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ON DEFORMATIONS OF HOMOMORPHISMS OF LOCALLY COMPACT GROUPS

BY

DONG HOON LEE

ABSTRACT. The rigidity of homomorphisms of compactly generated locally compact groups into Lie groups is investigated.

For locally compact groups G and H, let Hom (G, H) denote the space of all continuous homomorphisms of G into H under the compact-open topology. Then H acts continuously on the space Hom (G, H) under conjugation. We say that $f \in$ Hom (G, H) is *rigid* if the orbit of f in Hom (G, H) under the action of H is an open neighborhood of f. In [6], Nijenhuis and Richardson have obtained the rigidity of homomorphisms of real analytic groups (or more generally, compactly generated Lie groups) into any Lie groups under certain cohomological conditions, thereby generalizing the well-known theorem of Weil [7], and conjectured that their result can be generalized to compactly generated, locally compact groups.

The main purpose of this paper is to prove this conjecture affirmatively. Our proof consists mainly of the reduction to the case of Lie groups. §1 carries basic definitions and notation which are standard throughout. In §2, we embed the space Hom (G, H) as a closed analytic subset of a manifold. In §3 we prove the result of Nijenhuis and Richardson, the proof of which does not seem to have appeared at the time of preparation of this paper. Finally §4 carries the proof of our main result together with some of its applications and an example.

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1. Basic definition and notation.

(1.1) Let G be a locally compact group and let ρ be a continuous representation of G in a finite-dimensional real vector space V. A continuous map $\phi: G \to V$ is called a *1-cocycle of G with values in V* (relative to ρ) if, for $x, y \in G$, $\phi(xy) = \phi(x) + \rho(x)(\phi(y))$. The set of all 1-cocycles with values in V forms a vector space, which we denote by $Z^1(G, V, \rho)$. A cocycle $\phi \in Z^1(G, V, \rho)$

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is called a 1-*coboundary*, if there exists $v \in V$ so that $\phi(x) = \rho(x)(v) - v$ for all $x \in G$. The set of all 1-coboundaries forms a subspace $B^1(G, V, \rho)$. Let $H^1(G, V, \rho) = Z^1(G, V, \rho)/B^1(G, V, \rho)$ and call it the 1-*cohomology space of* G *with coefficients in* V. For a detailed discussion, see [5].

(1.2) Let G and H be locally compact groups. Then the set Hom (G, H) becomes a topological space under the compact-open topology. To describe neighborhoods of $f_0 \in$ Hom (G, H), let C be a compact subset of G and let U be a 1-neighborhood in H. Then define

$$W(C, U; f_0) = \{f \in \mathrm{Hom}(G, H): f(x)f_0(x)^{-1} \in U \text{ for all } x \in C\}.$$

When C and U run over all compact subsets of G and all 1-neighborhoods in H, respectively, the sets $W(C, U, f_0)$ form a neighborhood basis of f_0 in Hom (G, H). If G and H are both connected Lie groups with their Lie algebras $\mathcal{G}$ and $\mathcal{H}$, respectively, then we have a natural embedding of Hom (G, H) into Hom $(\mathcal{G}, \mathcal{H})$ via $f \to df$, where df is the differential of f at 1, being identified with the Lie algebra homomorphism induced from f.

(1.3) The following notation is standard throughout. For any topological group H, let H_0, Aut (H) denote the 1-component, the automorphism group of H, respectively. Also for each $x \in H$, I_x denotes the inner automorphism of H induced by x, and, for $X \subset H$, $\mathrm{Int}_H(X) = \{I_x: x \in X\}$. If $X = H$, then we write Int (H) for $\mathrm{Int}_H(H)$. Then H acts on Hom (G, H) via $(t, f) \to I_x \circ f$, and $f \in$ Hom (G, H) is rigid if Int $(H) \cdot f$ is an open neighborhood of f in Hom (G, H). If H is a Lie group, Ad_H denotes the adjoint representation of H in the Lie algebra of H.

Finally, a Lie group and its Lie algebra are denoted by the same capital italic and capital English script letter, respectively. Thus, for example, if G is a Lie group, then $\mathcal{G}$ denotes the Lie algebra of G.

2. The embedding of Hom (G, H) into a manifold.

(2.1) Let G be a compactly generated Lie group. We choose $\gamma_1, \cdots, \gamma_m \in G$ once and for all so that the γ_j generate G modulo G_0. Let F be the free group on symbols $\tilde{\gamma}_1, \cdots, \tilde{\gamma}_m$ and let $\sigma: F \to G$ be the homomorphism determined by $\sigma(\tilde{\gamma}_j) = \gamma_j$.

For any Lie group H, let $i^* =$ Hom $(i, 1)$: Hom $(G, H) \to$ Hom (G_0, H) and $\sigma^* =$ Hom $(\sigma, 1)$: Hom $(G, H) \to$ Hom (F, H) be the dual maps induced by the inclusion $i: G_0 \to G$ and σ, respectively.

Then clearly $i^* \times \sigma^*$: Hom $(G, H) \to$ Hom $(G_0, H) \times$ Hom (F, H) is a continuous injection. To determine Im $(i^* \times \sigma^*)$, let $(f_0, g) \in$ Hom $(G_0, H) \times$ Hom (F, H) and define $f(x\sigma(\gamma)) = f_0(x)g(\gamma)$ for $(x, \gamma) \in G_0 \times F$. Noting that $G = G_0 \cdot \mathrm{Im}\, \sigma$, we have

(1) f defines a map $f: G \to H$ if and only if
$f_0\sigma(y) = g(y)$ whenever $\sigma(y) \in G_0$.

Since the γ_j generate F, f is a homomorphism if and only if

$$f_0 \cdot I_{\gamma_j} = I_{g(\gamma_j)} \cdot f_0 \quad (1 \le j \le m). \tag{2}$$

Let $(\omega_\lambda)_{\lambda \in \Lambda}$ be an indexed set of generators of the subgroup $\sigma^{-1}(G_0 \cap \operatorname{Im} \sigma)$ of F and put $a_\lambda = \sigma(\omega_\lambda)$, $\lambda \in \Lambda$.

Then the condition in (1) can be replaced by

$$f_0(a_\lambda) = g(\omega_\lambda) \quad \text{for } \lambda \in \Lambda. \tag{3}$$

Hence $(f_0, g) \in \operatorname{Im}(i^* \times \sigma^*)$ if and only if it satisfies conditions (3) and (2).

(2.2) Now we choose a basis $(X_i)_{1 \le i \le n}$ of $\mathfrak{G}$ once and for all and let $x_1, \cdots, x_n$ be a canonical system of coordinates for a fixed exponential map $\exp_G: \mathfrak{G} \to G_0$ defined on a 1-neighborhood V of G as in [1, p. 118]. Thus if $a \in V$, then $a = \exp(\Sigma_{k=1}^n x_k(a)X_k)$. Since V generates G_0, the elements $a_\lambda = \sigma(\omega_\lambda)$ may be written in the form

$$a_\lambda = a_{\lambda,1}, \cdots, a_{\lambda, l(\lambda)} \quad \text{with } a_{\lambda,i} \in V \; (1 \le i \le l(\lambda)).$$

Hence

$$a_\lambda = \prod_{i=1}^{l(\lambda)} \exp\left(\sum_{k=1}^{n} x_k(a_{\lambda,i})X_k\right).$$

On the other hand, every element ω_λ can be written in the form

$$\omega_\lambda = \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_{k(\lambda)}}^{\epsilon_{k(\lambda)}}, \quad \epsilon_i = \pm 1,$$

where $(i_1, \cdots, i_{k(\lambda)})$ is a subset of $(1, 2, \cdots, m)$.

We fix one such expression and define $W_\lambda: H^m \to H$ by $W_\lambda(h_1, \cdots, h_m) = h_{i_1}^{\epsilon_1} \cdots h_{i_{k(\lambda)}}^{\epsilon_{k(\lambda)}}$. Thus, using $\exp_H \cdot df = f \cdot \exp_G$, we may replace (3) by

$$\prod_{i=1}^{l(\lambda)} \exp_H\left(\sum x_k(a_{\lambda,i})\, df(X_k)\right) = W_\lambda(g(\gamma_1), \cdots, g(\gamma_m)). \tag{4}$$

(2.3) Let $r: \tilde{G}_0 \to G_0$ be the universal covering of G_0 and let $\tilde{x}_1, \cdots, \tilde{x}_n$ be a canonical system of coordinates defined on a 1-neighborhood $\tilde{V}$ of $\tilde{G}_0$ so that every $\tilde{a} \in \tilde{V}$ can be written in the form

$$\tilde{a} = \exp\left(\sum_{k=1}^{n} \tilde{x}_k(\tilde{a})X_k\right).$$

It is well known that $\ker r$ is finitely generated. Let $(\tilde{b}_1, \cdots, \tilde{b}_l)$ be a set of generators of $\ker r$. Since $\tilde{V}$ generates $\tilde{G}_0$, there exist $\tilde{b}_{j,1}, \cdots, \tilde{b}_{j,m(j)} \in V$ so that

$$\tilde{b}_j = \prod_{p=1}^{m(j)} \tilde{b}_{j,p} = \prod_{p=1}^{m(j)} \exp\left(\sum_{k=1}^{n} x_k(b_{j,p})X_k\right).$$

Now consider the dual map $r^* = \mathrm{Hom}\,(r, 1)$: $\mathrm{Hom}\,(G_0, H) \to \mathrm{Hom}\,(\tilde{G}_0, H)$. Clearly $f_0 \in \mathrm{Hom}\,(G_0, H)$ belongs to $\mathrm{Im}\, r^*$ if and only if $f_0(\mathrm{Ker}\, r) = 1$.

Hence we have $f_0 \in \mathrm{Im}\, r^*$ if and only if it satisfies

$$(5) \qquad \prod_{p=1}^{m(j)} \exp_H\left(\sum_{k=1}^{n} \tilde{x}_k(\tilde{b}_{j,p})\, df_0(X_k)\right) = 1 \qquad (1 \le j \le l).$$

(2.4) We now define ϵ: $\mathrm{Hom}\,(G, H) \to \mathfrak{H}^n \times H^m$ by

$$\epsilon(f) = (df(X_1), \cdots, df(X_m), f(y_1), \cdots, f(y_m)).$$

Note that ϵ is merely the composition of the following sequence of maps: $\mathrm{Hom}\,(G, H) \to \mathrm{Hom}\,(G_0, H) \times \mathrm{Hom}\,(F, H) \to \mathrm{Hom}\,(\tilde{G}_0, H) \times H^m \xrightarrow{\cong} \mathrm{Hom}\,(\mathfrak{G}, \mathfrak{H}) \times H^m \to \mathfrak{H}^n \times \mathfrak{H}^m$, where $\mathrm{Hom}\,(G_0, H) \cong \mathrm{Hom}\,(\mathfrak{G}, \mathfrak{H})$ is given by $f \to df$.

From this, it is easy to see that ϵ: $\mathrm{Hom}\,(G, H) \to \mathrm{Im}\,\epsilon$ is a homeomorphism.

Using the formulas (3), (4) and (5), we have

Lemma. $(Y_1, \cdots, Y_n; b_1, \cdots, b_m) \in \mathfrak{G}^n \times H^m$ *is in* $\mathrm{Im}\,\epsilon$ *if and only if it satisfies the following conditions*:

(i) $\Pi_{p=1}^{m(j)} \exp_H (\Sigma\, \tilde{x}_k(\tilde{b}_{j,p})Y_k) = 1$ $(1 \le j \le l)$.

(ii) $\Sigma_{k=1}^{n} C_{ijk} Y_k - [Y_i, Y_j] = 0$ $(1 \le i, j \le n)$, *where the* C_{ijk} *are the structural constants of* $\mathfrak{G}$.

(iii) $\mathrm{Ad}\,(b_j)(Y_i) - \Sigma_{k=1}^{n} r_{ik}^{j} Y_k = 0$ $(1 \le i \le n,\ 1 \le j \le m)$, *where* $(r_{ik}^{j})_{i,k}$ *is the matrix of* $\mathrm{Ad}\,(y_j)$: $\mathfrak{G} \to \mathfrak{G}$ *relative to the basis* X_i.

(iv) $\Pi_{i=1}^{l(\lambda)} \exp\,(\Sigma_k\, x_k(a_{\lambda,i})Y_k)\, W(b_1, \cdots, b_m)^{-1} = 1$ *for* $\lambda \in \Lambda$.

Define $\Phi_j(Y_1, \cdots, Y_n; b_1, \cdots, b_m)$, $\psi_{ij}(Y_1, \cdots, Y_n; b_1, \cdots, b_m)$, $\Omega_{ij}(Y_1, \cdots, Y_n; b_1, \cdots, b_m)$ and $\chi_\lambda(Y_1, \cdots, Y_n; b_1, \cdots, b_m)$ to be the expressions on the left-hand sides of the equations (i), (i), (ii), (iii) and (iv), respectively.

Then Φ_j, χ_λ and ψ_{ij}, Ω_{ij} are all C^∞-maps of $\mathfrak{H}^n \times H^m$ into H and $\mathfrak{H}$, respectively.

3. Deformations of Lie groups.

(3.1) Let ρ: $G \to \mathrm{Aut}\,(V)$ be a continuous representation in a finite-dimensional real vector space V. We define an embedding ϵ': $Z^1(G, V, \rho) \to V^n \times V^m$ by $\epsilon'(f) = (df(X_1), \cdots, df(X_m); f(y_1), \cdots, f(y_m))$. Here we identified V with the tangent linear space of V at 0.

We give a convenient description of $\mathrm{Im}\,(\epsilon')$ below. For this purpose, we take the semidirect product $V \times_\rho G$ of V by G relative to ρ.

For any map f: $G \to V$, define f': $G \to V \times_\rho G$ by $f'(x) = (f(x), x)$. Then

$f \in Z^1(G, V, \rho)$ if and only if $f' \in \operatorname{Hom}(G, V \times_\rho G)$. Applying the lemma in §2 to f' with $H = V \times_\rho G$, it is not difficult to obtain

Lemma. *$(v_1, \cdots, v_n; u_1, \cdots, u_m) \in V^n \times V^m$ is in $\operatorname{Im}(\epsilon')$ if and only if it satisfies the following*:

$$\text{(i)} \qquad \sum_{p=1}^{m(j)} \sum_{k=1}^{n} \beta_{j,p,k} = 0 \qquad (1 \le j \le l),$$

where $\beta_{j,p,k} = \tilde{x}_k(\tilde{b}_{j,p})\rho(\tilde{b}_{j,1} \cdots \tilde{b}_{j,p-1})v_k$ *for* $1 \le k \le n$, $1 \le p \le m(j)$.

$$\text{(ii)} \qquad \sum_{k=1}^{n} c_{ijk} X_k - d\rho(X_j)(v_i) + d\rho(X_i)(v_j) = 0 \qquad (1 \le i, j \le n).$$

$$\text{(iii)} \quad \rho(\gamma_j)v_i - \rho(\gamma_j) \cdot d\rho(X_i) \cdot \rho(\gamma_j)^{-1}(u_j) - \sum_{k=1}^{n} r^j_{ik} v_k = 0 \qquad (1 \le i \le n,\ 1 \le j \le m).$$

$$\text{(iv)} \qquad \sum_{j=1}^{k(\lambda)} u_{\lambda,j} - \sum_{j=1}^{l(\lambda)} \sum_{k=1}^{n} \alpha_{\lambda,j,k} = 0 \quad \text{for } \lambda \in \Lambda,$$

where $\alpha_{\lambda,j,k} = x_k(a_{\lambda,i})\rho(a_{\lambda,1} \cdots a_{\lambda,i-1})v_k$, *and*

$$\mu_{\lambda,j} = \begin{cases} \rho(\gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_{j-1}}^{\epsilon_{j-1}})\gamma_{i_j}^{\epsilon_j} & \text{if } \epsilon_j = 1, \\ -\rho(\gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_j}^{\epsilon_j})\gamma_{i_j}^{\epsilon_j} & \text{if } \epsilon_j = -1. \end{cases}$$

(Here $\omega_\lambda = \gamma_{i_1}^{\epsilon_1} \cdots \gamma_{i_{k(\lambda)}}^{\epsilon_{k(\lambda)}}$.*)*

(3.2) Now we prove the theorem of Nijenhuis and Richardson announced in [6].

Theorem 1. *Let G be a compactly generated Lie group and H any Lie group. Let $f_0 \in \operatorname{Hom}(G, H)$. If $H^1(G, \mathfrak{H}, \operatorname{Ad} \circ f_0) = 0$, then f_0 is rigid.*

Proof. We first identify the space $\operatorname{Hom}(G, H)$ with a closed subset of $\mathfrak{H}^n \times H^m$ under the map ϵ in §2, and let $f_0 = \epsilon(f_0) = (Y_1^0, \cdots, Y_n^0; b_1^0, \cdots, b_m^0)$. Next we identify the tangent linear space of $\mathfrak{H}^n \times H^m$ at f_0 with that of $\mathfrak{H}^n \times H^m$ at $(0, \cdots, 0, 1, \cdots, 1)$ by the right translation.

On the other hand, we identify the tangent linear space of $\mathfrak{H}^n \times H^m$ at $(0, \cdots, 0, 1, \cdots, 1)$ with $\mathfrak{H}^n \times \mathfrak{H}^m$.

Let ϵ': $Z^1(G, \mathfrak{H}, \operatorname{Ad} \cdot f_0) \to \mathfrak{H}^n \times \mathfrak{H}^m$ be defined as in the beginning of this section with $V = \mathfrak{H}$. Then a routine (but lengthy) computation shows that the tangent maps $d\Phi_j$, $d\psi_{ij}$, $d\Omega_{ij}$ and $d\chi_\lambda$ are given by the expressions on the left-hand sides of the equations (i), (ii), (iii) and (iv), respectively, of the lemma in (3.1). Hence from the lemma,

$$\epsilon'(Z^1(G, \mathcal{H}, \mathrm{Ad} \cdot f_0)) = \left(\bigcap_j \ker d\Phi_j\right) \cap \left(\bigcap_{i,j} \mathrm{Ker}\ d\psi_{ij}\right) \cap \left(\bigcap_{i,j} \ker d\Omega_{ij}\right) \cap \left(\bigcap_\lambda \mathrm{Ker}\ d\chi_\lambda\right).$$

Now we define $\Lambda: H \to \mathcal{H}^n \times H^m$ by $\Lambda(h) = I_h \cdot f_0 \in \mathrm{Hom}\,(G, H) \subset \mathcal{H}^n \times H^m$. Then it is easy to see that $\mathrm{Im}\,(d\Lambda) = \epsilon'(B^1(G, \mathcal{H}, \mathrm{Ad} \cdot f_0))$. Hence $H^1(G, \mathcal{H}, \mathrm{Ad} \cdot f_0) = 0$ implies that $\mathrm{Im}\,(d\Lambda)$ coincides with the intersection of the kernels of $d\Phi_j$, $d\psi_{ij}$, $d\Omega_{ij}$ and $d\chi_\lambda$. Hence by Lemma 1 of Weil [7], we can find a 1-neighborhood U of H so that $\Lambda(U)$ is open. Since $\Lambda(U) \subset \mathrm{Hom}\,(G, H)$, $\mathrm{Int}\,(H) \cdot f_0$ is a neighborhood of f_0 in $\mathrm{Hom}\,(G, H)$.

To show that $\mathrm{Int}\,(H) \cdot f_0$ is open, consider $f = I_h \cdot f_0 \in \mathrm{Int}\,(H) \cdot f_0$. Then $H^1(G, \mathcal{H}, \mathrm{Ad} \cdot f) \cong H^1(G, \mathcal{H}, \mathrm{Ad} \cdot f_0) = 0$. Hence $\mathrm{Int}\,(H) \cdot f$ is a neighborhood of f and is contained in $\mathrm{Int}\,(H) \cdot f_0$. Hence $\mathrm{Int}\,(H) \cdot f_0$ is open, proving the rigidity of f_0.

4. Proof of the main result and application. In this section, we extend the main result of the preceding section to locally compact groups.

(4.1) We begin with the following special case:

Proposition. *Let G be a compact group and let H be any Lie group. Then every $f_0 \in \mathrm{Hom}\,(G, H)$ is rigid.*

Proof. Let $K = \mathrm{Ker}\, f_0$. Then G/K is a compact Lie group and f_0 induces $\hat{f}_0 \in \mathrm{Hom}\,(G/K, H)$. Since H is a Lie group, there exists an open 1-neighborhood U of H which contains no subgroups other than $\{1\}$.

Next we consider the basic neighborhood $W(K, U; f_0)$. If $f \in W(K, U; f_0)$, then $f(K) \subset U$, and hence $f(K) = \{1\}$ by the choice of U. Thus each $f \in W(K, U; f_0)$ defines a unique $\hat{f} \in \mathrm{Hom}\,(G/K, H)$ so that $\hat{f} \cdot \pi = f$, where $\pi: G \to G/K$ is the canonical projection.

We now define $\sigma: W(K, U; f_0) \to \mathrm{Hom}\,(G/K, H)$ by $\sigma(f) = \hat{f}$.

Clearly σ is continuous and if $\pi^* = \mathrm{Hom}\,(\pi, 1)$: $\mathrm{Hom}\,(G/K, H) \to \mathrm{Hom}\,(G, H)$ is the dual map induced by π, then $\sigma \cdot \pi^* = 1$ on $\mathrm{Im}\,\sigma$ and $\pi^* \cdot \sigma = 1$ on $W(K, U; f_0)$. Hence $\pi^*: \mathrm{Im}\,\sigma \to W(K, U; f_0)$ is a homeomorphism.

Since G/K is a compact group, $H^1(G/K, \mathcal{H}, \mathrm{Ad} \cdot \sigma(f_0)) = (0)$. Hence, by the theorem in (3.2), $\sigma(f_0) = \hat{f}_0$ is rigid. From this, it is immediate that $\pi^*(\mathrm{Int}\,(H) \cdot \sigma(f_0)) = \mathrm{Int}\,(H) \cdot f_0$ is a neighborhood of f_0, proving the rigidity of f_0.

Remark. The above proposition was proved in [3] by using an entirely different method.

(4.2) The following lemma will be needed in (4.3).

Lemma. *Let M be a locally compact group and let K be a closed subgroup of M so that $M = M_0 \cdot K$. If K_1 is an open subgroup of K, then $M_0 \cdot K_1$ is open in M.*

Proof. Choose a compact 1-neighborhood U of M, and let L be the subgroup of M generated by U. As L is open in M, $L \supset M_0$. Hence $L = M_0 \cdot (K \cap L)$ and $K \cap L$ is open in K. $M_0 \cdot K_1$ would be open in M if we can show that $M_0 \cdot (K_1 \cap L)$ is open in $M_0 \cdot (K \cap L) = L$. Hence, replacing K by $K \cap L$ if necessary, we may assume that M (and hence K) is σ-compact.

Now choose a compact 1-neighborhood V in K_1. Since K is σ-compact, we can write K as $K = \bigcup_{i=1}^{\infty} Vx_i$ with $x_i \in K$. Then $M = M_0 \cdot K = \bigcup_{i=1}^{\infty} M_0 \cdot Vx_i$.

Since M is locally compact, one of the M_0Vx_i has a nonvoid interior. Hence M_0V has a nonvoid interior, which implies that M_0K_1 is open in M.

(4.3) Now we are ready to prove the main theorem.

Theorem 2. *Let G be a compactly generated locally compact group and let H be any Lie group. Let $f_0 \in \mathrm{Hom}\,(G, H)$ with $H^1(G, \mathfrak{H}, \mathrm{Ad} \cdot f_0) = 0$. Then f_0 is rigid.*

Proof. Let K be the kernel of f_0. Then $G_0 \cdot K$ is open in G. To see this, let L be an open subgroup of G containing G_0 with compact L/G_0. Then

$$(G_0K) \cap L = G_0 \cdot (K \cap L).$$

If $G_0 \cdot (K \cap L)$ is open in L, G_0K is open in G. Hence, replacing G by L if necessary, we may assume that G/G_0 is compact. For such a group, we can find a compact totally disconnected subgroup C of G such that $G = G_0 \cdot C$. (See [3, Theorem 2.13, p. 42].) Let $\pi: G \to G/K$ be the natural projection. Then $\pi(G) = \pi(G_0)\pi(C)$. As K is the kernel of f_0, G/K is a Lie group. Hence $\pi(C)$ is finite and hence $\pi(G_0)$ is open in $\pi(G) = G/K$, proving that G_0K/K in G is open in G/K. Hence G_0K is open.

Next we choose an open subgroup K_1 of K containing K_0 so that K_1/K_0 is compact. Then applying the lemma to $M = G_0K$, we see that $G_0 \cdot K_1$ is open in M (and hence in G). Choose a compact subgroup C_1 of K_1 so that $K_1 = K_0 \cdot C_1$. Then $G_0 \cdot C_1 (= G_0 \cdot K_1)$ is open in G.

By a theorem of Iwasawa [2], G_0 contains the maximal compact connected normal subgroup P so that G_0/P is a Lie group. Then $G_0/K \cap P$ is also a Lie group. Since $K \cap P$ is a normal subgroup of K, the compact subgroup C_1 of K as found above normalizes $K \cap P$.

Let $C = (K \cap P) \cdot C_1$. If N is any closed normal subgroup of G containing C, then G/N is a Lie group. In fact, $G_0 \cdot N$ is open in G because $G_0 \cdot N$

contains $G_0 \cdot C_1$ which is open in G. Also $G_0/N \cap G_0$ is a Lie group because $N \cap G_0$ contains $K \cap P$. Thus it follows that G/N is a Lie group.

Now let $i^* = \text{Hom}(i, 1)$: $\text{Hom}(G, H) \to \text{Hom}(C, H)$ be the dual map induced by the inclusion $i: C \to G$. Since C is compact, $i^*(f_0) = f_0|C \in \text{Hom}(C, H)$ is rigid by the proposition in (4.1). Hence $\text{Int}(H) \cdot i^*(f_0)$ is an open neighborhood of $i^*(f_0)$ and $W = (i^*)^{-1}(\text{Int } H \cdot i^*(f_0))$ is an open neighborhood of f_0 in $\text{Hom}(G, H)$. If $g \in W$, then there exists $h \in H$ so that $g(x) = hf_0(x)h^{-1}$ for all $x \in C$. From this, $\text{Ker } G \cap C = \text{Ker } f_0 \cap C$. But $C \subseteq \text{Ker } f_0 = K$. Hence $C \subseteq \text{Ker } g$ for all $g \in W$. Let N be the intersection for all $\text{Ker } g$, $g \in W$. Then $C \subseteq N$ and hence G/N is a Lie group.

Let $\delta: G \to G/N$ be the natural projection and let $\delta^* = \text{Hom}(\delta, 1)$: $\text{Hom}(G/N, H) \to \text{Hom}(G, H)$ be the dual map of δ. For each $g \in W$, there exists a unique $\hat{g} \in \text{Hom}(G/N, H)$ so that $g = \hat{g} \cdot \delta^*(g)$. If we define $\sigma(g) = \hat{g}$, then $\sigma: W \to \text{Hom}(G/N, H)$ is a continuous map and we have $\sigma \cdot \delta^* = 1$ on $\text{Im } \sigma$, $\delta^* \cdot \sigma = 1$ on W. Hence δ^*: $\text{Im } \sigma \to W$ is a homeomorphism. On the other hand, $H^1(G, \mathfrak{H}, \text{Ad} \cdot f_0) = 0$ implies that $H^1(G/N, \mathfrak{H}, \text{Ad} \cdot \hat{f}_0) = 0$.

Hence by the theorem in §3, f_0 is rigid. That is, $\text{Int}(H) \cdot \hat{f}_0$ is an open neighborhood of f_0. It follows that $\delta^*(\text{Int}(H) \cdot \hat{f}_0 \cap \text{Im } \sigma)$ is an open neighborhood of $\delta^*(\hat{f}_0) = f_0$ in W. From this, it is clear that $\text{Int}(H) \cdot f_0 = \delta^*(\text{Int}(H) \cdot f_0)$ is an open neighborhood of f_0 in $\text{Hom}(G, H)$, proving that f_0 is rigid.

(4.4) Example. The theorem is false without the condition that G is compactly generated. In fact, let G be the quasi cyclic group $Z(p^\infty)$ with the discrete topology and let H be the circle group. If f_0 is the zero map, then $H^1(G, \mathfrak{H}, \text{Ad} \cdot f_0) = \text{Hom}(G, \mathfrak{R}) = 0$. Hence $\text{Hom}(G, H)$ would be a discrete group. However $\text{Hom}(G, H) = \hat{G}$, the character group of G, is well known to be the group of p-adic integers which is not discrete.

(4.5) *Application.* Using the main result, we characterize compactly generated groups which do not assume any nontrivial real characters.

Theorem. *Let G be a compactly generated locally compact group. Then the following are equivalent:*

(1) *There is no nontrivial continuous homomorphism of G into $\mathfrak{R}$, the additive group of real numbers.*

(2) *The trivial homomorphism from G into any Lie group H is an isolated point in the space* $\text{Hom}(G, H)$.

Proof. (1) *implies* (2). (1) states that $\text{Hom}(G, \mathfrak{R}) = 0$. Let H be any Lie group and let $f_0: G \to H$ be the trivial homomorphism. Then $\text{Ad}_H \cdot f_0 = 0$. Hence $H^1(G, \mathfrak{H}, \text{Ad}_H \cdot f_0) = \text{Hom}(G, \mathfrak{H}) = 0$ because $\text{Hom}(G, \mathfrak{R}) = 0$. Hence by the main result, $\{f_0\}$ is an open neighborhood of f_0, proving (2).

(2) *implies* (1). First we note that Hom $(G, \mathfrak{R}) \cong$ Hom $(G/G', \mathfrak{R})$ where G' denotes the closure of the commutator subgroup of G. By (2), Hom $(G/G', \mathfrak{R})$ is a discrete topological group.

On the other hand, G/G' is a compactly generated abelian group. Hence G/G' is isomorphic with the group of the form $\mathfrak{R}^n \times Z^m \times T$, where T is the maximal compact subgroup. Hence Hom $(G, \mathfrak{R}) \cong$ Hom $(G/G', \mathfrak{R}) \cong$ Hom $(\mathfrak{R}^n, \mathfrak{R})$ $\times$ Hom $(Z^m, \mathfrak{R})$. But Hom $(\mathfrak{R}^n, \mathfrak{R})$, and Hom $(Z^m, \mathfrak{R})$, are all connected. Since Hom $(G, \mathfrak{R})$ is discrete, it follows that Hom $(G, \mathfrak{R}) = 0$, proving (1).

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DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO 44106