bundles modeled on Hilbert cube manifolds; thus we generalize previous results for Q -manifolds (Anderson-Chapman). This analogy, as in the case of manifolds, requires a consideration of proper maps and proper homotopies. The approach to the present problem is similar to that considered in our previous papers. Bear in mind several distinct difficulties occur in our setting.

1. The purpose of this paper is to continue the study of homeomorphisms in the setting of bundles modeled on Hilbert cube manifolds (that is, a connected separable metric space having an open cover by sets homeomorphic to open subsets of the Hilbert cube $Q = [-1, 1]^\infty$). Similar theorems are obtained under the following headings: characterization, separation of sets, mapping replacements and extending homeomorphisms. In the present setting we consider proper maps and proper homotopies. A map $f: X \rightarrow Y$ is *proper* provided the inverse images of compact sets in Y are compact sets in X . A homotopy $F = \{f_t\}: X \times I \rightarrow Y$ is a *proper homotopy* provided the map F is proper.

Hypothesis. Throughout the following let M denote a connected Q -manifold and, for matter of convenience, assume that all spaces considered are metrizable.

Our notation and definitions follow that of [7] and [8].

2. **Separation of sets.** We recall that for a bundle $E = (E, p, B)$ and for a pair (U, \mathcal{U}) denoting an open set of E and an open cover of U , a (U, \mathcal{U}) -isotopy $\{\mu_t\}$ on E is a map $\mu = \{\mu_t\}: E \times I \rightarrow E$ such that $\mu_0 = \text{id}$, each μ_t is an isomorphism with support in U , and $\{\mu_t|_U\}$ is limited by \mathcal{U} . A closed set K in the product space $X \times Y$ is an X -projective Z -set provided the projection of K into X is contained in a closed Z -set of X . A closed subset K of E is a fibre Z -set provided $K \cap p^{-1}(b)$ is a Z -set in $p^{-1}(b)$ for each $b \in B$. The purpose of this section is to prove the following theorem.

Received by the editors May 4, 1971 and, in revised form, July 24, 1973.

AMS (MOS) subject classifications (1970). Primary 57A20; Secondary 55F10.

Key words and phrases. Hilbert cube manifolds, bundles, B -preserving homeomorphism, extension, mapping replacement, separation.

(¹) Research supported in part by NSF 14429.

(²) Research supported in part by NSF Grant GP20632.

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Theorem 2.1. *Let $E = (E, p, B)$ be given over a polyhedron B with fibre M . Suppose K, L_1, L_2, \dots , are closed fibre Z -sets of E ; then there is an isotopy $\{g_t\}$ of E such that $g_0 = \text{id}$ and $g_1(K) \cap (\bigcup_{i \geq 1} L_i) = \emptyset$.*

Moreover, if \mathcal{U} is any open cover of K , we may choose $\{g_t\}$ to be limited by \mathcal{U} .

To give a proof we need the following lemmas.

Lemma 2.2. *Let $E = (B \times M, p, B)$ be given over any space B and let K, L be closed M -projective Z -sets of $E = B \times M$. Suppose $K \setminus L \neq \emptyset$; then for any open cover \mathcal{U} of K , there is a fibre homotopy $G = \{g_t\}: K \rightarrow E$ limited by \mathcal{U} such that (1) $g_t|_{K \cap L} = \text{id}$ for all t , (2) $G(K \times I)$ is an M -projective Z -set, and (3) g_1 is a closed imbedding such that $g_1(K \setminus L) \cap (K \cup L) = \emptyset$.*

Moreover, if $M = Q$ and $K \subset B \times s$, we may choose G so that $G(K \times I) \subset B \times s$.

Proof. This follows from the Anderson-McCharen technique [5, §2] with minor modification.

Lemma 2.3. *Let $E = (B \times Q, p, B)$ be given over compact polyhedron B and let K_0, K_1 be closed fibre Z -sets in E . Then there is an isotopy $\{b_t\}$ on E such that $b_0 = \text{id}$, $b_t|_{K_0} = \text{id}$ for all t and $b_1(K_1 \setminus K_0) \cap (K_0 \cup K_1) = \emptyset$. Moreover, if \mathcal{U} is any open cover of K_1 , we may choose $\{b_t\}$ so that $\{b_t|_{K_1}\}$ is limited by \mathcal{U} .*

Proof. The proof follows more or less the same pattern as Lemma 3.9 of [7]. We outline as follows. We say D is a standard k -cell in $s \subset Q$ if D can be written in the form $D = \prod_{i=1}^k [a_i, b_i] \times (0, 0, \dots)$ where $-1 < a_i < b_i < 1$ and $[a_i, b_i]$ is contained in the i th-coordinate J_i of s . It follows from Lemma 3.8 of [7] that, for any open set U of E containing $B \times D$ and for any $\epsilon > 0$, there is an isomorphism f of E supported in U such that $f(B \times s) = B \times s$, $d(f, \text{id}) < \epsilon$ and $f(K) \cap (B \times D) = \emptyset$, where $K = K_0 \cup K_1$.

By an analogue of a procedure used in Lemma 7.1 of [2], the above statement implies that there is an isomorphism g of E such that $g(K)$ is a Q -projective Z -set. Let \mathcal{V} be an open cover of $g(K_1)$ such that $\text{St}^{(4)}(\mathcal{V})$ refines $g(\mathcal{U})$ (the open cover induced by g and \mathcal{U}). With respect to the map $g|_K: K \rightarrow E$ and to the open cover \mathcal{V} of $g(K_1)$, by Lemma 2.2 there is a homotopy $G = \{g_t\}$ of $K_1 \times I$ into E such that $g_1(K_1 \setminus K_0) \cap g(K) = \emptyset$. Using other properties of G provided by Lemma 2.2 and Lemma 3.7 of [7], there is a $(U, \text{St}^{(3)}(\mathcal{V}))$ -isotopy $\{\mu_t\}$ on E such that $\mu_1 g_0 = g_1$, where $U = \bigcup \{V \in \mathcal{V}\}$. $\{b_t = g^{-1} \mu_t\}$ is a desired isotopy.

Proof of Theorem 2.1. First suppose that E is trivial and B compact. Let

us illustrate a procedure that will separate K from L_1 . Let $\{b_i\}$ be a sequence of imbeddings of Q into M such that the union $\bigcup_{i \geq 1} b_i(Q \setminus W)$ is an open cover of M , where $W = \{(x_i) \in Q: x_1 = 1\}$. We may assume that the sequence $\{U_i = b_i(Q \setminus W)\}$ is ordered as in [4, Theorem 2]. Applying Lemma 2.3 there is a B -preserving isotopy $\{g_{1t}\}$ of $B \times b_1(Q)$ such that $g_{10} = \text{id}$, $g_{1t}|_{B \times b_1(W)} = \text{id}$ for all t , $g_{11}(K \cap (B \times U_1)) \cap (L_1 \cap (B \times U_1)) = \emptyset$ and $\{g_{1t}\}$ is limited by \mathcal{U}_1 , where \mathcal{U}_1 is an open cover of K such that $\text{St}^3(\mathcal{U}_1)$ refines \mathcal{U} . Since $B_1 \times b_1(W)$ is the boundary of $B \times \mathcal{U}_1$, we may assume that $\{g_{1t}\}$ is an isotopy of $B \times M$ which is the identity outside of $B \times U_1$. Applying the same procedure there is an isotopy $\{g_{2t}\}$ of $B \times M$ such that $g_{20} = \text{id}$, g_{2t} is the identity outside of $B \times U_2$ for all t , $g_{21}(g_{11}(K) \cap (B \times U_2)) \cap (L_1 \cap (B \times U_2)) = \emptyset$ and $\{g_{2t}\}$ is limited by \mathcal{U}_2 , where \mathcal{U}_2 is an open cover of K such that $\text{St}^3(\mathcal{U}_2)$ refines \mathcal{U}_1 . Continuing this process we take the isotopy $\{g_{1t}\}$ followed by $\{g_{2t}\}$, and then followed by $\{g_{3t}\}$, etc. We then arrive with a limit isotopy $\{g_t\}$ limited by \mathcal{U} such that $g_1(K) \cap L_1 = \emptyset$. The same idea also enables us to obtain an isotopy for the theorem.

A proof of the general case may be carried out in exactly the same manner as the last part of the proof in [8, Theorem 1].

3. Characterization. Using results of the last section we are able to characterize all fibre Z -sets in bundles $(B \times M, p, B)$. A proof may be given using Theorem 2.1 of this paper and Lemma 3.4 of [7] and will be the same as Theorem 1.1 of [7].

Theorem 3.1. *Let $E = (B \times M, p, B)$ be a bundle over a polyhedron B and let $K \subset E$ be closed. The following are equivalent statements:*

- (A) K is a fibre Z -set;
- (B) $K \cap p^{-1}(b)$ is Q -deficient in each $p^{-1}(b)$;
- (C) there is an isomorphism ϕ of $B \times M$ such that $\phi(K)$ is an M -projective Z -set; and
- (D) there is a B -preserving homeomorphism of $B \times M$ onto $B \times M \times Q$ which carries K into $B \times M \times \{0\}$.

Proof. The same as Theorem 1.1 of [7] (see also the remark above).

4. Mapping replacement.

Theorem 4.1. *Let A be a locally compact separable metric space and let $X \subset A$ be closed. Suppose $f: A \rightarrow E$ is a proper map into bundle $E = (E, p, B)$ over a polyhedron B with fibre M such that $f|_X: X \rightarrow E$ is a closed imbedding onto a fibre Z -set. Then for any open cover \mathcal{U} of $f(A)$ there is a proper imbedding $g: A \rightarrow E$, \mathcal{U} -close to f , such that $g|_X = f|_X$, $pg(x) = pf(x)$ for all x and $g(A)$ is a fibre Z -set.*

A proof may be given using the following lemma.

Lemma 4.2. *Let X, Y be locally compact spaces and let $f: X \rightarrow Y$ be a proper map. Then $f(X)$ is closed. Furthermore, there exists an open cover \mathcal{U} of Y such that if a map $g: X \rightarrow Y$ is \mathcal{U} -close to f , g is also proper.*

Proof. Elementary.

Proof of Theorem 4.1. In [8, Theorem 4] we prove a similar mapping replacement theorem in the setting of s -manifold bundles. The procedure of such a proof, together with Lemma 4.2, may be used here to provide a closed imbedding g for the theorem. In the following we outline the argument.

Let $\{V_i\}$ be a locally finite open cover of B such that $p^{-1}(\text{cl}(V_i))$ is trivial for all i . Without loss of generality we assume that each $\text{cl}(V_i)$ is itself a polyhedron. Choose a locally finite cover $\{A_k\}$ of A so that each A_k is a compact set contained in some member of $\{f^{-1}p^{-1}(V_i)\}$. Using Lemma 4.2, Theorem 3.1 and the compactness of A_1 we may prove a similar lemma as Lemma 2 of [8] so that there is a proper map $f_1: A \rightarrow E$ satisfying $f_1(x) = f(x)$ for $x \in X$, $pf(x) = pf_1(x)$ for all $x \in A$, $f_1|_{X \cup A_1}$ is a closed imbedding whose image is a fibre Z -set of E and f_1 is \mathcal{U}_1 -close to f , where \mathcal{U}_1 is an open cover of E such that $\text{St}^3(\mathcal{U}_1)$ refines \mathcal{U} . By the same manner we may construct proper mappings f_2, f_3, \dots of A into E inductively so that, for each n , $f_n(x) = f_{n-1}(x)$ for $x \in X \cup A_1 \cup \dots \cup A_{n-1}$, $f_n|_{X \cup A_1 \cup \dots \cup A_n}$ is a closed imbedding whose image is a fibre Z -set of E , $pf_n = pf$ and f_n is \mathcal{U}_n -close to f_{n-1} , where \mathcal{U}_n is an open cover such that $\text{St}^3(\mathcal{U}_n)$ refines \mathcal{U}_{n-1} . By virtue of the local finiteness of $\{A_k\}$ and Lemma 4.2, the construction of the mapping g for the theorem clearly follows.

5. Extending homeomorphisms. Our main result is

Theorem 5.1. *Let $E = (E, p, B)$ be given over a polyhedron B with fibre M . Suppose $f = \{f_t\}$ is a proper homotopy of a locally compact separable metric space A into E such that f_0, f_1 are imbeddings of A onto closed fibre Z -sets in E and $pf(\{a\} \times I) = \{\text{point}\}$ for all $a \in A$. Then there is an isotopy $\{g_t\}$ on E such that $g_0 = \text{id}$ and $g_1 f_0 = f_1$.*

Moreover, if E is trivial, then for any open neighborhood U of $f(A \times I)$ and any open cover \mathcal{U} of U by which f is limited, we may choose $\{g_t\}$ to be a $(U, \text{St}^4(\mathcal{U}))$ -isotopy.

Corollary 5.2. *Suppose K_1, K_2 are closed fibre Z -sets in $\Delta_n \times M$ of bundle $(\Delta_n \times M, p, \Delta_n)$ over an n -simplex Δ_n and suppose there is a proper homotopy $f = \{f_t\}$ of K_1 into $\Delta_n \times M$ such that $pf(\{x\} \times I) = \{\text{point}\}$ for all $x \in K_1$, $f_0 = \text{id}$, $f_t(x) = x$ for all $t \in I$, $x \in K_1 \cap p^{-1}(\text{Bd} \Delta_n)$ and f_1 is a homeomorphism of K_1*

onto K_2 . Then there is an isotopy $\{g_t\}$ on $B \times M$ such that $g_0 = \text{id}$, $g_t|_{p^{-1}(\text{Bd } \Delta_n)} = \text{id}$ for all t and $g_1|_{K_1} = f_1$.

To give a proof of the theorem we first establish several special cases designated below as lemmas.

Lemma 5.3. *Let A, B be closed Q -projective Z -sets of $B \times Q$ and suppose $b: A \rightarrow B$ is a B -preserving homeomorphism. Then b can be extended to a homeomorphism H of $B \times Q$.*

Proof. This follows from the Klee-Anderson techniques while we observe that the requirement for H to be B -preserving may be trivially achieved.

Lemma 5.4. *Let A, B be metric spaces and let $(B \times Q, p, B)$ be a bundle over B . Suppose that (1) $f: A \times I \rightarrow B \times Q$ is a closed imbedding such that $pf(\{a\} \times I) = \{\text{point}\}$ for all $a \in A$ and $f(A \times I)$ is a Q -projective Z -set; (2) B_0 is a closed set in B ; (3) U is a neighborhood of $f(A \times I)$; and (4) $\phi_1, \phi_2: B \rightarrow [0, 1]$ are maps satisfying $\phi_1|_{B_0} = \phi_2|_{B_0}$. For $i = 1$ or 2 , let*

$$C_i = \bigcup \{f(\{a\} \times [\frac{1}{2}\phi_i pf(a, 0), 1]) : a \in A\}$$

and let $l: C_1 \rightarrow C_2$ be the homeomorphism such that $f^{-1}l$ takes each $\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]$ linearly onto $\{a\} \times [\frac{1}{2}\phi_2 pf(a, 0), 1]$. Then there is an isotopy $\{g_t\}$ of $B \times Q$ such that $g_0 = \text{id}$, g_1 extends l and for all t , $g_t(x) = x$ for any $x \in (B_0 \times Q) \cup ((B \times Q) \setminus U)$.

Proof. Clearly there is a canonical B -preserving homeomorphism b_0 between $f(A \times I)$ and $f(A \times \{0\}) \times [1/3, 2/3]$ in $(B \times Q) \times [0, 1]$ taking each $f(\{a\} \times I)$ onto $\{f(a, 0)\} \times [1/3, 2/3]$. By Lemma 5.3 we may extend b_0 to a B -preserving homeomorphism b of $B \times Q$ onto $B \times Q \times I$. Now by employing a technique similar to Lemma 4.7 of [5] we can obtain an isotopy $\{g'_t\}$ on the bundle $(B \times Q \times I, p, B)$ which uses motions only in the I -direction so that the isotopy $\{g_t = b^{-1}g'_tb\}$ fulfills all the requirements of the lemma.

We now assert a stronger version of Lemma 5.4.

Lemma 5.5. *Suppose in Lemma 5.4 we assume that A is separable and locally compact. Then Lemma 5.4 holds when " Q " is replaced by Q -manifold " M ".*

Let $a \in A$ be any point. Since $pf(\{a\} \times I) = \{\text{point}\}$, $f(\{a\} \times I)$ is contained in the Q -manifold $M_a = p^{-1}(pf(\{a\} \times I))$. By Lemma 4.1 of [3] there is an imbedding $b_a: Q \rightarrow M_a$ such that $b_a(Q \setminus K)$ is an open set in M_a containing $f(\{a\} \times I)$, where $K = \{(x_i) \in Q: x_1 = 1\}$.

Since $f: A \times I \rightarrow B \times M$ is an imbedding, using the above remark and the fact A is σ -compact, it is an elementary exercise to show that there are open covers $\{U_i\}$ of A and $\{V_i \times b_i(Q \setminus K)\}$ of $f(A \times I)$, where each V_i is open in B , such that $f(\text{cl}(U_i) \times I) \subset V_i \times b_i(Q \setminus K) \subset U$ for all i and the collection $\{V_i \times b_i(Q \setminus K)\}$ is star-finite.

Let $j: V_i \rightarrow B$ be the inclusion map. Denote $j \times b_i: V_i \times Q \rightarrow B \times M$ by H_i . We may assume the collection $\{H_i(V_i \times (Q \setminus K))\}$ has been ordered as in Theorem 2 of Anderson-Henderson-West [4]. Choose a closed cover $\{A_i\}$ of A so that each $A_i \subset U_i$.

Let $\lambda_1: A \rightarrow [0, 1]$ be a map satisfying $\lambda_1(a) = 1$ for $a \in A_1$ and $\lambda_1(a) = 0$ for $a \notin U_1$. Let

$$D_1 = \bigcup \{f(\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]) : a \in A\}$$

and

$$E_1 = \bigcup \{f(\{a\} \times [\frac{1}{2}b_1(a), 1]) : a \in A\}$$

where $b_1(a) = (1 - \lambda_1(a))\phi_1 pf(a, 0) + \lambda_1(a)\phi_2 pf(a, 0)$. Let l_1 be the natural homeomorphism of D_1 onto E_1 so that $f^{-1}l_1f$ takes $\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]$ linearly onto $\{a\} \times [\frac{1}{2}b_1(a), 1]$. Note that $D'_1 = H_1^{-1}(D_1 \cap H_1(B \times Q))$ and $E'_1 = H_1^{-1}(E_1 \cap H_1(B \times Q))$ are closed Q -projective Z -sets in $B \times Q$. l_1 induces a homeomorphism of D'_1 onto E'_1 . It follows from Lemma 5.4 that there is an isotopy $\{g_{1t}\}$ of $B \times M$ so that $g_{10} = \text{id}$, g_{11} extends l_1 and $g_{1t}(a) = a$ for any $a \in (B_0 \times M) \cup ((B \times M) \setminus H_1(V_1 \times (Q \setminus K)))$.

Let $\lambda_2: A \rightarrow [0, 1]$ be a map so that $\lambda_2(a) = 1$ for $a \in A_2$ and $\lambda_2(a) = 0$ for $a \notin U_2$. Let

$$D_2 = E_1 \quad \text{and} \quad E_2 = \bigcup \{f(\{a\} \times [\frac{1}{2}b_2(a), 1]) : a \in A\}$$

where $b_2(a) = (1 - \lambda_2(a))b_1(a) + \lambda_2(a)\phi_2 pf(a, 0)$. Similarly there is an isotopy $\{g_{2t}\}$ of $B \times M$ so that $g_{20} = \text{id}$, $f^{-1}g_{21}f$ takes $\{a\} \times [\frac{1}{2}b_1(a), 1]$ linearly onto $\{a\} \times [\frac{1}{2}b_2(a), 1]$ and $g_{2t}(a) = a$ for any $a \in (B_0 \times M) \cup ((B \times M) \setminus H_2(V_2 \times (Q \setminus K)))$. Construct $\{g_{3t}\}, \{g_{4t}\}, \dots$ analogously. Let $\{g_t\}$ be the limit isotopy obtained by taking $\{g_{1t}\}$ followed by $\{g_{2t}\}$ and then followed by $\{g_{3t}\}$, etc. By the ordering of $\{H_i(V_i \times (Q \setminus K))\}$, $\{g_t\}$ exists and is an isotopy of $B \times M$ fulfilling all the requirements of the lemma.

Proof of Theorem 5.1. First suppose that $f_0(A) \cap f_1(A) = \emptyset$. Let us give a proof to the first part of Theorem 5.1. By Theorem 4.1 we may suppose that f is a closed imbedding. Without loss of generality we may also suppose that B is a countable lfsc. Clearly there is a star-finite open cover $\{V_i\}$ and a closed cover $\{B_i\}$ of B such that each pair (V_i, B_i) is a simplicial pair ($B_i \subset V_i$) and $p^{-1}(V_i)$

is trivial. Let $\phi_1: B \rightarrow [0, 1]$ be a map such that $\phi_1(b) = 1$ for $b \in B_1$ and $\phi_1(b) = 0$ for $b \notin V_1$. Let

$$C_1 = f(A \times I) \quad \text{and} \quad D_1 = \bigcup \{f(\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]) : a \in A\}.$$

Since $f(\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]) = f(\{a\} \times [0, 1])$ for any $a \in A$ such that $pf(a, 0) \notin V_1$, and since $p^{-1}(V_1)$ is trivial, by virtue of Theorem 3.1 and Lemma 5.5 we can obtain an isotopy $\{g_{1t}\}$ on E such that $g_{10} = \text{id}$ and $f^{-1}g_{11}f$ takes each $\{a\} \times I$ linearly onto $\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]$. In particular, $g_{11}f(a, 0) = \frac{1}{2}$ whenever $f(a, 0) \in p^{-1}(B_1)$.

Let $\phi_2: B \rightarrow [0, 1]$ be a map such that $\phi_2(b) = 1$ for $b \in B_2$ and $\phi_2(b) = 0$ for $b \notin V_2$. Let

$$C_2 = D_1 \quad \text{and} \quad D_2 = \bigcup \{f(\{a\} \times [\frac{1}{2}\kappa(a), 1]) : a \in A\}$$

where $\kappa(a) = (1 - \phi_2 pf(a, 0))\phi_1 pf(a, 0) + \phi_2 pf(a, 0)$. Similarly there is an isotopy $\{g_{2t}\}$ on E such that $g_{20} = \text{id}$ and $f^{-1}g_{21}f$ takes $\{a\} \times [\frac{1}{2}\phi_1 pf(a, 0), 1]$ linearly onto $\{a\} \times [\frac{1}{2}\kappa(a), 1]$. Note that $g_{21}g_{11}f(a, 0) = \frac{1}{2}$ whenever $f(a, 0) \in p^{-1}(B_1 \cup B_2)$.

Continuing this process we take the isotopy $\{g_{1t}\}$ followed by $\{g_{2t}\}$, and then by $\{g_{3t}\}$, etc. We then arrive with a limit isotopy $\{g'_t\}$ on E such that $g'_0 = \text{id}$ and $g'_1 f_0 = f_{1/2}$. The isotopy $\{g'_t\}$ for the theorem clearly follows.

To prove the second part of Theorem 5.1 we may assume, by means of Theorem 4.1, that f is a closed imbedding limited by \mathcal{U} . Since E is trivial, we may consider E as $B \times M$. Using the first part of Theorem 5.1 there is a B -preserving homeomorphism $b: B \times M \rightarrow B \times M \times [0, 1]$ which takes each $f(\{a\} \times I)$ linearly onto $f(a, 0) \times [1/3, 2/3]$. Let $U_1 = b(U)$ and \mathcal{U}_1 denote the open cover of U_1 induced by \mathcal{U} and b . Now by employing a technique of Anderson-McCharen we can obtain a (U_1, \mathcal{U}_1) -isotopy $\{g'_t\}$ on the bundle $E' = (B \times M \times I, p', B)$ using motions only in the I -direction so that $g'_0 = \text{id}$ and $(b^{-1}g'_1 b)f_0 = f_1$. The isotopy $\{g'_t = b^{-1}g'_t b\}$ is what we wanted.

For the general case suppose that $f_0(A) \cap f_1(A) \neq \emptyset$. By Theorem 2.1 there is an isotopy $\{\lambda_t\}$ on E (limited by \mathcal{U} when E is trivial) so that $\lambda_0 = \text{id}$ and $\lambda_1(f_0(A)) \cap f_1(A) = \emptyset$. Thus we may use the special case above to construct an isotopy $\{\mu_t\}$ on E (limited by $\text{St}^2(\mathcal{U})$ when E is trivial) so that $\mu_0 = \text{id}$ and $\mu_1 \lambda_1 f_0 = f_1$. $\{\lambda_t\}$ followed by $\{\mu_t\}$ clearly gives an isotopy for the theorem.

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SCHOOL OF MATHEMATICS, THE INSTITUTE FOR ADVANCE STUDY, PRINCETON, NEW JERSEY 08540

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106