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THE NORM OF THE L^p-FOURIER TRANSFORM ON UNIMODULAR GROUPS

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ABSTRACT. We discuss sharpness in the Hausdorff Young theorem for unimodular groups. First the functions on unimodular locally compact groups for which equality holds in the Hausdorff Young theorem are determined. Then it is shown that the Hausdorff Young theorem is not sharp on any unimodular group which contains the real line as a direct summand, or any unimodular group which contains an Abelian normal subgroup with compact quotient as a semidirect summand. A key tool in the proof of the latter statement is a Hausdorff Young theorem for integral operators, which is of independent interest. Whether the Hausdorff Young theorem is sharp on a particular connected unimodular group is an interesting open question which was previously considered in the literature only for groups which were compact or locally compact Abelian.

1. **Introduction.** Let G be a locally compact unimodular group with corresponding Lebesgue spaces $L^p(G)$, $1 \leq p \leq \infty$, relative to a fixed Haar measure dx. Let $\Gamma = (L^2(G), \mathfrak{L}, m)$ be the canonical dual gage space of G with corresponding Lebesgue spaces $L^p(\Gamma)$, $1 \leq p \leq \infty$ [9]. For a measurable function f on G let L_f denote the partially defined operator of left convolution by f on $L^2(G)$. If L_f is a measurable operator relative to Γ it is called the Fourier transform of f and will be denoted by $\hat{f}$. [2] In this context R. A. Kunze [9] has proved the following generalization of the Hausdorff Young theorem: If $1 < p < 2$ and $f \in L^p(G)$ then L_f is measurable relative to Γ, and in fact $\hat{f} = L_f \in L^{p'}(\Gamma)$ and $\|\hat{f}\|_{p'} \leq \|f\|_p$. Here, as throughout, p' denotes the index conjugate to p:

$$p' = p/(p-1) \quad \text{if } 1 < p < \infty, \qquad 1' = \infty, \qquad \infty' = 1.$$

The purpose of this paper is twofold. First we characterize functions for which equality holds in Kunze's Hausdorff Young theorem. These are called L^p-maximal functions and were studied by Hewitt and Hirschman for G Abelian [6, §43]; and by Hewitt and Ross for G compact [6, §43]. Their results extend verbatim to the unimodular case as follows: A function f on a locally compact group G is called a *subcharacter* if there is a compact open subgroup G_0 and a

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(2) This notation is consistent with the usual notation for the Fourier transform on Abelian groups.

continuous character χ_0 of G_0 such that $f(x) = \chi_0(x)$ for $x \in G_0$ and $f(x) = 0$ for $x \notin G_0$. The first result of this paper is the following theorem.

Theorem 1. *If G is a locally compact unimodular group and $f \in L^p(G)$ for some p, $1 < p < 2$, then $\|\hat{f}\|_{p'} = \|f\|_p$ if and only if f is equal almost everywhere to a multiple of a translate of a subcharacter of G.*

Now let $\mathfrak{T}_p(G)$ denote the L^p-Fourier transform on a locally compact unimodular group G, $1 < p < 2$, i.e. the map $f \to L_f$. As a linear transformation from $L^p(G)$ into $L^{p'}(\Gamma)$, $\mathfrak{T}_p(G)$ has norm at most 1 by Kunze's Hausdorff Young theorem. If G has a compact open subgroup, Theorem 1 shows that $\|\mathfrak{T}_p(G)\| = 1$ for all p, $1 < p < 2$. Our second purpose is to estimate the norm of $\mathfrak{T}_p(G)$ for groups G lacking compact open subgroups (e.g. connected noncompact groups). We provide two classes of examples of unimodular groups G with $\|\mathfrak{T}_p(G)\| < 1$ in §§3 and 5. The proofs will show that within these classes $\|\mathfrak{T}_p(G)\|$ can be arbitrarily small. We do not consider here the problem of computing the norm exactly. §4 is devoted to a Hausdorff Young theorem for integral operators which is needed in §5.

A brief history of the problem is the following. The Hausdorff Young theorem for $G =$ the circle group was proved by Young in 1912 for $p = 2k/(2k-1)$, k an integer ≥ 2, and by Hausdorff in 1923 for all p, $1 < p < 2$. The analog for Fourier integrals, i.e. $G = \mathbf{R}$ was established by Titchmarsh in 1924. For general locally compact Abelian groups the Hausdorff Young theorem was established by Weil in 1940. Kunze's result was new even for compact groups, except for a different form on compact groups [8]. Strong forms of the theorem are known on particular groups ([10], [11]).

The forerunners of Theorem 1, aside from the works of Hewitt, Hirschman, and Ross already mentioned, are the theorems of Hardy and Littlewood, 1926, stating that equality holds in the original Hausdorff Young theorem only for characters of the circle group, and the remarkable theorem of Babenko in 1961 showing that Titchmarsh's Hausdorff Young theorem on $\mathbf{R}$ is not sharp, to wit: If $p = 2k/(2k-1)$, k an integer ≥ 2, then $\|\hat{f}\|_{p'} \leq A_p\|f\|_p$ for all $f \in L^p(\mathbf{R})$, where $A_p = [p^{2-p}(p-1)^{p-1}]^{1/2p}$. This result of Babenko, as will be seen below, motivates and explains why Theorems 2 and 4 are true. For precise references to the original papers see [6, §43, Notes].

2. **L^p- maximal functions.** Let G be a locally compact unimodular group with corresponding Lebesgue spaces $L^p(G)$, $1 \leq p \leq \infty$, relative to a fixed Haar measure dx. Let $\mathfrak{L}$ be the von Neumann algebra generated by the left regular representation λ of G on $L^2(G)$. A regular gage m was defined by Segal [12] on the projections in $\mathfrak{L}$ as follows: If Q is a projection in $\mathfrak{L}$ set $m(Q) = \|f\|_2^2$ if $Q = L_f$ for some f in $L^2(G)$ and otherwise put $m(Q) = \infty$. The resulting gage space $\Gamma = (L^2(G), \mathfrak{L}, m)$ is called the *canonical dual gage space* of G. We refer to [9], [14] for properties of Γ.

Let Σ be the set of equivalence classes of unitary representations of a locally compact group G. Then $L^1(G)$ equipped with the norm $\|f\| = \sup_{\pi\in\Sigma}\|\pi(f)\|$ is a pre-C^*-algebra whose completion is called the C^*-algebra of G, denoted $C^*(G)$. The Banach space dual of $C^*(G)$ can be identified with the collection $B(G)$ of linear combinations of continuous positive definite functions on G. The set $B(G)$ is a commutative Banach algebra with unit under pointwise operations and is called the Fourier Stieltjes algebra of G. The Fourier algebra of G is the closed subalgebra $A(G)$ of $B(G)$ which is generated by the continuous positive definite functions with compact support. The study of $A(G)$ and $B(G)$ for an arbitrary locally compact group was initiated by Eymard [3].

Theorem 1. *If G is a locally compact unimodular group and $f \in L^p(G)$ for some p, $1 < p < 2$, then $\|\hat{f}\|_{p'} = \|f\|_p$ if and only if f is equal almost everywhere to a multiple of a translate of a subcharacter of G.*

Proof. The proof is patterned after that of Hewitt and Ross [6, Theorem 43.17].

Let G_0 be a compact open subgroup of G and χ_0 a continuous character of G_0. If f equals χ_0 on G_0 and is zero off G_0 then, for $1 \le p < \infty$, $\|f\|_p^p = \int_{G_0} dx = \text{meas}(G_0) = c^{-1}$, say. Since f is a subcharacter, it is easy to verify that cf is selfadjoint and idempotent so that L_{cf} is a projection in $\mathfrak{L}$. Thus

$$\|L_{cf}\|_p^p = m(|L_{cf}|^p) = m(L_{cf}) = \|cf\|_2^2 = c^2 \cdot c^{-1} = c.$$

So

$$\|L_f\|_p = c^{-1} \cdot \|L_{cf}\|_p = c^{-1} \cdot c^{1/p} = c^{-1/p'}.$$

Thus for $1 < p < 2$, $\|L_f\|_{p'} = c^{-1/p} = \|f\|_p$.

Now let $1 < p < 2$, $f \in L^p(G)$, $\|\hat{f}\|_{p'} = \|f\|_p = 1$. Define $h_z = |f|^{p(1+z)/2}\text{sgn}\, f$, $E(z) = V|\hat{f}|^{p'(1+z)/2}$, for $0 \le \text{Re}\, z \le 1$, where $\hat{f} = V|\hat{f}|$ is the polar decomposition of $\hat{f}$. Let $q = q(z) = 2/(1 + \text{Re}\, z) = 2/(1 + u)$ where $z = u + iv$ so that $1 \le q \le 2$ and $q' = 2/(1 - u)$. Then

$$|E(z)|^2 = |\hat{f}|^{p'(2+2u)/2} = |\hat{f}|^{2p'/q},$$

and therefore

$$\|E(z)\|_q^q = m(|E(z)|^q) = m(|\hat{f}|^{p'}) = \|\hat{f}\|_{p'}^{p'} = 1.$$

Also

$$\|h_z\|_q^q = \int (|f|^{p(1+u)/2})^q dx = \|f\|_p^p = 1.$$

Define $\Phi(z) = \langle \hat{h}_z, E(z)\rangle = m(\hat{h}_z E(z)^*)$ for $0 \le \text{Re}\, z \le 1$. One has $|\Phi(z)| \le \|\hat{h}_z\|_{q'}\|E(z)\|_q \le 1$ [9, Theorem 1].

We claim that Φ is analytic on $0 < \text{Re}\, z < 1$, continuous on $0 \le \text{Re}\, z \le 1$. To see this, let g be a simple function, $g = \sum \alpha_k \chi_{A_k}$, let

$$k_z = |g|^{p(1+z)/2}\text{sgn } g = \sum |\alpha_k|^{p(1+z)/2}\text{sgn } \alpha_k \chi_{A_k}$$

and let

$$G(z) = m(\hat{k}_z E(z)^*) = \sum |\alpha_k|^{p(1+z)/2}\text{sgn } \alpha_k m(\hat{\chi}_{A_k}|\hat{f}|^{p'(1+z)/2} V^*).$$

Suppose such a G is analytic. Then taking a sequence $\{f^{(n)}\}$ of simple functions converging to f as in [6, (43.11)], i.e. $|f^{(n)}(x)| \uparrow$ and $f^{(n)}(x) \to f(x)$ uniformly on $\{|f(x)| \leq l\}$, and letting $G_n(z) = m(h_z^{(n)\wedge} E(z)^*)$ where $h_z^{(n)} = |f^{(n)}|^{p(1+z)/2}\text{sgn } f^{(n)}$, then G_n is analytic and

$$\begin{aligned}|\Phi(z) - G_n(z)| &= |m((h_z - h_z^{(n)})^\wedge E(z)^*)| \leq \|(h_z - h_z^{(n)})^\wedge\|_{q'}\|E(z)\|_q \\ &\leq \|h_z - h_z^{(n)}\|_q \to 0\end{aligned}$$

uniformly on compact sets [6, (43.11)], so it will follow that Φ is analytic. Now G will be analytic if the function $H(z) = m(B \int_0^\infty \lambda^{p'(1+z)/2} dE_\lambda)$ is analytic where $B = V^* \hat{\chi}_A$, and $\{E_\lambda\}$ is the spectral resolution of $|\hat{f}|$. But $H(z) = \int_0^\infty \lambda^{p'(1+z)/2} d\mu(\lambda)$ where μ is the measure on $[0, \infty)$ given by the function $\lambda \to m(BE_\lambda)$ of bounded variation. Thus H is analytic by a standard application of Fubini's theorem and Morera's theorem.

Now if $\alpha = 2/p - 1$ then $0 < \alpha < 1$, $h_\alpha = f$, $E(\alpha) = V|\hat{f}|^{p'/p}$ and thus

$$\Phi(\alpha) = m(\hat{f}|\hat{f}|^{p'/p} V^*) = m(V|\hat{f}|^{1+p'/p} V^*) = m(|\hat{f}|^{p'}) = 1.$$

Thus, by the maximum modulus theorem, $\Phi(z) \equiv 1$ on $0 \leq \text{Re } z \leq 1$.

Let g_z be the inverse transform of $E(z)$ [9, Theorem 7]. Then $g_z \in L^{q'}(G)$, $\|g_z\|_{q'} \leq \|E(z)\|_q = 1$ and by the Parseval formula [9, Lemma 7.2]

$$\langle h_z, g_z \rangle = \langle \hat{h}_z, E(z) \rangle = 1 \quad \text{for all } 0 \leq \text{Re } z \leq 1.$$

For $z = 1 + iv$, $q(z) = 1$, $g_{1+iv} \in A(G)$ [3] and $\|g_{1+iv}\|_\infty \leq \|g_{1+iv}\|_A = \|E(1 + iv)\|_1 = 1$. Thus

$$\begin{aligned}1 = \langle h_{1+iv}, g_{1+iv} \rangle &= \int |f|^{p(2+iv)/2}\text{sgn } f\overline{g_{1+iv}}\, dx \leq \int |f|^p |g_{1+iv}|\, dx \\ &\leq \|g_{1+iv}\|_\infty \|f\|_p^p \leq 1,\end{aligned}$$

so [7, (12.29)]

$$|f|^{p(2+iv)/2}\text{sgn } f\overline{g_{1+iv}} = \alpha_v |f|^p |\text{sgn } f|\, |g_{1+iv}| = \alpha_v |f|^p |g_{1+iv}| = \alpha_v |f|^p$$

a.e. for each real v where $|\alpha_v| = 1$. From this we infer that

$$|f|^p |f|^{piv/2}\text{sgn } f\overline{g_{1+iv}} = \alpha_v |f|^p$$

a.e. for each v; that f is supported a.e. for each v on $\{x \in G: |g_{1+iv}(x)| = 1\}$; that on the set where $f(x) \neq 0$, $|f|^{piv/2}\text{sgn } f = \alpha_v g_{1+iv}$ a.e. each v and that $\|g_{1+iv}\|_\infty$

$= \|g_{1+iv}\|_A = 1$. In particular $\operatorname{sgn} f = \alpha_0 g_1$ a.e. on the set where $f(x) \neq 0$ and f is supported a.e. on the set $H = \{x: |g_1(x)| = 1\}$. By translating f as the statement of the theorem allows we may suppose that g_1 is a constant (modulus 1) multiple of a positive definite function. Hence [6, (32.7)] H is a subgroup of G, $g_1 \mid H$ is a constant multiple of a character of H and since g_1 vanishes at infinity, H is compact, and hence open.

We claim next that by adjusting f on a null set we may assume f continuous and that everywhere on G one has $|f|^{piv/2}\operatorname{sgn} f = g_{1+iv}$ for all real v.

For this, observe first that because $\langle h_z, g_z \rangle = 1$ we have $g_z = |h_z|^{q-1}\operatorname{sgn} h_z$ a.e. for each z with $0 \leq \operatorname{Re} z < 1$ [7, (13.5)]. Also $|h_z| = |f|^{p(1+u)/2}$, $\operatorname{sgn} h_z = |f|^{piv/2}\operatorname{sgn} f$ so that $|h_z|^{q-1} = |f|^{p/q'}$ and $g_z = |f|^{p/q'}|f|^{piv/2}\operatorname{sgn} f$ a.e. for each z with $0 \leq \operatorname{Re} z < 1$. Let λ_H be the left regular representation of H, Let $\mathfrak{L}(H)$ be the von Neumann algebra it generates on $L^2(H)$ and let $\mathfrak{N}$ be the von Neumann algebra generated on $L^2(G)$ by $\{\lambda(s): s \in H\}$. Since H is open, there is an isomorphism α of $\mathfrak{L}(H)$ onto $\mathfrak{N}$ which carries $\lambda_H(s)$ onto $(\lambda \mid H)(s)$ whenever $s \in H$, where the vertical bar denotes restriction. One checks that also $\alpha(\lambda_H(h)) = (\lambda \mid H)(h)$ whenever $h \in L^1(H)$. Now observe that if $k \in L^1(G)$ and k is supported on H then $\lambda(k) = (\lambda \mid H)(k \mid H)$, and consequently for such k, $\alpha(\lambda_H(k \mid H)) = (\lambda \mid H)(k \mid H) = \lambda(k)$. From this follows the crucial observation that f, h_z and g_z, with the exception possibly of g_{1+iv}, all have their transforms in $\mathfrak{N}$. Now making use of the compactness of H we have that $\mathfrak{N}$ is isomorphic to $\prod_{\sigma \in \hat{H}} \mathfrak{B}(\mathfrak{K}_\sigma)$ [9, Theorem 8] where $\mathfrak{K}_\sigma$ is finite dimensional for each unitary equivalence class σ of irreducible representations of H. It follows easily that $m \mid \mathfrak{N} = \sum_{\sigma \in \hat{H}} d_\sigma \operatorname{tr}(\cdot)$ for positive numbers d_σ where $\operatorname{tr}(\cdot)$ denotes the trace on $\mathfrak{B}(\mathfrak{K}_\sigma)$. Since $\hat{f}$ belongs to $\mathfrak{N}$, so does V and it follows that, writing $T = (T_\sigma)$ for operators T in $\mathfrak{N}$, one has

$$\begin{aligned} 1 = m(\hat{h}_1 E(1)^*) &= \sum_{\sigma \in \hat{H}} d_\sigma \operatorname{tr}((\hat{h}_1)_\sigma E(1)_\sigma^*) \leq \sum_{\sigma \in \hat{H}} d_\sigma \|(\hat{h}_1)_\sigma\|_\infty \|E(1)_\sigma\|_1 \\ &\leq \|\hat{h}_1\|_\infty \sum_{\sigma \in \hat{H}} d_\sigma \|E(1)_\sigma\|_1 = \|\hat{h}_1\|_\infty \|E(1)\|_1 \leq 1. \end{aligned}$$

Thus if $E(1)_\sigma \neq 0$ for a certain σ then $\|(\hat{h}_1)_\sigma\|_\infty = \|\hat{h}_1\|_\infty$ for that σ. But $\{\sigma: \|(\hat{h}_1)_\sigma\| \geq \varepsilon\}$ is finite for every $\varepsilon > 0$ [6, (28.40)]. Thus $E(1)$, $\hat{f}$ and therefore $E(z)$ are all operators of finite rank in $\mathfrak{L}$ and it follows easily that

$$\|E(1 + iv) - E(u + iv)\|_1 \to 0$$

as $u \uparrow 1$. Thus

$$\|g_{1+iv} - g_{u+iv}\|_\infty \leq \|g_{1+iv} - g_{u+iv}\|_A = \|E(1 + iv) - E(u + iv)\|_1 \to 0.$$

For a fixed v taking a subsequence $u_j \uparrow 1$ we have

$$g_{1+iv} = \lim_{j\to\infty} g_{u_j+iv} = \lim_{j\to\infty} |f|^{p(1-u_j)/2}|f|^{piv/2}\operatorname{sgn} f = |f|^{piv/2}\operatorname{sgn} f.$$

This holds a.e. for each v where we have discarded countably many null sets. In particular $g_1 = \operatorname{sgn} f$ a.e. Moreover $f \mid H$ is a trigonometric polynomial on H almost everywhere by [6, (28.39)(ii)]. Thus f is continuous a.e. on G and it follows that, assuming f continuous everywhere, both $h_z = |f|^{p(1+z)/2}\operatorname{sgn} f$ and thus $|h_z|^{q-1}\operatorname{sgn} h_z$ are continuous everywhere. But $g_z = |h_z|^{q-1}\operatorname{sgn} h_z$ a.e. and g_z is continuous being in $A(G)$. It follows that $g_z = |h_z|^{q-1}\operatorname{sgn} h_z$ everywhere and this establishes the claim.

The proof of the theorem will be completed by showing that $|f|$ is constant a.e. on H. To see this, multiply f by a scalar to get $f(e) > 0$ and let $k_v = f(e)^{-piv/2} g_{1+iv}$. Then $k_v(e) = 1$ and $\|k_v\|_\infty = \|k_v\|_A = k_v(e)$ so that k_v is a positive definite function. Now $\hat{k}_v = f(e)^{-piv/2}E(1+iv) = f(e)^{-piv/2}V|\hat{f}|^{p'(2-iv)/2}$. But $\hat{k}_v$ is positive in $L^1(\Gamma)$ so $\hat{k}_v = |\hat{k}_v| = |\hat{f}|^{p'}$. Thus $\hat{k}_v = \hat{k}_0$ for all v and $\operatorname{sgn} f = k_0 = k_v = f(e)^{-piv/2}|f|^{piv/2}\operatorname{sgn} f$. So on H, $|f|^{is}$ is constant depending on real s. The proof is complete.

Remark. vs in [6, §43] it is possible to consider L^p-maximal functions for $2 < p < \infty$. Namely $f \in L^p(G)$, $p > 2$, is L^p-maximal if f also belongs to $L^r(G)$ for some r, $1 \le r \le 2$, if $\hat{f}$ (= the L^r-Fourier transform of f) belongs to $L^{p'}(\Gamma)$ and $\|\hat{f}\|_{p'} = \|f\|_p$. Using Theorem 1 it is easy to show that the L^p-maximal functions, $2 < p < \infty$, are precisely the same as the L^p-maximal functions, $1 < p < 2$, i.e. constant multiples of translates of subcharacters. To see this observe that $\hat{f} \in L^{p'}(\Gamma) \cap L^{r'}(\Gamma)$, and since $p' \le 2 \le r'$, $\hat{f} \in L^2(\Gamma)$. Thus f is the inverse transform of $\hat{f}$. Let $g = |f|^{p/p'}\operatorname{sgn} f$. We claim g is $L^{p'}$-maximal. Indeed

$$\langle f,g\rangle = \int f|f|^{p/p'}\overline{\operatorname{sgn} f}\,dx = \int |f|^{1+p/p'}\,dx = \int |f|^p\,dx = 1$$

without loss of generality. Thus

$$1 = \langle f,g\rangle = \langle \hat{f},\hat{g}\rangle \le \|\hat{f}\|_{p'}\|\hat{g}\|_p \le \|g\|_{p'} = \|f\|_p^{p/p'} = 1$$

and so g is $L^{p'}$-maximal. Now since $1 < p' < 2$, Theorem 1 implies that $|f|^{p/p'}\operatorname{sgn} f$ is a.e. a translate of a constant multiple of a subcharacter and so the same holds for f.

3. Direct products.

Theorem 2. *If H is an arbitrary unimodular locally compact group then* $\|\mathfrak{T}_p(\mathbf{R} \times H)\| < 1$ *for all* p, $1 < p < 2$.

Proof. In the proof an equal sign is sometimes used to denote unitary equivalence. If G is a direct product $\mathbf{R} \times H$ then $\lambda_G(s,x) = \lambda_{\mathbf{R}}(s) \otimes \lambda_H(x)$, $s \in \mathbf{R}$, $x \in H$. By Stone's theorem in direct integral form $\lambda_{\mathbf{R}} = \int_{\mathbf{R}}^{\oplus} \chi_t\,dt$, i.e. $\lambda_{\mathbf{R}}(s) = \int_{\mathbf{R}}^{\oplus} e^{its}\,dt$ where dt is Lebesgue measure divided by $(2\pi)^{1/2}$ and $\chi_t(s) = e^{its}$ is the operator of multiplication by e^{its} on a one-dimensional Hilbert space. Identifying $\mathfrak{L}(\mathbf{R})$ with $L^\infty(\mathbf{R})$ we can write $\mathfrak{L}(\mathbf{R}) = \int_{\mathbf{R}}^{\oplus} \mathcal{C}_t\,dt$ where $\mathcal{C}_t$ is the complex numbers for each t. Thus

$$\mathfrak{L}(G) = \mathfrak{L}(\mathbf{R}) \otimes \mathfrak{L}(H) = \int_{\mathbf{R}}^{\oplus} (\mathcal{C}_t \otimes \mathfrak{L}(H))\,dt$$

and

$$\lambda_G(s,x) = \left(\int_{\mathbf{R}}^{\oplus} e^{its}\,dt\right) \otimes \lambda_H(x) = \int_{\mathbf{R}}^{\oplus} (e^{its} \otimes \lambda_H(x))\,dt$$

(cf. [2, Chapter II]).

It follows from Fubini's theorem that $\hat{f} = \int_{\mathbf{R}}^{\oplus} \hat{f}(\chi_t \otimes \lambda_H)\,dt$ for any continuous function f on G with compact support. A routine calculation shows that $\hat{f}(\chi_t \otimes \lambda_H) = \hat{g}_t$ where $g_t(x) = [f(\cdot,x)]^{\hat{}}(-t)$, $x \in H$, $t \in \mathbf{R}$. By [2, p. 211] the gage m on G has the form $m = \int_{\mathbf{R}}^{\oplus} \varphi_t\,dt$ where φ_t is a faithful, normal semifinite trace on $\mathcal{C}_t \otimes \mathfrak{L}(H)$. If we identify $\mathcal{C}_t \otimes \mathfrak{L}(H)$ with $\mathfrak{L}(H)$ then φ_t is almost everywhere the canonical gage m_H on H. To see this, let $F \in L^\infty(\mathbf{R})$ and $T \in \mathfrak{L}(H)$ be positive. Then $m(F \otimes T) = \int_{\mathbf{R}} \varphi_t(F(t)T)\,dt = \int_{\mathbf{R}} F(t)\varphi_t(T)\,dt$. But m is the product gage $m_{\mathbf{R}} \times m_H$ [14, §9] so $m(F \otimes T) = m_{\mathbf{R}}(F) \cdot m_H(T)$. Thus

$$\int_{\mathbf{R}} F(t)[m_H(T) - \varphi_t(T)]\,dt = 0$$

for all $F \in L^\infty(\mathbf{R})$, so $\varphi_t = m_H$ a.e. Now

$$\begin{aligned}
\|\hat{f}\|_{p'}^{p'} &= m(|\hat{f}|^{p'}) = \int_{\mathbf{R}} \varphi_t(|\hat{g}_t|^{p'})\,dt = \int_{\mathbf{R}} m_H(|\hat{g}_t|^{p'})\,dt \\
&= \int_{\mathbf{R}} \|\hat{g}_t\|_{p'}^{p'}\,dt \le \int_{\mathbf{R}} \|g_t\|_{p}^{p'}\,dt = \int_{\mathbf{R}} \left[\int_H |[f(\cdot,x)]^{\hat{}}(-t)|^p\,dx\right]^{p'/p} dt \\
&\le \left[\int_H \left[\int_{\mathbf{R}} |[f(\cdot,x)]^{\hat{}}(-t)|^{p'}\,dt\right]^{p/p'} dx\right]^{p'/p} \\
&= \left[\int_H \|[f(\cdot,x)]^{\hat{}}\|_{p'}^{p}\,dx\right]^{p'/p} \le A_p^{p'}\left[\int_H \|f(\cdot,x)\|_p^p\,dx\right]^{p'/p} \\
&= A_p^{p'}\left[\int_H \left[\int_{\mathbf{R}} |f(s,x)|^p\,ds\right] dx\right]^{p'/p} = A_p^{p'}\|f\|_p^{p'},
\end{aligned}$$

where we have used Minkowski's integral inequality [13, p. 271] and A_p denotes $\|\mathfrak{F}_p(\mathbf{R})\|$. The proof is complete.

Corollary. *Let G be a central topological group, i.e. G/Z is compact where Z is the center of G (cf.* [4]*). The following statements are equivalent*:

(1) *G has no compact open subgroups*;

(2) $\|\mathfrak{F}_p(G)\| < 1$ *for all p*, $1 < p < 2$;

(3) $\|\mathfrak{F}_p(G)\| < 1$ *for some p*, $1 < p < 2$.

The corollary is a simple consequence of the structure theorem for central topological groups [4, Theorem 4.4] and the log convexity of the function $p \to \|\mathfrak{F}_p(G)\|$, valid for any unimodular group [9, Corollary 3.1].

The idea for the proof of Theorem 2 came from a consideration of the Abelian case which is considerably more elementary.

The author wishes to thank Masamichi Takesaki for indicating how to drop assumptions of separability and type I in a previous version of Theorem 2.

4. The Hausdorff Young theorem for integral operators. A plausible conjecture for a locally compact unimodular group G is: G has no compact open subgroups if and only if $\|\mathfrak{T}_p(G)\| < 1$ for some (hence all) p, $1 < p < 2$.

In §3 we established this conjecture for central topological groups. In the next section the conjecture is established for any locally compact group G which is a semidirect product $A \times_\varphi X$ of an Abelian group A and a compact group X (acting on A), e.g. the groups of rigid motions of Euclidean space.

As in the proof of Theorem 2 use will be made of Babenko's theorem (i.e. that $\|\mathfrak{T}_p(\mathbf{R})\| < 1$) and elementary direct integral decompositions. However, a new element is needed, namely a Hausdorff Young theorem for integral operators, which we state as a separate theorem because of its independent interest. I am indebted to E. M. Stein for essentially stating this theorem and for the reference [1].

Let X and Y be σ-finite measure spaces with measures denoted by dx and dy. For a square summable function k on $X \times Y$ we consider the integral operator K: $L^2(X) \to L^2(Y)$ defined (a.e.) by $Kf(y) = \int_X k(x,y)f(x)\,dx$; and the norms

$$\|k\|_{p,q} = \left(\int_Y \left(\int_X |k(x,y)|^p\,dx\right)^{q/p} dy\right)^{1/q}, \qquad 1 \le p, q < \infty.$$

We note that $\|K\|_r = (\operatorname{Tr}(K^*K)^{r/2})^{1/r}$ is well defined for $1 \le r < \infty$ (possibly $+\infty$), and we let k^*: $Y \times X \to \mathbf{C}$ be defined by $k^*(y,x) = \overline{k(x,y)}$.

Theorem 3. *Let* $1 < p < 2$, $p' = p/(p-1)$ *and let* $k \in L^2(X \times Y)$. *If* K *is the integral operator with kernel* k *then*

$$\|K\|_{p'} \le (\|k\|_{p,p'} \cdot \|k^*\|_{p,p'})^{1/2}.$$

Proof. By a density argument which is outlined below it is sufficient to establish the theorem for simple functions k of the form $k = \sum \alpha_i \chi_{A_i \times B_i}$ where $\{A_i\}$ (resp. $\{B_i\}$) is a finite disjoint family of measurable subsets of X (resp. Y) of finite measure. For notation's sake let $f_i = \chi_{A_i}$, $g_i = \chi_{B_i}$ and let $|A|$ denote the measure of a set A. Then $K^*K = \sum |\alpha_i|^2 \|f_i\|_2^2 \|g_i\|_2^2 P_i$ where $\{P_i\}$ is a finite family of mutually orthogonal one-dimensional projections on $L^2(X)$. It follows that

$$\|K\|_{p'} = (\sum (|\alpha_i|\,\|f_i\|_2 \|g_i\|_2)^{p'})^{1/p'} = (\sum |\alpha_i|^{p'} |A_i|^{p'/2} |B_i|^{p'/2})^{1/p'}.$$

But

$$\|k\|_{p,p'} = (\sum |\alpha_i|^{p'} |A_i|^{p'/p} |B_i|)^{1/p'}$$

and

$$\|k^*\|_{p,p'} = \left(\sum |\alpha_i|^{p'} |A_i|\, |B_i|^{p'/p}\right)^{1/p'}.$$

Schwarz's inequality now gives the result if we notice that

$$(|\alpha_i|^{p'/2} |A_i|^{p'/2p} |B_i|^{1/2})(|\alpha_i|^{p'/2} |A_i|^{1/2} |B_i|^{p'/2p}) = |\alpha_i|^{p'} |A_i|^{p'/2} |B_i|^{p'/2}$$

because $p'/p + 1 = p'$.

Suppose now that $k \in L^2(X \times Y)$ and that $\|k\|_{p,p'}$ and $\|k^*\|_{p,p'}$ are both finite. The function k belongs to the Banach spaces determined by finiteness of the norms $\|k\|_{p,p'}$, $\|k^*\|_{p,p'}$, $\|k\|_{2,2}$. There is a simple function s on $X \times Y$ such that $\|k - s\|$ is small in all three norms. Here we have used a bounded convergence theorem for the spaces $L^{p,q}$ with mixed norm [1, p. 302]. Write $s = \sum \alpha_i \chi_{E_i}$ with $\{E_i\}$ a mutually disjoint family of measurable subsets of $X \times Y$. For each E_i choose a measurable set F_i which is a disjoint union of measurable rectangles with the measure of $E_i \Delta F_i$ small. (3) If we let $t = \sum \alpha_i \chi_{F_i}$ then $\|s - t\|$ is small in each norm. Here we may assume that $\{F_j\}$ is a disjoint family. Therefore $\|k - t\|$ is small in each norm, say less than ε, and t is a simple function of the type considered in the first part of the proof. If T denotes the integral operator with kernel t we have $\|K - T\|_{p'} \le \|K - T\|_2 = \|k - t\|_2 < \varepsilon$ and thus

$$\begin{aligned} \|K\|_{p'} &\le \varepsilon + \|T\|_{p'} \le \varepsilon + (\|t\|_{p,p'} \cdot \|t^*\|_{p,p'})^{1/2} \\ &\le \varepsilon + (\|k\|_{p,p'} + \varepsilon)^{1/2} (\|k^*\|_{p,p'} + \varepsilon)^{1/2}. \end{aligned}$$

This completes the proof.

Remarks. 1. The cases $p = 1$ and $p = 2$ of Theorem 3 are well-known results and we expected Theorem 1 to follow by interpolation.

2. Equality holds in Theorem 3 for $k = \chi_{A \times B}$, i.e. the result is sharp.

3. If X and Y are discrete with the same mass at each point, say a for X and b for Y, then $\|K\|_{p'} \le (ab)^{1/2 - 1/p} \|k\|_p$ holds. This can also be shown by interpolation. Conversely this inequality for arbitrary X and Y easily implies that the measures of nonnull sets are bounded away from zero.

4. If G is a locally compact group and $\varphi \in L^1(G)$ then $L_\varphi \colon g \to \varphi * g$ is an integral operator on $L^2(G)$ with kernel $k(x,y) = \varphi(xy^{-1})$. In case G is compact Theorem 3 yields an elementary proof (modulo the Peter Weyl theorem) of the Hausdorff Young theorem for compact groups ([6, (31.22)], [9]).

5. **L^p-Fourier transforms on semidirect products.** It is possible to avoid separability assumptions and induced representations by employing the following device which is due to Godement.

(3) This result is well known to probabilists but I lack the reference.

Lemma 1 [5]. *Let G be a locally compact group, A an Abelian closed subgroup of G. For each character χ in the dual $\hat{A}$ of A there is a representation U^χ of G on a Hilbert space H_χ. We have $\lambda = \int_{\hat{A}}^{\oplus} U^\chi d\chi$, where λ denotes the left regular representation of G, $(\lambda(s)f)(t) = f(s^{-1}t), f \in L^2(G), s, t \in G$.*

Proof. We sketch Godement's argument since the construction will be needed. Let da and $d\chi$ denote normalized Haar measures on A and $\hat{A}$. For $f, g \in \mathcal{K}(G)$ (= continuous functions with compact support) and $\chi \in \hat{A}$ let $\varphi_{f,g}(\chi) = \int_A (\rho(a)f \mid g)\overline{\chi(a)}\,da$, where ρ is the right regular representation of G, $\rho(s)f(t) = f(ts)$, $f \in L^2(G)$, $s, t \in G$. By Fourier inversion $(\rho(a)f \mid g) = \int_{\hat{A}} \varphi_{f,g}(\chi)\chi(a)\,d\chi$. The Hilbert space H_χ is the completion of $\mathcal{K}(G)/N_\chi$ with inner product $(f_\chi \mid g_\chi) = \varphi_{f,g}(\chi)$ ($f_\chi =$ equivalence class of f and $N_\chi = \{f \in \mathcal{K}(G): \varphi_{f,f}(\chi) = 0\}$). For $s \in G$, let $U_s^\chi f_\chi = (\lambda(s)f)_\chi$. The above Fourier inversion formula yields

$$(\rho(a)\lambda(s)f \mid g) = \int_{\hat{A}} (U_s^\chi f_\chi \mid g_\chi)\chi(a)\,d\chi$$

which proves the lemma.

Let now G be a (topological) semidirect product $A \times_\varphi X$ of an additively written Abelian locally compact group A and a locally compact unimodular group X (acting on A). The product in G will be denoted by $(a,x)\cdot(b,y) = (a + x(b), xy)$, where $x(b)$ denotes the action of the automorphism $x \in X$ on $b \in A$. G is unimodular with Haar measure $ds = da \cdot dx$, $s = (a,x)$.

Lemma 2. *Let G be a semidirect product $A \times_\varphi X$ of an Abelian locally compact group A and a locally compact unimodular group X acting on A by measure-preserving automorphisms. For $\chi \in \hat{A}$ the map*

$$W_\chi h(x) = \int_A h(a,x)\chi(x^{-1}(-a))\,da \qquad (h \in \mathcal{K}(G))$$

sets up a unitary equivalence of H_χ with $L^2(X)$ which for $f \in \mathcal{K}(G)$ transports U_f^χ into an integral operator on $L^2(X)$ with kernel $k_\chi(x,y) = f(y(\cdot), yx^{-1})\hat{\ }(\chi)$ (i.e. $k_\chi(x,y)$ is the Fourier transform of the function $a \to f(y(a), yx^{-1})$ evaluated at the character χ).

Proof.

$$|W_\chi h(x)|^2 = \int_A \int_B h(a,x)\overline{h(b,x)}\chi(x^{-1}(b-a))\,db\,da$$

$$\overset{b \to a + x(b)}{=} \int_A \int_B h(a,x)\overline{h(a + x(b),x)}\chi(b)\,db\,da$$

so that

$$\|W_\chi h\|^2 = \int_X |W_\chi h(x)|^2\,dx = \int_X \int_A \int_B h(a,x)\overline{h(a + x(b),x)}\chi(b)\,db\,da\,dx.$$

On the other hand

$$\begin{aligned}\|h_\chi\|^2 &= \varphi_{h,h}(\chi) = \int_B (\rho(b)h \mid h)\overline{\chi(b)}\,db\\ &= \int_B\int_A\int_X h((a,x)(b,e))\overline{h(a,x)\chi(b)}\,da\,dx\,db\\ &= \int_X\int_A\int_B h(a+x(b),x)\overline{h(a,x)\chi(b)}\,db\,da\,dx\\ &= \|W_\chi h\|^2.\end{aligned}$$

Thus W_χ maps $\mathcal{K}(G)/N_\chi$ isometrically into $L^2(X)$ and it is trivial that the range of W_χ is dense in $L^2(X)$. Let $W = W_\chi$ denote the unitary operator thus defined on H_χ onto $L^2(X)$. For $s = (b,x) \in G$ let $V_s^\chi = WU_s^\chi W^{-1}$. Let $g \in L^2(X)$ be such that $h_\chi = Wg^{-1} \in \mathcal{K}(G)/N_\chi$. Then

$$V_s^\chi g = WU_s^\chi W^{-1}g = WU_s^\chi h_\chi = W((\lambda(s)h)_\chi)$$

so

$$\begin{aligned}V_s^\chi g(y) &= \int_A (\lambda(b,x)h)(a,y)\chi(y^{-1}(-a))\,da\\ &= \int_A h(x^{-1}(a-b),x^{-1}y)\chi(y^{-1}(-a))\,da.\end{aligned}$$

Hence

$$\begin{aligned}(V_f^\chi g \mid g) &= \int_G f(s)(V_s^\chi g \mid g)\,ds\\ &= \int_B\int_X f(b,x)\int_Y\int_A h(x^{-1}(a-b),x^{-1}y)\chi(y^{-1}(-a))\,da\,\overline{g(y)}\,dy\,dx\,db\end{aligned}$$

and thus

$$\begin{aligned}V_f^\chi g(y) &= \int_B\int_X f(b,x)\int_A h(x^{-1}(a-b),x^{-1}y)\chi(y^{-1}(-a))\,da\,dx\,db\\ &\overset{a\to x(a)+b}{=} \int_B\int_X f(b,x)\int_A h(a,x^{-1}y)\chi(y^{-1}x(-a))\chi(y^{-1}(-b))\,da\,dx\,db\\ &\overset{x\to yx^{-1}}{=} \int_B\int_X f(b,yx^{-1})\int_A h(a,x)\chi(x^{-1}(-a))\chi(y^{-1}(-b))\,da\,dx\,db\\ &\overset{b\to y(b)}{=} \int_X\int_B f(y(b),yx^{-1})\int_A h(a,x)\chi(x^{-1}(-a))\,da\,\overline{\chi(b)}\,db\,dx\\ &= \int_X\int_B f(y(b),yx^{-1})g(x)\overline{\chi(b)}\,db\,dx\\ &= \int_X f(y(\cdot),yx^{-1})^\wedge(\chi)g(x)\,dx.\end{aligned}$$

Theorem 4. *Let G be a locally compact group which is a semidirect product of an Abelian locally compact group A and a compact group X acting on A. Then G has* *no compact open subgroups if and only if* $\|\mathfrak{T}_p(G)\| < 1$ *for some (hence all)* p, $1 < p < 2$.

We note first that G is unimodular and that X, being compact, acts as measure-preserving automorphisms of A.

Proof. It is sufficient to prove that $\|\mathfrak{T}_p(G)\| \leq \|\mathfrak{T}_p(A)\|$ for a single value of p, $1 < p < 2$. For if G had no compact open subgroups, neither would A since X is compact so that $\|\mathfrak{T}_p(A)\| < 1$ by the corollary to Theorem 2. The converse is contained in Theorem 1.

We fix on the value $p = 4/3$ so that $p' = 4$. If $f \in \mathcal{K}(G)$, then, from Lemma 2, $\hat{f} = L_f = \int_{\hat{A}}^{\oplus} U_f^{\chi}\, d\chi$ where U_f^{χ} can be taken to be an integral operator on $L^2(X)$ with kernel $k_{\chi}(x,y) = f(y(\cdot),yx^{-1})\hat{\ }(\chi)$. This entails $\|U_f^{\chi}\|_2^2 = \|k_{\chi}\|_2^2$ so that

$$\int_{\hat{A}} \|U_f^{\chi}\|_2^2\, d\chi = \int_{\hat{A}}\int_X\int_Y |f(y(\cdot),yx^{-1})\hat{\ }(\chi)|^2\, dx\, dy\, d\chi$$
$$= \int_X\int_Y \|f(y(\cdot),yx^{-1})\hat{\ }\|_2^2\, dx\, dy = \|f\|_2^2$$

since the Haar measure on X is normalized in the usual way to have total mass one.

For notation's sake let $g = f^* * f$, $h = g^* * g$. Then

$$\|L_f\|_4^4 = m(|L_f|^4) = m(L_g^2) = m(L_h) = h(e) = \|g\|_2^2$$
$$= \int_{\hat{A}} \|U_g^{\chi}\|_2^2\, d\chi = \int_{\hat{A}} \|U_f^{\chi}\|_4^4\, d\chi \leq \int_{\hat{A}} (\|k_{\chi}\|_{p,p'} \cdot \|k_{\chi}^*\|_{p,p'})^{p'/2}\, d\chi$$
$$\leq \left(\int_{\hat{A}} \|k_{\chi}\|_{p,p'}^{p'}\, d\chi\right)^{1/2}\left(\int_{\hat{A}} \|k_{\chi}^*\|_{p,p'}^{p'}\, d\chi\right)^{1/2}.$$

But

$$\int_{\hat{A}} \|k_{\chi}\|_{p,p'}^{p'}\, d\chi = \int_{\hat{A}}\int_Y \left(\int_X |f(y(\cdot),xy^{-1})\hat{\ }(\chi)|^p\, dx\right)^{p'/p} dy\, d\chi$$
$$\leq \int_Y \left(\int_X \left(\int_{\hat{A}} |f(y(\cdot),xy^{-1})\hat{\ }(\chi)|^{p'}\, d\chi\right)^{p/p'} dx\right)^{p'/p} dy$$
$$= \int_Y \left(\int_X \|f(y(\cdot),xy^{-1})\hat{\ }\|_{p'}^p\, dx\right)^{p'/p} dy$$
$$\leq \int_Y \left(\int_X \|\mathfrak{T}_p(A)\|^p \|f(y(\cdot),xy^{-1})\|_p^p\, dx\right)^{p'/p} dy = \|\mathfrak{T}_p(A)\|^{p'}\|f\|_p^{p'}.$$

Similarly $\int_{\hat{A}} \|k_{\chi}^*\|_{p,p'}^{p'}\, d\chi \leq \|\mathfrak{T}_p(A)\|^{p'}\|f\|_p^{p'}$ and so $\|L_f\|_{p'} \leq \|\mathfrak{T}_p(A)\|\,\|f\|_p$ with $p = 4/3$. This completes the proof.

A slightly simpler and more transparent proof of Theorem 4 can be given if one assumes separability of G and uses the language of induced representations for the separable case as follows: If λ is the left regular representation of G then by inducing in stages, $\lambda = U^{\lambda_A}$ where λ_A is the regular representation of A. By Stone's theorem $\lambda_A = \int_{\hat{A}}^{\oplus} \chi\, d\chi$ and since inducing commutes with direct integration, $\lambda = \int_{\hat{A}}^{\oplus} U^{\chi} d\chi$. By Fubini, $L_f = \int_{\hat{A}}^{\oplus} U_f^{\chi} d\chi$ for $f \in \mathcal{K}(G)$. Next the Hilbert space $\mathcal{H}(U^{\chi})$ of the induced representation U^{χ} is mapped onto $L^2(X)$ by $Wg(x) = g(e,x)$, $g \in \mathcal{H}(U^{\chi})$ and a computation, not unlike that for Lemma 2, yields U_f^{χ} as an integral operator with an appropriate kernel.

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