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REPRESENTING MEASURES AND TOPOLOGICAL TYPE OF FINITE BORDERED RIEMANN SURFACES(1)

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ABSTRACT. A finite bordered Riemann surface $\mathfrak{R}$ with s boundary components and interior genus g has first Betti number $r = 2g + s - 1$. Let a be any interior point of $\mathfrak{R}$ and e_a denote evaluation at a on the usual hypo-Dirichlet algebra associated with $\mathfrak{R}$. We establish some connections between the topological and, more strongly, the conformal type of $\mathfrak{R}$ and the geometry of $\mathfrak{M}_a$, the set of representing measures for e_a. For example, we show that if $\mathfrak{M}_a$ has an isolated extreme point, then $\mathfrak{R}$ must be a planar surface. Several questions posed by Sarason are answered through exhausting the possibilities for the case $r = 2$.

1. Let R be a finite open Riemann surface with boundary ∂R consisting of a finite nonzero number s of disjoint Jordan curves (see Ahlfors and Sario [4, p. 117]). The first Betti number of R, which we denote by r, equals $2g + s - 1$, where g is the interior genus of R. We take ∂R to be oriented with the induced orientation from R. Let A be the hypo-Dirichlet algebra (see [2], [15]) of functions continuous on ∂R and possessing continuous extensions to $\overline{R} = R \cup \partial R$ which are holomorphic on R. Fix a point $a \in R$ and let e_a denote evaluation on A at a; i.e., $e_a(f) = f(a)$ for $f \in A$. Let $\mathfrak{M}_a$ be the compact convex set of representing measures for e_a. A measure $m \in \mathfrak{M}_a$ is a positive Borel measure of total mass one supported on ∂R such that $f(a) = \int_{\partial R} f\,dm$ for all $f \in A$.

In this paper we shall show some connections between the topological and, more strongly, the conformal type of R and the geometry of $\mathfrak{M}_a$. We will let S^2 denote the Riemann sphere and Re be the real axis. The set $\mathfrak{M}_a$ will be called *strictly convex* if $\partial\mathfrak{M}_a$ consists entirely of extreme points. The principal results are stated below.

Theorem 1.1. *If $\mathfrak{M}_a$ has an isolated extreme point, then R is conformally equivalent to S^2 minus a finite number of slits contained in* Re.

The next theorem exhausts the possibilities for $\mathfrak{M}_a$ when $r = 2$, and thus answers Questions 1 and 3 on p. 376 of [12].

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Theorem 1.2. *If $g = 0$ and $s = 3$, then $\mathfrak{M}_a$ has precisely four extreme points if a lies on one of three distinguished analytic arcs, and $\mathfrak{M}_a$ is strictly convex if a lies off these arcs. If $g = s = 1$, then $\mathfrak{M}_a$ is strictly convex for all $a \in R$.*

The geometry of $\mathfrak{M}_a$ is not yet completely known for all R's with $r > 2$, but some results are presented for particular cases. Other relevant material can be found in [12].

The proofs given are of an elementary nature and rely primarily on properties of zero sets of representing measures.

2. We shall begin with some basic facts about $\mathfrak{M}_a$ and V, the space of real annihilating measures of A.

Let m_a denote harmonic measure for e_a. This measure has the representation $dm_a = *d_p g(p,a)/-2\pi$ on ∂R, where $g(p,a)$ is the Green's function for $\bar{R}$ with singularity at a, d_p is exterior differentiation on $g(p,a)$ as a function of p and $*$ is the Hodge star operator (see [13, pp. 217–218]). Fix a nowhere vanishing abelian differential ω which is holomorphic on $\bar{R}$. Then there is a function Q_{m_a} which is holomorphic on ∂R such that $*d_p g(p,a)/-2\pi = Q_{m_a}\omega$ where defined. If m is any element of $\mathfrak{M}_a$, then $m - m_a \in V$. The space V is r dimensional and is spanned by Schottky differentials $\omega_1, \ldots, \omega_r$, which are real on ∂R. Thus $dm = Q_{m_a}\omega + \sum_{i=1}^r a_i\omega_i$ for some real numbers a_i. It follows that there is a function Q_m which is holomorphic on ∂R such that $dm = Q_m\omega$. From the known structure of $*d_p g(p,a)$ and of the ω_i's, it can be seen that Q_m can be regarded as a meromorphic differential on $\bar{R}$ whose only pole is a simple pole at a.

Since $Q_m\omega$ is real on ∂R, it can be extended by the reflection principle to a meromorphic differential on $\hat{R}$, the (compact) doubled Riemann surface associated with R. The sum of the orders of the zeros minus the sum of the orders of the poles of a nontrivial meromorphic differential on a compact Riemann surface is $2G - 2$, where G is the genus of the surface (see [13, p. 267]). The genus of $\hat{R}$ is r, the first Betti number of R, so if $Q_m\omega$ extended to $\hat{R}$ has p zeros on R and q zeros on ∂R, counting multiplicities, then $Q_m\omega$ has $2p + q$ zeros on $\hat{R}$, which along with two simple poles on $\hat{R}$ implies $2r - 2 = 2p - q - 2$. We have established

Theorem 2.1. *Q_m has r zeros on $\bar{R}$ provided zeros on ∂R are counted with one-half their multiplicities,*

Definition 2.2. Let $Z(Q_m)$ denote the set of zeros of Q_m in $\bar{R}$. For convenience we shall identify m and $Q_m\omega$ and call $Z(Q_m)$ the zero set of m.

In the case of harmonic measure, $dm_a = *d_p g(p,a)/-2\pi = Q_{m_a}\omega$, it happens that $*d_p g(p,a)$ is never zero on ∂R; therefore, Q_{m_a} has no zeros on ∂R. We have shown

Theorem 2.3. *The measure m_a has r zeros on R.*

Theorem 2.4. *If $m \in \mathfrak{M}_a$ and $dm = Q_m\omega$ and $p \in Z(Q_m) \cap \partial R$, then p has even multiplicity.*

Proof. The conclusion follows from the fact that $dm = Q_m \omega$ is nonnegative on ∂R. □

An argument similar to the one used to prove Theorem 2.1 establishes the following.

Theorem 2.5. *Each nonzero $\eta \in V$ has $r - 1$ zeros on $\overline{R}$ provided zeros on ∂R are counted with one-half their multiplicities.*

If the elements of $\mathfrak{M}_a$ are regarded as meromorphic differentials on $\overline{R}$, then it is clear that $\partial\mathfrak{M}_a$ consists precisely of those elements of $\mathfrak{M}_a$ with zeros on ∂R.

3. The following, through Lemma 3.1, paraphrases material in Ahern [1]. If P is an $r - 1$ dimensional hyperplane in the vector space of Schottky differentials on $\overline{R}$ such that $\mathfrak{M}_a \cap P = \partial\mathfrak{M}_a \cap P$ and is not empty, then P is called a *supporting hyperplane*. If the interior of $\partial\mathfrak{M}_a \cap P$ is not empty, then this interior is called a *face* of $\mathfrak{M}_a$. If m belongs to a face, then there is a unique supporting hyperplane containing m. If m_1 and $m_2 \in \mathfrak{M}_a$, we define

$$[m_1, m_2] = \{\lambda m_1 + (1 - \lambda)m_2 \mid 0 \leq \lambda \leq 1\}.$$

Lemma 3.1. *Suppose $\mathfrak{M}_a$ has faces F_1 and F_2, and $m_1 \in F_1$ and $m_2 \in F_2$. Then $F_1 = F_2$ if and only if $[m_1, m_2] \subset \partial\mathfrak{M}_a$.*

Proof. If $F_1 = F_2$, then it is obvious that $[m_1, m_2] \subset \partial\mathfrak{M}_a$. If $[m_1, m_2] \subset \partial\mathfrak{M}_a$, then by [6, p. 5] there is a supporting hyperplane P which contains m_1 and m_2. Since m_1 and $m_2 \in P$ and m_1 and $m_2 \in F_2$, the uniqueness property of supporting hyperplanes implies the interior of $P \cap \partial\mathfrak{M}_a = F_1 = F_2$. □

Let $C_1, \ldots, C_s$ be the components of ∂R and $\phi_j(p) = r_j \exp(i\theta_j)$ be a uniformizer which maps a neighborhood of $C_j \subset \hat{R}$ onto a neighborhood of the unit circle T in $\mathbf{C}$ such that $\phi_j(C_j) = T$, $1 \leq j \leq s$. Each $\mu \in \mathfrak{M}_a$ is represented on T by $(d\mu/d\theta_j)(\theta_j)d\theta_j$, where $(d\mu/d\theta_j)(\theta_j) \geq 0$ and is analytic. Similarly, each $\eta \in V$ is represented on T by $(d\eta/d\theta_j)(\theta_j)d\theta_j$, where $(d\eta/d\theta_j)(\theta_j)$ is real and analytic.

Lemma 3.2. *A measure $\mu \in \partial\mathfrak{M}_a$ is not an extreme point if and only if there is an $\eta \in V$ such that each zero of μ on ∂R is covered by a zero of η with at least the same multiplicity.*

Proof. If $\eta \in V$ satisfies the condition stated, then $\mu + \varepsilon\eta \in \partial\mathfrak{M}_a$ for all ε in some sufficiently small neighborhood of 0, and μ fails to be extreme.

If μ is not extreme, then $\mu = (\mu_1 + \mu_2)/2$ for some $\mu_1 \neq \mu_2 \in \partial\mathfrak{M}_a$. The functions $d\mu_1/d\theta_j$ and $d\mu_2/d\theta_j$, $1 \leq j \leq s$, are nonnegative on T; therefore, each zero of μ on ∂R is a zero of μ_1 and μ_2 with at least as high multiplicity. The measure $\eta = \mu_1 - \mu_2$ satisfies the conclusion. □

Corollary 3.3. *If $\mu \in \mathfrak{M}_a$ has all its zeros on ∂R, then μ is extreme.*

Proof. If not, there is an $\eta \in V$ with r zeros on ∂R. This contradicts 2.5. □

Lemma 3.4. *Let $K \subset \partial\mathfrak{M}_a$ be a nonempty convex set. Then there is a $P \in \partial R$ such that $\mu(P) = 0$ for all $\mu \in K$.*

Proof. Fix some $m \in K$ and let $P_1, \ldots, P_n$, $1 \leq n \leq r$, be the zero points of m on ∂R. Suppose for each P_i there is some $\mu_i \in K$ such that $\mu_i(P_i) \neq 0$. Then the measure μ defined by $(\mu_1 + \cdots + \mu_n)/n$ is in K and has no zeros in common with m. It follows that $(\mu + m)/2$ has no zeros on ∂R. But this is impossible since $(\mu + m)/2 \in K \subset \partial\mathfrak{M}_a$. We conclude that for some P_j, $1 \leq j \leq n$, $\mu(P_j) = 0$ for all $\mu \in K$. □

We shall call $K \subset \partial\mathfrak{M}_a$ a *maximal convex subset* of $\partial\mathfrak{M}_a$ if it is convex and is not properly contained in any convex subset of $\partial\mathfrak{M}_a$.

Lemma 3.5. *Let $m \in \partial\mathfrak{M}_a$ and $Z = \{P_1, \ldots, P_n\}$, $1 \leq n \leq r$, be the zeros of m on ∂R. Then m is contained in at most n distinct maximal convex subsets of $\partial\mathfrak{M}_a$.*

Proof. Suppose K_1 and K_2 are distinct maximal convex subsets of $\partial\mathfrak{M}_a$ which contain m. Then by Lemma 3.4 for some $P_i, P_j \in Z$ we have $\mu_1(P_i) = 0$ for all $\mu_1 \in K_1$ and $\mu_2(P_j) = 0$ for all $\mu_2 \in K_2$. We claim that $P_i \neq P_j$. Suppose $P_i = P_j$ and fix some $\mu \in K_2 \backslash K_1$. Then the convex set $C = \{\lambda\mu_1 + (1 - \lambda)\mu \mid \mu_1 \in K_1, 0 \leq \lambda \leq 1\}$ (the convex hull of $K_1 \cup \{\mu\}$) is contained in $\partial\mathfrak{M}_a$ since all measures in C vanish at P_i. But $K_1 \subsetneq C$, a contradiction. Thus $P_i \neq P_j$ and the desired conclusion follows. □

Lemma 3.6. *If the boundary of an n dimensional compact convex body has at most $n + 1$ maximal convex subsets, then the convex body is an n-simplex.*

Proof. The lemma is trivially true for $n = 0$. Assume it is true for $n = k$. Let K be a $k + 1$ dimensional compact convex body with boundary containing at most $k + 2$ maximal convex subsets. Denote these subsets by $F_1, \ldots, F_j$, where $j \leq k + 2$. Let H be a hyperplane which contains an interior point of K but which does not intersect F_1. Then $H \cap K$ is a k dimensional compact convex body. Each maximal convex subset of $\partial(H \cap K)$ is the intersection of $\partial(H \cap K)$ with some F_i, $2 \leq i \leq j$; therefore, by the induction hypothesis, $H \cap K$ is a k-simplex. Since a k-simplex has $k + 1$ faces, it follows that $j = k + 2$ and that the faces of $H \cap K$ are $H \cap K \cap F_i$, $2 \leq i \leq k + 2$. By the preceding it is clear that for any hyperplane H containing an interior point of K but disjoint from F_1, the faces of $\partial(H \cap K)$ are $H \cap K \cap F_i$, $2 \leq i \leq k + 2$. This implies that each F_i, $2 \leq i \leq k + 2$, is k dimensional. By a similar argument, using hyperplanes disjoint from F_2 and passing through interior points of K, we can conclude F_1 is also k dimensional.

Now let H_i denote the supporting hyperplane of F_i, $1 \leq i \leq k + 2$. Each H_i is the boundary of a closed half space S_i such that the interior of $S_i \cap K$ is nonempty. Introduce coordinates $x = (x_1, \ldots, x_r)$ so that $S_1 = \{x \mid x_1 \geq 0\}$. Let $H(t)$ denote the hyperplane $\{x \mid x_1 = t\}$. Then $H(0) = H_1$. Let $t_0 = \sup\{t \mid H(t) \cap K \text{ is nonempty }\}$. Since K is $k + 1$ dimensional and compact,

$0 < t_0 < \infty$. Let $S = S_1 \cap \cdots \cap S_{k+2}$. Then $K \subset S$. By the first paragraph, for any t such that $0 < t < t_0$, we have $H(t) \cap K = H(t) \cap S$. If $H(0) \cap K \neq H(0) \cap S$, then the same inequality would hold for sufficiently small $t > 0$, a contradiction. Hence $H(0) \cap S = H(0) \cap K = F_1$, and F_1 is a face of S. Similarly, F_j for $2 \leq j \leq k+2$ is a face of S; consequently, $\partial S \subset \partial K$. Since $K \subset S$, it follows that $K = S$, a $(k+1)$-simplex. □

Lemma 3.7 *If μ is an isolated extreme point of $\mathfrak{M}_a$, then μ has r distinct double zeros on ∂R, and there is an $m \in \mathfrak{M}_a$ with $r-1$ double zeros on ∂R and one simple zero in R.*

Proof. Since μ is isolated, there is a closed half space S such that μ is the only extreme point of $\mathfrak{M}_a$ in $\mathfrak{M}_a \cap S$, and $\mu \notin$ the hyperplane ∂S. We claim that if $m' \in \partial\mathfrak{M}_a \cap S$ and C is a maximal convex subset of $\partial\mathfrak{M}_a$ which contains m', then any extreme point of C is extreme in $\mathfrak{M}_a$. To establish the claim let m_1 be extreme in C. By Lemma 3.4 all measures in C possess a common zero on ∂R, say P. If m_1 were not extreme in $\mathfrak{M}_a$, then there would exist $m_2, m_3 \in \mathfrak{M}_a$, $m_2 \neq m_3$, such that $m_1 = (m_2 + m_3)/2$. Then $m_2(P) = m_3(P) = 0$, and by maximality, m_2 and $m_3 \in C$, which implies m_1 is not extreme in C, a contradiction.

Since C is the closed convex hull of its extreme points and $C \cap S$ is nonempty, it follows that μ is an extreme point of C. Thus $\partial\mathfrak{M}_a \cap S \subset \bigcup_{K\in\mathcal{C}} K$, where $\mathcal{C}$ is the class of maximal convex subsets of $\partial\mathfrak{M}_a$ which contain μ. By Lemma 3.5 it follows that $\mathcal{C}$ has at most r elements. Hence, since $\partial(\mathfrak{M}_a \cap S) = (\partial\mathfrak{M}_a \cap S) \cup (\partial S \cap \mathfrak{M}_a)$ and since $\partial S \cap \mathfrak{M}_a$ is clearly a maximal convex subset of $\partial(\mathfrak{M}_a \cap S)$, we see that $\partial(\mathfrak{M}_a \cap S)$ has at most $r+1$ maximal convex subsets. Now Lemma 3.6 shows that $\mathfrak{M}_a \cap S$ is an r-simplex.

By the preceding it follows that μ is an intersection of r hyperplanes, $P_1, \ldots, P_r$ each of which contains a face of $\mathfrak{M}_a$, and these P_i's determine r affinely independent line segments, $E_1, \ldots, E_r$ which are one dimensional edges in $\partial\mathfrak{M}_a$ with μ as a common endpoint. (The case $r = 3$ may prove illuminating.) Furthermore, the interior of the convex hull of every $r-1$ E_i's is contained in a face. If m_1 and m_2 belong respectively to two distinct faces, then by Lemma 3.1 it follows that $[m_1, m_2]$ is not contained in $\partial\mathfrak{M}_a$. Hence m_1 and m_2 have no common zeros. It follows that each measure $\mu_i \in E_i$ has at least $r-1$ distinct zeros on ∂R, and these, of course, must also be zeros of μ. If $\mu_i \neq \mu$, then μ_i has precisely $r-1$ zeros in common with μ. This implies that if $\mu_i \in$ interior of E_i and $\mu_j \in$ interior of E_j, $i \neq j$, then μ_i and μ_j cannot have precisely the same zeros in common with μ. Consequently, μ must have r distinct zeros on ∂R. These zeros must be double by Theorem 2.4.

Choose an m from the interior of any E_i. Then m has $r-1$ double zeros on ∂R and one simple zero in R, for if m had all of its r zeros on ∂R, it would be extreme by Corollary 3.3. □

The next lemma is due to Gamelin and Voichick (see [7, p. 924]).

Lemma 3.8. *If a function h is meromorphic on $\overline{R}$, real on ∂R and has just one pole, a simple pole, in R, then h is a conformal homeomorphism of R onto S^2 minus a finite number of slits contained in* Re.

Theorem 1.1 now follows if we take μ and m as in Lemma 3.7 and set $h = \mu/m$ in Lemma 3.8. □

We next discuss an example in which $\mathfrak{M}_a$ has isolated extreme points.

Let R_r be any bounded region in $\mathbf{C}$ which is bounded by $r + 1$ disjoint circles, $C_0, C_1, \ldots, C_r$ with respective centers $a_0, a_1, \ldots, a_r$ on the real axis Re. We shall assume $a_1 < a_2 < \cdots < a_r$ and that C_0 is the boundary of the unbounded component of $\mathbf{C} \backslash R_r$. Let $C_0 \cap \mathrm{Re} = \{a_{-1}, a_{r+1}\}$ with $a_{-1} < a_{r+1}$.

Theorem 3.9. *If $a \in R_r \cap$ Re, then $\mathfrak{M}_a$ has $2r$ faces and at most 2^r extreme points.*

Remarks. We shall see that $\mathfrak{M}_a$ has four extreme points when $a \in R_2 \cap$ Re. A somewhat tedious exercise shows that $\mathfrak{M}_a$ has eight extreme points for $a \in R_3 \cap$ Re. It would be interesting to know if $\mathfrak{M}_a$ has precisely 2^r extreme points for all r when $a \in R_r \cap$ Re. Before giving a proof of Theorem 3.9, we shall present some relevant material.

Let Ω_j be harmonic measure for C_j, $1 \leq j \leq r$, i.e., Ω_j is the solution to the Dirichlet problem for $\overline{R}_r$ with boundary values 1 on C_j and 0 on C_k, $j \neq k$. A basis for V is given by $*d\Omega_j - i d\Omega_j$, $1 \leq j \leq r$, and since each Ω_j is symmetric with respect to Re, elements of V must also possess this symmetry. The next lemma follows easily.

Lemma 3.10. (i) *If $\eta \not\equiv 0$ is in V and has a zero in $\partial R_r \cap$ Re, then this zero has even order.*

(ii) *If η is in V and has a zero $z \in \partial R_r$, then it has a zero at $\bar{z}$ of the same order.*

The next two lemmas hold for arbitrary R.

Lemma 3.11. *Suppose $z_1, \ldots, z_q \in R$ and $q < r/2$. Then there is a nonzero $\eta \in V$ with zeros at z_i, $1 \leq i \leq q$.*

Proof. Let ω be a nowhere zero holomorphic differential on $\overline{R}$ and ω_i, $1 \leq i \leq r$, be a basis of Schottky differentials for V. Define Q_i by $\omega_i = Q_i \omega$, $1 \leq i \leq r$, and let $v_i = (Q_i(z_1), \ldots, Q_i(z_q))$, $1 \leq i \leq r$. Each v_i can be viewed as an element of complex q space. Since the number of v_i's is r and $2q < r$, it follows that there exist $\alpha_1, \ldots, \alpha_r \in$ Re, not all zero, such that $\sum_{i=1}^{r} \alpha_i v_i = 0$. The differential $\eta = \sum_{i=1}^{r} \alpha_i \omega_i$ satisfies the conclusion of the lemma. □

Lemma 3.12. *Let $z_1, \ldots, z_k$ be distinct points on ∂R and $q_1, \ldots, q_k$ be positive integers whose sum is less than r. Then there is a nonzero $\eta \in V$ such that η has a zero at each z_i of at least order q_i.*

A proof of Lemma 3.12 for planar R, which modifies easily for general R, can be found in [12, p. 375]. We shall need Lemma 3.12 for planar R only.

Lemma 3.13 *Let $a \in R_r \cap \mathrm{Re}$ and suppose, without loss of generality, that $a \in R_r \cap (a_1, a_2)$. Then each $\mu \in \mathfrak{M}_a$ has a zero on each of the r components of $\overline{R}_r \cap \mathrm{Re}\backslash(a_1, a_2) = Y$. We shall denote this property by ψ and let the components of Y be given by $[c_1, c_1'], \ldots, [c_r, c_r']$.*

Proof. Let $dm_a = *dg/-2\pi = (-1/2\pi)(\partial g/\partial n)d\sigma$ be harmonic measure, where $\partial/\partial n$ denotes differentiation along the outer unit normal to ∂R_r and σ denotes arc length. Since $g(z,a)$, the Green's function for $\overline{R}_r$ with singularity a, vanishes at each boundary point of Y, the interior of each $[c_i, c_i']$ contains at least one zero of $\partial g/\partial x$. The function $\partial g/\partial y$ is zero on $R_r \cap \mathrm{Re}$ since $g(z,a)$ is symmetric about Re; therefore, m_a has property ψ. Let μ be any element of $\mathfrak{M}_a$ and consider the homotopy

$$\mu_t = tm_a + (1-t)\mu, \qquad 0 \le t \le 1.$$

Let $U = \{t \in [0,1] \mid \mu_t \text{ has property } \psi\}$. Then U is nonempty since $1 \in U$. If $t_0 \in U$, then for t sufficiently near t_0, μ_t has a zero P_j in each of the r open strips $\{x + yi \mid \alpha_j < x < \alpha_{j+1}, -\infty < y < +\infty\}, j = -1, 2, 3, \ldots, r$. The measure m_a is symmetric with respect to Re; hence, any μ_t, being the sum of m_a and an element of V, is symmetric with respect to Re. Since no μ_t can have more than r zero points, the r P_j's must therefore lie on Re. The set U is thus open. It is easy to see that U is closed. Consequently, $U = [0,1]$ and $\mu_0 = \mu$ has property ψ. □

Lemma 3.14. *If $\mu \in \mathfrak{M}_a$, $a \in Y \cap R_r$, is extreme, then it must have all its zeros in $\partial R_r \cap \mathrm{Re}$.*

Proof. If not, then by Lemma 3.12 and Lemma 3.10(i) it is easy to produce a nonzero $\eta \in V$ such that η has a zero at each zero of μ on $\partial R_r \cap \mathrm{Re}$ of at least as high order. The measure μ would then fail to be extreme by Lemma 3.2. □

Lemma 3.14 shows that the zeros of each extreme point of $\mathfrak{M}_a$ must lie in the finite set ∂Y; consequently, $\mathfrak{M}_a$ has only finitely many extreme points and Lemma 3.7 implies each extreme point has r double zeros on ∂R. These remarks and Lemma 3.13 show that $\mathfrak{M}_a$ has at most 2^r extreme points.

Let $\mu \in \mathfrak{M}_a$ be extreme. Without loss of generality we can assume μ has zeros at $c_1, c_2, \ldots, c_r$. The measure μ is in the intersection of the closure of some r faces. Each face must have elements with at least one zero in common with μ. There are r possible zeros, $c_1, c_2, \ldots, c_r$, and by Lemma 3.1, no two faces can have elements possessing more than one of these zeros. Consequently, each zero, $c_1, c_2, \ldots, c_r$, corresponds to one and only one face. We claim that given any $z \in \partial Y$, there is an $m \in \partial\mathfrak{M}_a$ with a zero at z. The notation will be that of Lemma 3.7. Each E_i must have an extreme point μ_i with precisely $r-1$ zero points in common with μ. If, say, the zero set of μ_i is $\{c_1, c_2, \ldots, c_{k-1}, c_k', c_{k+1}, \ldots, c_r\}$ and the zero set of $\mu_j \in E_j$ is $\{c_1, c_2, \ldots, c_{n-1}, c_n', c_{n+1}, \ldots, c_r\}$, $i \neq j$, then c_k' and c_n' cannot lie in the same component of Y. If they did, then either $c_k' = c_q$, or $c_n' = c_q$ for some q, $1 \leq q \leq r$. Then either the zeros of μ and μ_i, or the zeros of μ and μ_j coincide, a contradiction. The claim follows easily.

Each $z \in \partial Y$ must therefore be a zero of an extreme point and can correspond to one and only one face. The set ∂Y has $2r$ elements, so $\mathfrak{M}_a$ must have $2r$ faces. The proof of Theorem 3.9 is now complete. □

Corollary 3.15. *If $a \in R_2 \cap$ Re, then $\mathfrak{M}_a$ has precisely four extreme points.*

Let R be arbitrary, $W = \{z \in \partial R \mid$ no element of V has a simple zero at $z\}$ and $Z = \{z \in \partial R \mid$ there is an $m \in \mathfrak{M}_a$ with a zero at $z\}$.

Theorem 3.16. *If $a \in R$, then the number of faces of $\mathfrak{M}_a \leq$ the cardinality of $W \cap Z$.*

Proof. If $\mathfrak{M}_a$ has no faces, then Theorem 3.16 is trivially true. Otherwise, let F be a face. Then F is $r - 1$ dimensional and is contained in an $r - 1$ dimensional hyperplane. Since the zero measure is not in F, there are r elements in F, $\mu_1, \ldots, \mu_r$ such that $\mu_i - \mu_1$, $2 \leq i \leq r$, are linearly independent. Since the μ_i's belong to a face, they must have at least one common zero, say z, which is of at least order two and, of course, $z \in Z$. It is easy to find an $\eta \in V$ which is nowhere zero on ∂R (see, e.g., [1, pp. 4–5]). Then $\{\mu_i - \mu_1 \mid 2 \leq i \leq r\} \cup \{\eta\}$ is a basis for V; consequently, $z \in W$. Lemma 3.1 implies no two faces can have common elements, and Theorem 3.16 follows. □

It is easy to show using Lemmas 3.12 and 3.10 that for $z \in \partial R_r \setminus \mathrm{Re}$ there is an $\eta \in V$ with a simple zero at z and that $W \cap Z = Y$ for this case. Lemma 3.14 then shows that Theorem 3.16 gives the best possible estimate of the number of faces. It is not known yet whether equality always holds in Theorem 3.16.

Lemma 3.17. *If $a \in R_r \setminus \mathrm{Re}$ and $r \geq 3$, then no μ in $\mathfrak{M}_a$ can have r double zeros on $\partial R \cap$ Re.*

Proof. Suppose there is such a μ. We have $\mu = m_a + \eta$ for some $\eta \in V$. The measure dm_a can be expressed by $(-1/2\pi)(\partial g/\partial n)(z, a)d\sigma$ on ∂R (see Lemma 3.13), and it is easy to verify that harmonic measure for $e_{\bar{a}}$ is given on ∂R by $dm_{\bar{a}} = (-1/2\pi)(\partial g/\partial n)(\bar{z}, \bar{a})d\sigma$. It follows from our hypothesis that each zero z_i of μ on C_i is real. In some neighborhood of z_i on C_i there are coordinates such that the following representations hold (the a_{ij}'s and b_i's are certain real numbers):

$$
\begin{aligned}
dm_a(\theta_i) &= (a_{i0} + a_{i1}\theta_i + a_{i2}\theta_i^2 + \cdots)d\theta_i, \\
d\eta(\theta_i) &= (-a_{i0} - a_{i1}\theta_i + b_i\theta_i^2 + \cdots)d\theta_i \quad \text{and} \\
dm_{\bar{a}}(\theta_i) &= dm_a(-\theta_i) = (a_{i0} - a_{i1}\theta_i + a_{i2}\theta_i^2 + \cdots)d\theta_i,
\end{aligned}
$$

where $\theta_i = 0$ corresponds to z_i. Then $d\mu(\theta_i) = [(a_{i2} + b_i)\theta_i^2 + \cdots]d\theta_i$ and μ_1 defined by $\mu_1 = m_{\bar{a}} + \eta$ satisfies $d\mu_1(\theta_i) = [(a_{i2} + b_i)\theta_i^2 + \cdots]d\theta_i$. Let $\nu = \mu - \mu_1$. Then ν is real on ∂R and has a zero of at least order three at each of the r zero points of μ. Therefore, ν has at least $3r$ zeros on ∂R if r is even, and since ν

must have an even number of zeros on ∂R, it has at least $3r + 1$ zeros on ∂R if r is odd. The measure ν has just two poles, simple poles at a and $\bar{a}$.

The argument principle shows that if $n =$ the number of zeros of ν minus its number of poles, where zeros on ∂R are counted with one-half their multiplicities, then $n = r - 1$. The considerations of the preceding paragraph yield a contradiction for $r \geq 3$. □

Theorem 3.18. *If $r \geq 2$ and $a \in R_r \backslash \mathrm{Re}$, then $\mathfrak{M}_a$ has no isolated extreme points.*

Proof. Suppose $\mathfrak{M}_a$ has an isolated extreme point μ. Then, by Lemma 3.7, μ has double zeros on ∂R. Refer to the proof of Lemma 3.7. If μ_i and μ_j belong to the interiors of E_i and E_j respectively, $i \neq j$, then each has precisely $r - 1$ double zeros in common with μ. The measures $\mu - \mu_i$ and $\mu - \mu_j \in V$, and each has precisely $r - 1$ double zeros on ∂R. Elements of V are symmetric with respect to Re and can have no more than $r - 1$ double zeros; therefore, $\mu - \mu_i$ and $\mu - \mu_j$ each has precisely $r - 1$ double zeros in $\partial R \cap \mathrm{Re}$. The zero sets of μ_i and μ_j, and thus those of $\mu - \mu_i$ and $\mu - \mu_j$, are not equal for $i \neq j$. It follows that μ has r double zeros on $\partial R \cap \mathrm{Re}$. A contradiction results from Lemma 3.17 if $r \geq 3$. The case $r = 2$ is covered by Theorem 1.2. □

Theorem 3.19. *If $r \geq 3$ and is odd and $a \in R_r$, then $\mathfrak{M}_a$ is not strictly convex.*

Proof. Let m_a have zeros $z_1, \ldots, z_r$ in R_r (see Theorem 2.3). Lemma 3.11 gives a nonzero $\eta \in V$ with zeros at $z_1, \ldots, z_{(r-1)/2}$. For some ε, $m_a + \varepsilon\eta = m \in \partial\mathfrak{M}_a$, and since m has zeros at $z_1, \ldots, z_{(r-1)/2}$, m can have no more than $r + 1$ zeros on ∂R_r counting multiplicities. The zeros on ∂R_r are of even order, so if m has less than $r + 1$ zeros on ∂R_r, it has at most $r - 1$. Then Lemmas 3.12 and 3.2 imply m is not extreme.

Suppose m has $r + 1$ zeros on ∂R_r. Let $P_1, \ldots, P_n$ be the zero points of m on ∂R_r with respective multiplicities $2v_i$, v_i a positive integer. Using Lemmas 3.12 and 3.10 we can find an $\eta \in V$ which has zeros of order $2v_i$ at P_i, $1 \leq i \leq n - 1$, and either no zero or a zero of even order at P_n. In both cases for some $\varepsilon > 0$, we have $m + \delta\eta \in \partial\mathfrak{M}_a$ for either $0 \leq \delta < \varepsilon$ or $-\varepsilon < \delta \leq 0$. Consequently, $\partial\mathfrak{M}_a$ contains a line segment and $\mathfrak{M}_a$ is not strictly convex. □

It is conjectured that Theorem 3.19 is true for r even and greater than two. Theorem 1.2, which we prove next, shows Theorem 3.19 is false for $r = 2$.

Proof of Theorem 1.2. Let R be such that $g = 0$ and $s = 3$. Then it is easy to show that R is conformally equivalent to some R_2, so without loss of generality for this part of the proof we can and shall assume $R = R_2$.

Let $a \in R_2$ and suppose $\mathfrak{M}_a$ is not strictly convex. Then $\partial\mathfrak{M}_a$ contains an open line segment E. Let $\mu \in E$. Then μ has at least one double zero z on ∂R_2 since $\mu \in \partial\mathfrak{M}_a$. If it had two distinct zeros on ∂R, it would be extreme by Corollary 3.3. Therefore, μ has a simple zero at some $b \in R_2$. Let $\mu_1 \in E$ and $\mu_1 \neq \mu$. Then $\mu_1(z) = 0$ and η defined by $\mu_1 - \mu$ is a nonzero element of V.

By Lemma 3.8 the function $h = \eta/\mu$ is a conformal homeomorphism of R_2 onto a domain D_1 in S^2 such that ∂D_1 consists of three slits in Re. It is easy to check that the class of real valued functions which are harmonic on D_1 and constant on each slit is carried by h^{-1} onto the class of real valued functions which are harmonic on R_2, continuous on $\overline{R}_2$ and constant on each component of ∂R_2. The first order complex differentials $*d\Omega - id\Omega$ for Ω in these classes have one zero point each, so by symmetry these points must lie on Re. It is easy to verify that each element of $D_1 \cap \mathrm{Re}$ and $R_2 \cap \mathrm{Re}$ is a zero of some differential of the type mentioned above; consequently, $h^{-1}(D_1 \cap \mathrm{Re}) = R_2 \cap \mathrm{Re}$. Hence $a \in \mathrm{Re}$ since $h(a) = 0$. Corollary 3.13 shows that $\mathfrak{M}_a$ has precisely four extreme points for any a on the three segments which comprise $R_2 \cap \mathrm{Re}$.

Now suppose $g = 1$ and $s = 1$. If $\mathfrak{M}_a$ is assumed to be not strictly convex, then by Lemma 3.2 there is some $\eta \in V$ with a double zero z on ∂R. We have $r = 2$, which implies η has no other zeros. Hence, since ∂R has only one component, η is either strictly positive or strictly negative on $\partial R \setminus \{z\}$. But then η would fail to annihilate A. Thus $\mathfrak{M}_a$ is strictly convex and the proof of Theorem 1.2 is complete. □

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