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SOME THIN SETS IN DISCRETE ABELIAN GROUPS(1)

BY

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ABSTRACT. Let Γ be a discrete abelian group, and $E \subset \Gamma$. For $F \subset E$, we say that $F \in \mathcal{P}(E)$, if for all Λ, finite subsets of Γ, $0 \notin \Lambda, \Lambda + F \cap F$ is finite. Having defined the Banach algebra, $\tilde{A}(E) = c(E) \cap B(E)$, we prove the following: (i) $E \subset \Gamma$ is a Sidon set if and only if every $F \in \mathcal{P}(E)$ is a Sidon set; (ii) $E \in \mathcal{P}(\Gamma)$ is a Sidon set if and only if $\tilde{A}(E) = A(E)$.

0. Introduction. Let Γ denote a discrete abelian group, and let G denote its dual. We let G_d denote the abelian group G endowed with the discrete topology; $(G_d)\hat{} = \tilde{\Gamma}$ is the Bohr compactification of Γ. Γ is dense in $\tilde{\Gamma}$, and $C(\tilde{\Gamma})$ is identified naturally with the almost periodic functions on Γ. In what follows below, we use standard notation and facts as presented in Chapters 1 and 2 of [9].

For $E \subset \Gamma$, set

$$A(E) = L^1(G)\hat{}/\{f \in L^1(G): \hat{f} = 0 \text{ on } E\}$$

and

$$B(E) = M(G)\hat{}/\{\mu \in M(G): \hat{\mu} = 0 \text{ on } E\},$$

where the quotients are the usual Banach algebra quotients. Let $c(E)$ denote all bounded functions on E which vanish at infinity. Clearly, $A(E) \subset c(E)$, and $B(E) \subset l^\infty(E)$; $A(E)$ is norm dense in $c(E)$, and $\|g\|_{B(E)} \geq \|g\|_\infty$ for all $g \in B(E)$. We set

$$\tilde{A}(E) = c(E) \cap B(E).$$

Equipped with the $B(E)$-norm, and pointwise multiplication $\tilde{A}(E)$ is a Banach algebra. Since $(C_E(G))^* = B(E)$, and $(A(E))^* = L^\infty_E(G)$, $A(E)$ is isometrically imbedded in $B(E)$ and, therefore, in $\tilde{A}(E)$. (If $R(G)$ is any subspace of $L^1(G)$, $R_E(G)$ denotes all functions in $R(G)$ whose spectrum is in E.) Detailed studies of algebras related to $\tilde{A}(E)$ appear in [5], [7], [8], [11], and [12].

When $A(E) = c(E)$, we say that E is a Sidon set, and define the Sidon con-

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stant of E to be

$$b(E) = \inf\{\|f\|_\infty/\|f\|_{A(E)} : f \in A(E),\ f \neq 0\}.$$

The following are easily seen to be equivalent: (i) $E \subset \Gamma$ is a Sidon set, (ii) $A_E(G) = C_E(G)$, (iii) $b(E) > 0$, (iv) $\tilde{A}(E) = c(E)$. A natural question, the answer to which is still not known, is the following: Can we have a non-Sidon set $E \subset \Gamma$, and $\tilde{A}(E) = A(E)$? If E is an element of the coset ring of Γ, then $\tilde{A}(E) \supsetneqq A(E)$ (e.g., cf. [10]). Our main result, for whose proof we extend and generalize methods in [1] and [7], deals with the case where E is a "thin" set (cf. [2]):

Definition 1. Let $E \subset \Gamma$. We say that the pace of E tends to infinity, if given any finite set $\Lambda \subset \Gamma$, and $0 \notin \Lambda$, then $\Lambda + E \cap E$ is finite. We set

$$\mathcal{P}(E) = \{S \subset E : \text{pace of } S \to \infty\}.$$

Theorem A. *Let Γ be a discrete abelian group.* (i) *If G is metrizable, then E is a Sidon set if and only if every $F \in \mathcal{P}(E)$ is a Sidon set.* (ii) *If $E \in \mathcal{P}(\Gamma)$ is a non-Sidon set, then $\tilde{A}(E) \supsetneqq A(E)$.*

In §1 we collect the necessary tools, and in §2 we prove Theorem A. We then deduce as a corollary (2.3) a generalization of a result of Katznelson and McGehee (cf. [7, Theorem 3.1]). In §3 we list open questions.

1. Some lemmas.

Definition 1.1. Let Γ be a discrete abelian group. $F \subset \Gamma$ is said to be partitioned with respect to the supremum norm if there exists $\{F_j\}$, a family of finite, mutually disjoint sets, such that $\bigcup F_j = F$, and

$$\bigoplus_{l^1} C_{F_j}(G) \approx C_F(G).$$

We prove in [1] that every non-Sidon set $E \subset \Gamma$ contains a non-Sidon set F which can be partitioned with respect to the sup norm.

Lemma 1.2. *Let $E \subset \Gamma$ be a non-Sidon set. Then, there exists a non-Sidon set $F \subset E$, so that $\tilde{A}(F) \supsetneqq A(F)$.*

Proof. Let $E \subset \Gamma$ be a non-Sidon set. Let $F \subset E$ be a non-Sidon set so that

$$\bigoplus_{l^1} C_{F_i}(G) \approx C_F(G),$$

where $\{F_j\}$ is as above. But, then

$$B(F) \approx \bigoplus_{l^\infty} B(F_j). \tag{1}$$

Since F is non-Sidon, $\underline{\lim}\, h(F_j) = 0$. Let $f_j \in B(F_j)$, so that $\overline{\lim}\, \|f_j\|_{B(F_j)} = 1$, and $\|f_j\|_\infty < 1/j$. Define $f \in c(F)$: $f|_{F_j} = f_j$. By (1),

$$f \in B(F) \cap c(F) = \tilde{A}(F).$$

But a necessary condition for $f \in A(F)$ is $\lim_{j\to\infty} \|f\|_{B(F_j)} = 0$ (cf. [9, 2.6.3] or [3]). □

Let Γ be a countable group. Then, $\hat{\Gamma} = G$ is a compact metrizable group, and, therefore, there exists D, a countable dense subgroup of G. Consider D as a discrete abelian group, and let $\phi : \Gamma \to \hat{D}$ be the natural injective map: $(\phi(y), d) = (y, d)$, for $y \in \Gamma$ and $d \in D$. We shall say that $F \subset \hat{D}$ is a Sidon set if F is a Sidon set in $(\hat{D})_d$.

Lemma 1.3. *Let Γ, D, ϕ be as above. Then, $E \subset \Gamma$ is Sidon if and only if $\phi(E)$ is Sidon. Furthermore $h(E) = h(\phi(E))$.*

Proof. See Lemma 2.2 of [1].

Lemma 1.4. *Let $E \subset \Gamma$ be a non-Sidon set. Then, there exists $x_0 \in \overline{\phi(E)}$ = closure of $\phi(E)$ in $\hat{D}$, so that if U is any neighborhood of x_0, then $\phi(E) \cap U$ is a non-Sidon set.*

Proof. Suppose that for each $x \in \overline{\phi(E)}$ there exists U_x, an open neighborhood of x, so that $\phi^{-1}(U_x \cap E)$ is a Sidon set. By the compactness of $\overline{\phi(E)}$, there exist $x_1, \cdots, x_n \in \overline{\phi(E)}$, so that $\bigcup_{i=1}^n U_{x_i} \supset \overline{\phi(E)}$. But, by Drury's theorem (cf. [4]), $E = \bigcup_{i=1}^n \phi^{-1}(U_{x_i} \cap E)$ is a Sidon set, and we reach a contradiction. □

Lemma 1.5. *$\tilde{A}(E) = A(E)$ if and only if $\tilde{A}(\phi(E)) = A(\phi(E))$. ($\tilde{A}(\phi(E))$ and $A(\phi(E))$ are defined with respect to $M(\tilde{D})$ and $L^1(\tilde{D})$, respectively.)*

Proof. Finitely supported functions on E ($\phi(E)$) are norm-dense in $A(E)$ ($A(\phi(E))$), and any $f \in \tilde{A}(E)$ ($\tilde{A}(\phi(E))$) can be realized as the pointwise limit of some sequence $\langle\psi_n\rangle \subset A(E)$ ($A(\phi(E))$), where the ψ_n's are finitely supported, and $\sup_n \|\psi_n\|_{A(E)} < \infty$ ($\sup_n \|\psi_n\|_{\tilde{A}(\phi(E))} < \infty$). Therefore, to prove the lemma, it suffices to show

$$\|g\|_{A(E)} = \|g \circ \phi^{-1}\|_{A(\phi(E))}, \tag{1}$$

where g is any finitely supported function on E: Let $\{a_i\}_{i=1}^n$ be any finite set of complex numbers, and $\{y_i\}_{i=1}^n$ any set in E. Since D is dense in $\tilde{D}$, we obtain

$$\sup_{y \in \tilde{D}} \left| \sum_{i=1}^n a_i(\phi(y_i), y) \right| = \sup_{x \in D} \left| \sum_{i=1}^n a_i(\phi(y_i), x) \right|.$$

But, D was chosen dense in G, and hence

$$\sup_{x\in G}\left|\sum_{i=1}^{n} a_i(\gamma_i, x)\right| = \sup_{y\in\tilde{D}}\left|\sum_{i=1}^{n} a_i(\phi(\gamma_i), y)\right|.$$

Therefore,

$$\sup_{p\in C_E(G),\ \|p\|_\infty=1} |(g, \hat{p})| = \sup_{p\in C_{\phi(E)}(\tilde{D}),\ \|p\|_\infty=1} |(g\circ\phi^{-1}, \hat{p})|;$$

(1) now follows, and the lemma is proved. □

Remark 1.6. Let G be a compact abelian group, $\hat{G} = \Gamma$. If $f \in L^1(\Gamma)$, and g is a trigonometric polynomial in $C(\tilde{\Gamma})$, then the following hold:

(i) $\hat{f}_\Gamma = \hat{f}_{\tilde{\Gamma}}$, where $\hat{f}_\Gamma$ and $\hat{f}_{\tilde{\Gamma}}$ denote the Fourier transform of $f \in L^1(\Gamma)$, and the Fourier-Stieltjes transform of $f \in M(\tilde{\Gamma})$, respectively.

(ii) $f *_{L^1(\Gamma)} g = f *_{M(\tilde{\Gamma})} g$ on Γ.

Part (i) follows from the definition of $\tilde{\Gamma}$, and Parseval's formula. (ii) follows from (i).

Lemma 1.7. *Let G be a compact abelian group, $\hat{G} = \Gamma$. Let $\epsilon > 0$, and $V \subset G$ be a symmetric neighborhood of 0. There exists K, a neighborhood of 0 in G, such that if H is a finite set in K, then there is a trigonometric polynomial $g \in C_{V+V}(\tilde{\Gamma})$, so that the following hold:*

(i) $\hat{g}(0) = \hat{g}(h) = 1$ *for all* $h \in H$.

(ii) $\|g\|_{L^1(\tilde{\Gamma})} < 1 + \epsilon$.

(iii) *Suppose g_i, $i = 1, \cdots, n$, are trigonometric polynomials as above, corresponding to ϵ_i, V_i, K_i, H_i, $i = 1, \cdots, n$, respectively. Furthermore, assume that if $x_i \in$ support $\hat{g}_i$, and $\sum_{i=1}^n x_i = 0$, then $x_i = 0$ for $i = 1, \cdots, n$. Then*

$$\|\hat{g}_1 * \cdots * \hat{g}_n\|_{A(G_d)} < \prod_{i=1}^{n} (1 + \epsilon_i).$$

Proof. We shall first construct a function $R \in L^1(\Gamma)$ so that $\hat{R} = 0$ outside $V + V$, $\hat{R} = 1$ on K, a neighborhood of 0 in G, and $\|R\|_1 < \sqrt{1+\epsilon}$. Furthermore, R will be constructed so that $R = k + \lambda$, where

(1) $$k \geq 0 \quad\text{and}\quad \hat{k}(0) = 1,$$

(2) $$\|\lambda\|_\infty < \epsilon/3.$$

Given H, a finite set in K, we shall construct a trigonometric polynomial $g_0 \in C(\tilde{\Gamma})$, so that $\hat{g}_0 = 1$ on $H \cup \{0\}$, and $\|g_0\|_{L^1(\tilde{\Gamma})} < \sqrt{1+\epsilon}$. We shall have $g_0 = p + h$, where p and h are trigonometric polynomials in $C(\tilde{\Gamma})$ such that

(3) $$p \geq 0 \quad\text{and}\quad \hat{p}(0) = 1,$$

(4) $$\|h\|_\infty \leq \|\hat{h}\|_{M(G_d)} < \epsilon/3.$$

We set $g = g_0 * R$. g satisfies parts (i) and (ii) of the lemma. To prove part (iii), it suffices to display the proof for $n = 2$. Suppose that g_1 and g_2 are as in the hypothesis of (iii).

$$\|\hat{g}_1 * \hat{g}_2\|_{A(G_d)} = \|((\hat{p}_1 + \hat{h}_1)(\hat{k}_1 + \hat{\lambda}_1)) * ((\hat{p}_2 + \hat{h}_2)(\hat{k}_2 + \hat{\lambda}_2))\|_A.$$

By the assumption on support $(\hat{g}_i)$, $i = 1, 2$, we have that

$$(\hat{p}_1\hat{k}_1) * (\hat{p}_2\hat{k}_2)(0) = 1,$$

and since

$$(p_1 * k_1)(p_2 * k_2) \geq 0,$$

we have that

$$\|(\hat{p}_1\hat{k}_1) * (\hat{p}_2\hat{k}_2)\|_A = 1.$$

By (1), (2), (3), and (4), we have that

$$\|(p_1 * k_1)(p_2 * \lambda_2)\|_1 \leq \|p_1 * k_1\|_1 \|p_2\| \, \|\lambda_2\|_\infty \leq \epsilon_2/3$$

and

$$\|(p_1 * \lambda_1)(h_2 * k_2)\|_1 \leq \|p_1\|_1 \|\lambda_1\|_\infty \|h_2\|_\infty \|k_2\|_1 \leq \epsilon_1\epsilon_2/9.$$

The norms of the other terms are similarly estimated, and part (iii) follows.

Our constructions of R and g_0 are based on [9, 2.6]. Set $k = (\chi_V * \chi_V)\hat{}/m(V)$. Clearly, $\hat{k} = 0$ outside $V + V$, $\hat{k}(0) = 1$, and $\hat{k} \geq 0$. Let W be a neighborhood of 0 in G, so that for $g \in W$

(5) $$|1 - \hat{k}(g)| < \sqrt{\epsilon/3}.$$

Let $\theta \in L^1(\Gamma)$ be so that $\hat{\theta} = 1 - \hat{k}$ on B, where $B \subset W$ is an open neighborhood of 0 such that

(6) $$m(B) < \sqrt{\epsilon/3}.$$

Choose $\psi \in L^1(\Gamma)$ such that $\hat{\psi} = 0$ outside B, $\|\hat{\psi}\|_\infty = 1$, and $\hat{\psi} = 1$ on K, where K is a neighborhood of 0. Set $R = k + \theta * \psi$. It follows from (5) and (6) that

$$\|\theta * \psi\|_1 \leq \|\theta * \psi\|_\infty \leq m(B)\sqrt{\epsilon/3} \leq \epsilon/3.$$

By the construction, $\hat{R} = 0$ outside $V + V$, $\hat{R} = 1$ on K, and $\|R\|_1 < \sqrt{1 + \epsilon}$.

Let H be a finite set in K. Select $\{y_1, \cdots, y_N\}$, a linearly independent set in G, so that $H \subset \mathrm{gp}(y_1, \cdots, y_N)$. Assume first that all the y_i's are torsion free. Let

$$E_n = \left\{ x \in \mathrm{gp}(y_1, \cdots, y_N) : x = \sum_{j=1}^{N} \alpha_j y_j,\ |\alpha_j| \leq n \right\}.$$

Then, $m_d(E_n) = \mathrm{card}(E_n) = (2n+1)^N$. Let

$$p_n = (1/m_d(E_n))(\chi_{E_n} * \chi_{E_n})^{\wedge}.$$

Choose n sufficiently large so that

$$\sum_{x \in H} |1 - \hat{p}_n(x)| < \epsilon/3. \tag{7}$$

Set $p = p_n$, and let h be the trigonometric polynomial in $C_{V+V}(\tilde{\Gamma})$ defined by $\hat{h}(x) = (1 - \hat{p}(x))\chi_H(x)$. Let $g_0 = p + h$. Clearly, g_0 satisfies (3) and (4). Suppose that not all the y_i's are torsion free. We then add sufficiently many linearly independent elements, $\{y_{N+1}, \cdots, y_M\}$, to $\{y_1, \cdots, y_N\}$, so that if we set

$$n = \max_{x \in H} \left\{ \max_{1 \leq j \leq N} |\alpha_j| : x = \sum_{j=1}^{N} \alpha_j y_j \right\},$$

and

$$E_n = \left\{ x \in \mathrm{gp}(y_1, \cdots, y_M) : x = \sum_{j=1}^{M} \alpha_j y_j,\ |\alpha_j| \leq n \right\},$$

then card (E_n) is sufficiently large for (7) to follow. The construction of $g = R * g_0$ is now complete. □

Lemma 1.8. *Let $E \subset \Gamma$ be a non-Sidon set. Given $1 > \epsilon > 0$, and F, a finite set in Γ, there exists a finitely supported function $f \in A(E)$ so that $H =$ support $f \subset E \setminus F$, $\|f\|_{A(E)} = 1$, and $\|f\|_\infty < \epsilon$.*

Proof. Let θ be a finitely supported function in $A(E)$ so that $\|\theta\|_{A(E)} = 2$, and $\|\theta\|_\infty \leq \frac{1}{2}\epsilon b(E \cap F)$. We have

$$\|\theta\|_{A(E)} \leq \|\chi_{E \setminus F}\theta\|_{A(E)} + \|\theta\|_\infty / b(E \cap F),$$

and therefore,

$$2 - \epsilon/2 \leq \|\chi_{E\setminus F}\theta\|_{A(E)}.$$

Choose $k \in L^1(G)$ so that $\hat{k} = 1$ on F, $\hat{k}$ has finite support, and $\|(1-\hat{k})\chi_{E\setminus F}\theta\|_{A(E)} < \epsilon/2$. We then have

$$\|\chi_{E\setminus F}\theta\|_{A(E)} \leq \|\hat{k}\chi_{E\setminus F}\theta\|_{A(E)} + \|(1-\hat{k})\chi_{E\setminus F}\theta\|_{A(E)},$$

and therefore, $\|\hat{k}\chi_{E\setminus F}\theta\|_{A(E)} > 1$. Set

$$f = \hat{k}\chi_{E\setminus F}\theta / \|\hat{k}\chi_{E\setminus F}\theta\|_{A(E)}. \quad \square$$

Proof of Theorem A. We prove our theorem by transferring the problem from a countable group Γ to a compact metrizable group $\hat{D}$ via ϕ, where D and ϕ are as in the remark preceding Lemma 1.3. We note that the transfer depends on Drury's theorem (see also Remark 2.1 below). The assumption that $E \in \mathcal{P}(\Gamma)$ and the use of Lemma 1.7 illustrate well the structural nature of the proof of part (ii), and the difficulty of the problem in general.

Proof of part (i). Let $E \subset \Gamma$ be a non-Sidon set. By Lemma 1.4, we select a non-Sidon set $F \subset E$, so that $\phi(F)$ has only one limit point x_0 in $\hat{D}$. Without loss of generality we assume $x_0 = 0$. Let S be any finite set in Γ, $0 \notin S$. Let U be an open neighborhood of 0 so that

$$(\phi(y) + U) \cap \phi(F) = \begin{cases} \emptyset, & \text{if } \gamma \in S\setminus(S \cap F), \\ \{y\}, & \text{if } \gamma \in S \cap F. \end{cases}$$

Then $\phi(S) + \phi(F) \cap \phi(F)$ is finite; by the definition of ϕ, $S + F \cap F$ is finite, and hence $F \in \mathcal{P}(E)$.

Proof of part (ii). We first assume that Γ is a countable group. Let $E \in \mathcal{P}(\Gamma)$ be a non-Sidon set. By Lemmas 1.3 and 1.5, we can assume that $E \subset (\hat{D})_d$.

The proof will proceed as follows: We shall construct inductively $\{F_j\}_{j=1}^{\infty}$, a family of mutually disjoint finite subsets of E, and $\{f_j\}_{j=1}^{\infty} \subset A(E)$, so that the following hold:

(1) $\bigoplus_{l^1} C_{F_j}(\tilde{D}) \approx C_F(\tilde{D})$, where $F = \bigcup_{j=1}^{\infty} F_j$;

(2) (a) support $f_j \subset F_j$,
(b) $\|f_j\|_\infty < 1/j$,
(c) $\|f_j\|_{A(F)} \leq 1$ and $\|f_j\|_{A(F_j)} \geq ½$;

(3) There exists a measure $\mu \in M(\tilde{D})$, such that $\hat{\mu} = 1$ on the support of $f = \Sigma f_j$, and $\hat{\mu} = 0$ on $E\setminus F$.

From (1) and (2), it will follow, as in the proof of Lemma 1.2, that $f \in \tilde{A}(F)\setminus A(F)$. But if $\lambda \in M(\tilde{D})$ such that $\hat{\lambda}|_F = f$, then, by (3), $\widehat{\mu\lambda}|_E \in \tilde{A}(E)\setminus A(E)$.

Let $\langle \epsilon_j \rangle_{j=1}^{\infty}$ be so that $\epsilon_j > 0$ and $\Pi_{j=1}^{\infty} (1 + \epsilon_j) < 2$. Let V_1 be any symmetric neighborhood of 0 in $\hat{D}$. Find K_1, a neighborhood of 0 in $\hat{D}$, corresponding to ϵ_1 and V_1 in the conclusion of Lemma 1.7. By Lemma 1.4, $K_1 \cap E$ is not a Sidon set, and hence, we can choose a finitely supported function f_1,

$$H_1 = \text{support } f_1 \subset K_1 \cap E\setminus\{0\}, \quad \|f\|_{A(E \cap K_1)} = 1,$$

and $\|f_1\|_\infty < 1$. Let g_1 be a trigonometric polynomial in $C_{V_1+V_1}(\tilde{D})$ corresponding to ϵ_1, and H_1 as in Lemma 1.7. Set F_1 = support $\hat{g}_1 \cap E\setminus\{0\}$. Clearly, $f_1 = 0$ on $(K_1 \cap E)\setminus F_1$. We proceed by induction: Let $k > 1$ and assume that V_j, K_j, f_j, H_j, g_j, and F_j were selected for $j \geq k - 1$. Set

$$Q_k = \left\{ \sum_{j=1}^{k-1} \epsilon_j x_j : x_j \in \text{support } \hat{g}_j, \epsilon_j = -1, 0, 1, j = 1, \cdots, k-1 \right\}.$$

By the assumption on E, $Q_k + E \cap E$ is finite. Therefore, we can find U_k, a symmetric neighborhood of 0 in $\hat{D}$, so that $U_k + U_k \subset V_{k-1}$, and for all $x \in Q_k$,

$$(x + U_k) \cap (E \setminus \{x\}) = \emptyset. \tag{4}$$

Let V_k be an open symmetric neighborhood of 0, so that

$$V_k + V_k \subset U_k. \tag{5}$$

Select K_k corresponding to ϵ_k and V_k, as in Lemma 1.7. Set $F^{(k)} = \bigcup_{j=1}^{k-1} F_j$. By Lemma 1.4, $E \cap K_k$ is a non-Sidon set, and by Lemma 1.8, choose a finitely supported function f_k, so that $H_k = \text{support } f_k \subset (K_k \cap E) \setminus F^{(k)}$, $\|f_k\|_{A((E \cap K_k) \cup F^{(k)})} = 1$ and $\|f_k\|_\infty < 1/k$. Again by Lemma 1.7, find $g_k \in C_{V_k + V_k}(\hat{D})$ corresponding to ϵ_k and H_k, and set

$$F_k = \text{support } \hat{g}_k \cap (E \setminus \{0\}). \tag{6}$$

Our selection process is now complete. Clearly, (2)(a), (b) hold for all k. We now verify (2)(c):

$$\|f_k\|_{A(F_k)} = \sup_{\nu \in l^1(F_k)} |(f_k, \nu)| / \|\hat{\nu}\|_\infty$$

and by (6)

$$= \sup_{\nu \in l^1((E \cap K_k) \cup F^{(k)})} |(f_k, \nu)| / \|g_k * \hat{\nu}\|_\infty.$$

But, $\|\hat{\nu} * g_k\|_\infty \le \|g_k\|_1 \|\hat{\nu}\|_\infty \le 2\|\hat{\nu}\|_\infty$. Therefore,

$$\|f_k\|_{A(F_k)} \ge \sup_{\nu \in l^1((E \cap K_k) \cup F^{(k)})} \tfrac{1}{2} |(f_k, \nu)| / \|\hat{\nu}\|_\infty$$

$$= \tfrac{1}{2} \|f_k\|_{A((E \cap K_k) \cup F^{(k)})} = \tfrac{1}{2}.$$

The proof of part (1) is handled in the same way as the proof of Theorem 1.2 of [1]: When we select V_k, in addition to the constraint in (5), we insist that V_k be sufficiently small in order that the methods of [1] apply.

We now prove (3). (4) and (5) easily imply that if $\sum_{i=1}^M x_i \in E$, where $x_i \in F_i \cup \{0\}$, then all, except for possibly one of the summands, equal 0. Therefore, $\langle *_{j=1}^n \hat{g}_j \rangle$ converges pointwise to a function on E, which equals 1 on $\bigcup_{j=1}^\infty H_j =$ support f, and equals 0 on $E \setminus F$. Furthermore, since the hypothesis of part (iii) in Lemma 1.7 holds, we obtain

$$\left\| \mathop{*}_{j=1}^{n} \hat{g}_j \right\|_{A((\hat{D})_d)} < \prod_{j=1}^{n} (1 + \epsilon_j) < 2, \quad \text{for all } n.$$

By the metrizability of the weak* topology on bounded sets in $B(E) = C_E(\tilde{D})$, and by the weak* compactness of bounded sets in $B(E)$, it follows that there exists a measure $\mu \in M(\tilde{D})$ so that $\hat{\mu} = 1$ on support f and $\hat{\mu} = 0$ on $E \setminus F$. The proof of the theorem in the case that Γ is countable is now complete.

Let Γ be any discrete abelian group, and $E \in \mathcal{P}(\Gamma)$ a non-Sidon set. Let $E' \subset E$ be a countable non-Sidon set in Γ; set $F = E \cap \mathrm{gp}(E')$. It is easy to see that F is a Sidon set in Γ if and only if it is a Sidon set in $\mathrm{gp}(E')$. Therefore, since $\mathrm{gp}(E')$ is countable, $\tilde{A}(F) \supsetneq A(F)$ (the algebras are defined with respect to $M(\mathrm{gp}(E')^\wedge)$); i.e., there exists $\mu \in M(\mathrm{gp}(E'))$ so that $\hat{\mu}|_F \in c_0(F) \setminus A(F)$. By Theorem 2.7.2 of [9], there exists $\nu \in M(G)$ so that $\hat{\nu}|_F = \hat{\mu}|_F$, and since $\chi_{\mathrm{gp}(E')} \in M(G)^\wedge$, we have

$$f = \chi_{\mathrm{gp}(E')} \hat{\nu}|_E \in \tilde{A}(E).$$

Since $\hat{\mu}|_F \notin A(F)$, by Theorem 2.7.4 of [9], it follows that $f \notin A(E)$. □

Remark 2.1. When $\Gamma = \mathbf{Z}$, part (i) of Theorem A follows from the following elementary fact: Let $E \subset \mathbf{Z}$ be a non-Sidon set. Given any $N > 0$, there exists $0 \leq l < N$ so that $(N\mathbf{Z} + l) \cap E$ is non-Sidon. Part (ii) may also be obtained, though the arguments are essentially the same as in the proof of the theorem, without the use of Lemma 1.4, which rests on [4]. We do not know how to prove our theorem for general groups without resorting to Drury's theorem.

We now proceed to deduce a generalization of Theorem 3.1 in [7] (Corollary 2.3 below).

Let G be a compact abelian group, and $E \subset G$ a closed set. We define, as in [7],

$$D(E) = \left\{ f \in C(E) : \sup_{\mu \in M(E),\ \mu \neq 0} \left| \int f \, d\mu \right| \Big/ \|\hat{\mu}\|_\infty < \infty \right\}.$$

If $K(E)$ is any space of functions on E, we let

$$K_0(E) = \{ f \in K(E) : f(0) = 0 \}.$$

Let $E \subset G$ be a closed countable set with 0 as its only accumulation point; then, $c_0(E_d) = C_0(E)$. Also, it follows easily from the Hahn-Banach theorem, and Parseval's formula that $D_0(E) = \tilde{A}_0(E_d)$.

Lemma 2.2. *Let $E \subset G$ be as above. Then $A_0(E) = A_0(E_d)$ ($A(E)$ and $A(E_d)$ are the restriction algebras defined with respect to $L^1(\Gamma)$ and $L^1(\tilde{\Gamma})$, respectively).*

Proof. Finitely supported functions on E (on E_d) are norm dense in $A_0(E)$ ($A_0(E_d)$). It suffices, therefore, to prove that if f is finitely supported on $\underset{\sim}{E}$, then $\|f\|_{A(E)} = \|f\|_{A(E_d)}$. Since E is a closed and countable set, $(A(E))^* = C_E(\Gamma)$. The proof now proceeds as in Lemma 1.5. □

Corollary 2.3. *Let $E \subset G$ be a closed countable non-Helson set with a finite number of limit points, then $D(E) \supsetneqq A(E)$.*

Proof. Without loss in generality, we assume that 0 is the only limit point of E. Since E is a closed countable non-Helson set, E_d is a non-Sidon set in G_d. It is also clear that $E_d \in \mathcal{P}(G_d)$. The conclusion now follows from the remark preceding 2.2, Lemma 2.2, and Theorem A. □

3. **Open questions.** Let $E \in \mathcal{P}(\Gamma)$.

(a) We recall that $E \subset \Gamma$ is a Sidon set if and only if there exists $\beta > 0$ so that $\|f^2\|_{A(E)} \geq \beta \|f\|^2_{A(E)}$ (cf. [6, 8.3.8]). Therefore, in the proof of Theorem A, when choosing f_j at the jth step, in addition to requiring that $\|f_j\|_{A(E)} = 1$, and $\|f_j\|_\infty < 1/j$, we may also demand that

$$\|f_j^2\|_{A(E)} < 1/j^2. \tag{1}$$

It then follows, as in the proof of Theorem B that, $f = \Sigma f_j \in \mathfrak{X}(E) \setminus A(E)$. But, since support $f_i \cap$ support $f_j = \emptyset$ whenever $i \neq j$, $f^2 = \Sigma f_i^2$; by (1), $f^2 \in A(E)$.

Question. For any $n > 1$, can we find $f \in \mathfrak{X}(E)\setminus A(E)$, so that $f, \cdots, f^{n-1} \in \mathfrak{X}(E)\setminus A(E)$, but $f^n \in A(E)$?

(b) Consider $A'(E) = \{f \in \mathfrak{X}(E): f^2 \in A(E)\}$. Let $f = \Sigma f_j$ be as above, and $\epsilon = (\epsilon_j)_{j=1}^\infty$, where $\epsilon_j = -1$ or 1. It is clear that $f_\epsilon = \Sigma \epsilon_j f_j \in \mathfrak{X}(E)\setminus A(E)$, $f_\epsilon^2 = f^2 \in A(E)$. Furthermore, since $\|f_j\|_{A(F_j)} \geq \frac{1}{2}$, it follows that $\|f_\epsilon - f_{\epsilon'}\|_{\mathfrak{X}(E)} \geq \frac{1}{2}$, where $\epsilon \neq \epsilon'$. Therefore, $A'(E)$ is a closed nonseparable subalgebra of $\mathfrak{X}(E)$. On the other hand, it follows, as in the proof of Theorem 1 of [5], that the maximal ideal space of $A'(E)$ is precisely E.

Question. Does there exist a non-Sidon set $E \in \mathcal{P}(\Gamma)$ so that the maximal-ideal space of $\mathfrak{X}(E)$ is precisely E (see also [8])?

(c) Let $E \subset \Gamma$ be a non-Sidon set. Does $\mathfrak{X}(E) \supsetneqq A(E)$?

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REFERENCES

1. R. C. Blei, *On trigonometric series associated with separable, translation invariant subspaces of $L^\infty(G)$*, Trans. Amer. Math. Soc. 173 (1972), 491–499.

2. M. Déchamps-Gondim, *Ensembles de Sidon topologiques*, Ann. Inst. Fourier (Grenoble) 22 (1972).

3. R. Doss, *On the transform of a singular or an absolutely continuous measure*, Proc. Amer. Math. Soc. 19 (1968), 361–363. MR 36 #5619.

4. S. W. Drury, *Sur les ensembles de Sidon*, C. R. Acad. Aci. Paris Sér. A–B 271 (1970), A162–A163. MR 42 #6530.

5. C. Graham, *On a Banach algebra of Varopoulos*, J. Functional Analysis 4 (1969), 317–328. MR 40 #4768.

6. Y. Katznelson, *An introduction to harmonic analysis*, Wiley, New York, 1968. MR 40 #1734.

7. Y. Katznelson and C. McGehee, *Measures and pseudomeasures on compact subsets of the line*, Math. Scand. 23 (1968), 57–68. MR 40 #4688.

8. ———, *Some Banach algebras associated with quotients of $L^1(R)$*, Indiana Univ. Math. J. 21 (1971), 419–436. MR 44 #7224.

9. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR 27 #2808.

10. N. Varopoulos, *Sets of multiplicity in locally compact abelian groups*, Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 2, 123–158. MR 35 #3379.

11. ———, *On a problem of A. Beurling*, J. Functional Analysis 2 (1968), 24–30. MR 39 #3317.

12. ———, *Groups of continuous functions in harmonic analysis*, Acta Math. 125 (1970), 109–154. MR 43 #7868.

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