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INEQUALITIES FOR POLYNOMIALS WITH A PRESCRIBED ZERO

BY

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ABSTRACT. Inequalities for the derivative and for the maximum modulus on a larger circle of a polynomial with a given zero on the unit circle are obtained in terms of its degree and maximum modulus on the unit circle; examples are given to show that these are sharp with respect to the degree (best constants are not known). Inequalities for L^p norms, in particular L^2 norms, are also derived. Also certain functions of exponential type are considered and similar inequalities are obtained for them. Finally, the problem of estimating $P_n(r)$ (with $0 < r < 1$) given $P_n(1) = 0$ is taken up.

1. Introduction and statement of results. If $P_n(z)$ is a polynomial of degree n such that $\max_{|z|=1} |P_n(z)| = 1$ then

$$\max_{|z|=1} |P_n'(z)| \leq n, \tag{1.1}$$

$$\max_{|z|=R>1} |P_n(z)| \leq R^n. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [18]). Inequality (1.2) is a simple deduction from the maximum principle (see [15, p. 346], or [12, vol. 1, p. 137, Problem III 269]).

In both (1.1), (1.2) equality holds only for $P_n(z) = e^{i\gamma} z^n$, i.e. when all the zeros of $P_n(z)$ lie at the origin. Erdös conjectured and later Lax [11] verified that if $P_n(z) \neq 0$ in $|z| < 1$ then (1.1) can be replaced by

$$\max_{|z|=1} |P_n'(z)| \leq n/2 \tag{1.3}$$

and in (1.3) equality holds if all the zeros of $P_n(z)$ lie on $|z| = 1$. Ankeny and Rivlin [2] used (1.3) to prove that if $|P_n(z)| \leq 1$ for $|z| = 1$ and $P_n(z) \neq 0$ in $|z| < 1$ then

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(1.4) $$\max_{|z|=R>1} |P_n(z)| \le (R^n + 1)/2$$

which is much better than (1.2). Besides, in (1.4) equality is possible and in fact holds for $P_n(z) = \alpha + \beta z^n$ where $|\alpha| = |\beta| = \frac{1}{2}$.

Several years ago Professor R. P. Boas, Jr. asked one of us, namely Rahman, as to what can be said about $\max_{|z|=1} |P_n'(z)|$, $\max_{|z|=R>1} |P_n(z)|$ if we assume $P_n(z)$ to have precisely k zeros in $|z| \ge 1$ instead of all the zeros as Erdős did. In particular, what happens if $P_n(1) = 0$ or (more generally) if $|P_n(1)| = a$? We are thus led to consider the class $\mathcal{P}_{n,a}$ of all polynomials $P_n(z)$ of degree n such that $\max_{|z|=1} |P_n(z)| = 1$, $\min_{|z|=1} |P_n(z)| \le a$ where $0 \le a \le 1$. It is clear that every polynomial $P_n(z)$ with $\max_{|z|=1}|P_n(z)| = 1$ belongs to $\mathcal{P}_{n,a}$ for some a.

In view of (1.3), one might expect that if $P_n(z) \in \mathcal{P}_{n,0}$ then $\max_{|z|=1}|P_n'(z)| \le n - c$ where c is a constant, possibly equal to $\frac{1}{2}$. But this is far from the truth as the next two theorems show.

Theorem 1. *If* $P_n(z) \in \mathcal{P}_{n,a}$ *then for* $|z| \le 1$

(1.5) $$|P_n'(z)| \le n - (1 - a)\{1 - a - \sin(1 - a)\}/4\pi n.$$

Theorem 2. *There exists an absolute constant* $c_1 > 0$ *such that*

(1.6) $$\max_{P_n(z)\in \mathcal{P}_{n,a}} \left(\max_{|z|=1} |P_n'(z)|\right) \ge n - (1 - a)c_1/n.$$

We also prove

Theorem 3. *If* $P_n(z) \in \mathcal{P}_{n,a}$ *then for* $R > 1$

(1.7) $$\max_{|z|=R} |P_n(z)| \le R^n \left\{1 - \frac{1}{([\pi/(1 - a)] + 1)^2 n} (1 - e^{-\pi/2})(1 - R^{-1})^2\right\}.$$

Theorem 4. *There exists an absolute constant* $c_2 > 0$ *such that for* $R > n^2$

(1.8) $$\max_{P_n(z)\in \mathcal{P}_{n,a}} \left(\max_{|z|=R} |P_n(z)|\right) > R^n\{1 - c_2(1 - a)/n\}.$$

Theorem 1 says in particular that if $P_n(z) \in \mathcal{P}_{n,0}$ then

(1.5′) $$\max_{|z|=1} |P_n'(z)| \le n - (1 - \sin 1)/4\pi n.$$

But in this special case namely for polynomials $P_n(z) \in \mathcal{P}_{n,0}$ we obtain the better estimate:

$$\max_{|z|=1} |P_n'(z)| \le n - (2-\sqrt{2})/4n. \tag{1.5''}$$

We are also able to replace

$$\max_{P_n(z)\in \mathcal{P}_{n,0}} \left(\max_{|z|=R>1} |P_n(z)|\right) \le R^n \left\{1 - \frac{1}{16n}(1-e^{-\pi/2})(1-R^{-1})^2\right\} \tag{1.7'}$$

obtainable from Theorem 3 by

$$\max_{P_n(z)\in \mathcal{P}_{n,0}} \left(\max_{|z|=R>1} |P_n(z)|\right) \le R^n \left\{1 - \frac{2-\sqrt{2}}{2n}(1-R^{-1})^2\right\}. \tag{1.7''}$$

Coming back to the original question of Boas, namely what happens if $p_n(z)$ has k zeros in $|z| \ge 1$, we can prove the following theorem:

Theorem 5. *Corresponding to every $\epsilon > 0$, there exist a $\delta > 0$ and an integer n_0 such that for all $n > n_0$ there is a polynomial $P_n(z) \in \mathcal{P}_{n,0}$ which has at least $\delta\sqrt{n}$ zeros in $|z| \ge 1$ and is such that $\max_{|z|=1} |P_n'(z)| > n - \epsilon$.*

Theorem 2 gives us an idea as to how large $\max_{|z|=1} |P_n'(z)|$ can be if $P_n(z)$ is a polynomial of degree n such that $\max_{|z|=1}|P_n(z)| = 1$ and $|P_n(1)| = a$. It is natural to ask how small $\max_{|z|=1}|P_n'(z)|$ can be under these conditions. This question turns out to be easy. Indeed, from

$$P_n(z) = P_n(1) + \int_1^z P_n'(\zeta)\, d\zeta$$

we obtain, for $|z| \le 1$,

$$|P_n(z)| \le a + |z-1| \max_{|z|\le 1} |P_n'(z)| \le a + 2 \max_{|z|\le 1} |P_n'(z)|$$

that is

$$\max_{|z|\le 1} |P_n'(z)| \ge (1-a)/2. \tag{1.9}$$

We may consider the polynomial

$$P_n(z) = a + \frac{1-a}{2(k+1)}(1-z)\{k + (-z)^{n-1}\} \tag{1.10}$$

with sufficiently large positive k to see that the bound $(1-a)/2$ is best possible. In fact, $P_n(1) = a$, $\max_{|z|=1} |P_n(z)| = 1$ and, for every given $\epsilon > 0$,

$$\max_{|z|=1} |P_n'(z)| = (1-a)\left(\frac{1}{2} + \frac{n-1}{k+1}\right) < \frac{1-a}{2} + \epsilon$$

if $k > (1-a)(n-1)/\epsilon - 1$.

Applying Theorem 3 to the polynomial $z^n P_n(1/z)$ we deduce

$$(1.11)\quad \max_{P_n(z)\in \mathcal{P}_{n,a}} \left(\max_{|z|=\rho<1} |P_n(z)|\right) \le 1 - \frac{1}{([\pi/(1-a)]+1)^2 n}(1-e^{-\pi/2})(1-\rho)^2.$$

For polynomials $P_n(z) \in \mathcal{P}_{n,0}$ we may use (1.7″) to get

$$(1.11')\quad \max_{|z|=\rho<1} |P_n(z)| \le 1 - \frac{2-\sqrt{2}}{2n}(1-\rho)^2.$$

Recently, but in quite a different context Halász asked how large $\min_{|z|=1-(\omega/n)} |P_n(z)|$ can be for a given ω in $(0, n]$ if $P_n(z) \in \mathcal{P}_{n,0}$. It has been shown by Rahman and Stenger [14] that for given

$$\lambda > \frac{1}{\pi}\int_{-\infty}^{\infty} |\log(1 - \sin^2 u/u^2)|\, du$$

there exists a positive number $A(\lambda)$ depending on λ such that

$$(1.12)\quad \mu(\omega, n) = \max_{P_n(z)\in\mathcal{P}_{n,0}} \left\{\max_{|z|=1-(\omega/n)} |P_n(z)|\right\} > 1 - \frac{\lambda}{\omega}$$

provided $\omega > A(\lambda)$. On the other hand they showed that if ω is large then for $n \ge \omega$

$$(1.13)\quad \mu(\omega, n) \le 1 - 1/(e\omega) + o(1/\omega).$$

Here we shall prove the following theorem which gives a better estimate for $\mu(\omega, n)$ than (1.13). Besides, in lieu of requiring $\max_{|z|=1}|P_n(z)| \le 1$ we only assume $|P_n(\exp(ij\pi/n))| \le 1$ for $j = 1, 2, 3, \cdots, 2n-1$ if 1 is the point on $|z| = 1$ where $P_n(z)$ vanishes.

Theorem 6. *If $P_n(z)$ is a polynomial of degree n such that $P_n(1) = 0$, $|P_n(\exp(ij\pi/n))| \le 1$ for $j = 1, 2, 3, \cdots, 2n-1$ then, for $0 < \omega \le n$,*

$$(1.14)\quad \left|P_n\left(1-\frac{\omega}{n}\right)\right| \le 1 - \left(\frac{1}{\omega} - \frac{1}{2n}\right) + \left(\frac{1}{\omega} - \frac{1}{2n}\right)\left(1 - \frac{\omega}{n}\right)^n.$$

Inequalities (1.1), (1.2) can be obtained by letting $p \to \infty$ in the inequalities

$$(1.15)\quad \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n'(e^{i\theta})|^p\, d\theta\right)^{1/p} \le n\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^p\, d\theta\right)^{1/p}, \quad p \ge 1,$$

and

$$(1.16)\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(Re^{i\theta})|^p\, d\theta\right)^{1/p} \le R^n\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^p\, d\theta\right)^{1/p}, \quad p > 0,\ R > 1,$$

respectively. Inequality (1.15) is due to Zygmund [20] who proved it for all trigonometric polynomials of degree n and not only for those which are of the form $P_n(e^{i\theta})$. As for (1.16) it is difficult to trace its origin. We can deduce it from a well-known result of G. H. Hardy [10] according to which for every function $f(z)$ analytic in $|z| < \rho_0$ and, for every $p > 0$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta\right)^{1/p}$$

is a nondecreasing function of ρ for $0 < \rho < \rho_0$. If $P_n(z)$ is a polynomial of degree n, then $f(z) = z^n \overline{P_n(1/\bar{z})}$ is again a polynomial, i.e. an entire function and by Hardy's result

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^p \, d\theta\right)^{1/p} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta\right)^{1/p}, \quad p > 0,$$

for $\rho = R^{-1} < 1$. This is equivalent to (1.16).

In both (1.15), (1.16) equality holds only if $P_n(z)$ is a constant multiple of z^n.

It was shown by de Bruijn [8] that if $P_n(z) \neq 0$ in $|z| < 1$ then (1.15) can be replaced by

$$(1.17) \quad \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n'(e^{i\theta})|^p \, d\theta\right)^{1/p} \leq nC_p \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^p \, d\theta\right)^{1/p}, \quad p \geq 1,$$

where

$$C_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{i\phi}|^p \, d\phi\right)^{-1/p}.$$

The corresponding refinement of (1.16) namely

$$(1.18) \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(Re^{i\theta})|^p \, d\theta\right)^{1/p} \leq K_p \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(Re^{i\theta})|^p \, d\theta\right)^{1/p}, \quad p \geq 1, \ R > 1,$$

where

$$K_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + R^n e^{in\phi}|^p \, d\phi\right)^{1/p} \Bigg/ \left(\frac{1}{2\pi}\int_0^{2\pi} |1 + e^{in\phi}|^p \, d\phi\right)^{1/p}$$

if $P_n(z) \neq 0$ in $|z| < 1$ was proved by Boas and Rahman [7].

Both inequalities (1.17), (1.18) are sharp. Equality holds for all polynomials of the form $\lambda + \mu z^n$ with $|\lambda| = |\mu|$.

If we let p tend to infinity in (1.17), (1.18) we get (1.3), (1.4) respectively.

In the special case $p = 2$ inequalities (1.17), (1.18) reduce to

$$(1.17') \qquad \frac{1}{2\pi}\int_0^{2\pi} |P_n'(e^{i\theta})|^2 \, d\theta \leq \left(\frac{n^2}{2}\right) \frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta,$$

$$(1.18') \qquad \frac{1}{2\pi}\int_0^{2\pi} |P_n(Re^{i\theta})|^2 \, d\theta \leq \frac{R^{2n}+1}{2} \frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta$$

respectively, i.e. the hypothesis $P_n(z) \neq 0$ in $|z| < 1$ allows us to put $n^2/2$ instead of n^2 on the right-hand side of (1.17′) and $(R^{2n}+1)/2$ instead of R^{2n} on the right-hand side of (1.18′). It may be asked what happens if we simply assume $P_n(z)$ to have a zero on $|z|=1$. The answer is provided by Corollaries 1 and 2 of the following theorem.

Theorem 7. *If* $P_n(z)=\sum_{k=0}^n a_k z^k$ *is a polynomial of degree* n *with* $P_n(e^{i\theta_0})=0$ *for some* θ_0 *in* $[0, 2\pi)$ *and* $\lambda_0, \lambda_1, \cdots, \lambda_n$ *are nonnegative numbers such that* $\lambda_j > \lambda_k$ *for* $k=0, 1, \cdots, j-1, j+1, \cdots, n$, *then*

$$\sum_{k=0}^{n} \lambda_k |a_k|^2 \le (\lambda_j - \lambda) \sum_{k=0}^{n} |a_k|^2 \tag{1.19}$$

where λ *is the unique root of the equation*

$$\sum_{k=0}^{n} \frac{1}{\lambda_j - \lambda_k - x} = 0 \tag{1.20}$$

in the interval $(0, \Lambda = \min_{0\le k\le n; k\ne j}(\lambda_j - \lambda_k))$.

That equation (1.20) has one and only one root in the interval $(0, \Lambda)$ follows on writing it as $f'(x)/f(x)=0$ where

$$f(x) = \prod_{k=0}^{n} \{x - (\lambda_j - \lambda_k)\}.$$

Since all the zeros of the *polynomial* $f(x)$ are real and $0, \Lambda$ are consecutive zeros, Rolle's theorem shows that $f'(x)$ vanishes once and only once in $(0, \Lambda)$.

Inequality (1.19) is sharp and becomes an equality if (and only if) $P_n(z)$ is a constant multiple of $\sum_{k=0}^n (e^{-i\theta_0}z)^k/(\lambda_j - \lambda_k - \lambda)$.

The following two corollaries of Theorem 7 are obtained on setting $\lambda_k = k^2$ $(0\le k\le n)$, $\lambda_k = R^{2k}$ $(0\le k\le n)$ respectively.

Corollary 1. *If* $P_n(z)$ *is a polynomial of degree* n *with* $P_n(e^{i\theta_0})=0$ *for some* θ_0 *in* $[0, 2\pi)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} |P_n'(e^{i\theta})|^2\, d\theta \le (n^2 - \alpha_n)\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2\, d\theta \tag{1.21}$$

where α_n *is the unique root of the equation*

$$\sum_{k=0}^{n} \frac{1}{n^2 - k^2 - x} = 0 \tag{1.22}$$

in the interval $(0, 2n-1)$.

Corollary 2. *If* $P_n(z)$ *is a polynomial of degree* n *with* $P_n(e^{i\theta_0})=0$ *for some* θ_0 *in* $[0, 2\pi)$, *then for* $R>1$

(1.23) $$\frac{1}{2\pi}\int_0^{2\pi} |P_n(Re^{i\theta})|^2\, d\theta \le (R^{2n} - \beta_n)\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2\, d\theta$$

where β_n *is the unique root of the equation*

(1.24) $$\sum_{k=0}^{n} \frac{1}{R^{2n} - R^{2k} - x} = 0$$

in the interval $(0, R^{2n} - R^{2n-2})$.

Although we do not know the explicit values of α_n, β_n we can show that (i) $\alpha_n \sim 2n/(\log n)$ as $n \to \infty$, (ii) $\beta_n \sim R^{2n}/(n+1)$ as $R \to \infty$ and n is fixed.

(i) That $\alpha_n \sim 2n/(\log n)$ as $n \to \infty$ may be deduced from the fact that

(1.25) $$\sum_{k=0}^{n-1} \frac{1}{n^2 - k^2 - 2n/(\log n)} \sim \frac{\log n}{2n} \quad \text{as } n \to \infty.$$

Indeed, if $F_n(x)$ denotes the sum $\Sigma_{k=0}^{n-1}[1/(n^2 - k^2 - x)]$ then by virtue of (1.25) we have for every $\epsilon > 0$

$$F_n\left(\frac{(2+\epsilon)n}{\log n}\right) > F_n\left(\frac{2n}{\log n}\right) > \frac{\log n}{(2+\epsilon)n}$$

and

$$F_n\left(\frac{(2-\epsilon)n}{\log n}\right) < F_n\left(\frac{2n}{\log n}\right) < \frac{\log n}{(2-\epsilon)n}$$

provided n is sufficiently large. Thus $F_n(x) - 1/x$ changes sign between $(2-\epsilon)n/\log n$ and $(2+\epsilon)n/\log n$, i.e. $(2-\epsilon)n/\log n < \alpha_n < (2+\epsilon)n/\log n$. In other words, $\alpha_n \sim 2n/(\log n)$ as $n \to \infty$.

In order to prove (1.25) we write $\Sigma_{k=0}^{n-1}[1/(n^2 - k^2 - 2n/\log n)]$ as

$$\frac{1}{2(n^2 - 2n/\log n)^{1/2}} \sum_{k=0}^{n-1} \left\{\frac{1}{(n^2 - 2n/\log n)^{1/2} - k} + \frac{1}{(n^2 - 2n/\log n)^{1/2} + k}\right\}$$

which is equal to

$$\frac{1}{2n(1 - 2/n\log n)^{1/2}} \left\{\sum_{k=0}^{n-1}\left(\frac{1}{n-k} + \frac{1}{n+k}\right) + r_n\right\}$$

where the positive number r_n remains bounded as $n \to \infty$. In fact, the inequalities

$$\log(N+1) < \sum_{l=1}^{N} \frac{1}{l} < 1 + \log N$$

show that (for sufficiently large n)

$$0<\sum_{k=0}^{n-1}\frac{1}{(n^2-2n/\log n)^{1/2}-k}-\sum_{k=0}^{n-1}\frac{1}{n-k}$$

$$\le\frac{n-n(1-2/n\log n)^{1/2}}{1-n+n(1-2/n\log n)^{1/2}}\sum_{k=0}^{n-1}\frac{1}{n-k}$$

$$<\frac{2}{\log n}\,\frac{1+\log n}{1-n+n(1-2/n\log n)^{1/2}}=O(1)$$

and that

$$0<\sum_{k=0}^{n-1}\frac{1}{(n^2-2n/\log n)^{1/2}+k}-\sum_{k=0}^{n-1}\frac{1}{n+k}$$

$$\le\frac{n-n(1-2/n\log n)^{1/2}}{n(1-2/n\log n)^{1/2}}\sum_{k=0}^{n-1}\frac{1}{n+k}$$

$$<\frac{2}{\log n}\,\frac{1+\log(2n-1)}{n(1-2/n\log n)^{1/2}}=o(1).$$

Thus

$$\sum_{k=0}^{n-1}\frac{1}{n^2-k^2-2n/\log n}=\frac{1}{2n(1-2/n\log n)^{1/2}}\left(\sum_{l=0}^{2n-1}\frac{1}{l}+\frac{1}{n}+r_n\right)$$

$$\sim\frac{\log n}{2n}\quad\text{as } n\to\infty.$$

This proves (1.25), and the verification of the claim that "$\alpha_n\sim 2n/(\log n)$ as $n\to\infty$" is complete.

(ii) That, for fixed n, $\beta_n\sim R^{2n}/(n+1)$ as $R\to\infty$ can be deduced from the trivial fact:

$$\sum_{k=0}^{n-1}\frac{1}{R^{2n}-R^{2k}-R^{2n}/(n+1)}\sim\frac{n+1}{R^{2n}}\quad\text{as } R\to\infty.$$

Remark 1. Theorem 7 is easily seen to be equivalent to the following:

Theorem 7′. *If* $f(s)=\Sigma_{k=0}^{n}a_k e^{s\sqrt{\lambda_k}}$ $(s=\sigma+it)$ *where* $0\le\lambda_0<\lambda_1<\cdots<\lambda_{n-1}<\lambda_n$ *and* $f(\sigma_0)=0$, *then*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f'(\sigma_0+it)|^2\,dt\le(\lambda_n-\lambda)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma_0+it)|^2\,dt$$

where λ *is the smallest root of the equation*

$$\sum_{k=0}^{n} \frac{1}{\lambda_n - \lambda_k - x} = 0.$$

Remark 2. Instead of assuming $P_n(e^{i\theta_0}) = 0$ we may assume $P_n(\rho e^{i\theta_0}) = 0$ for some $\rho > 0$ and ask what becomes of (1.19).

Theorem 7″. *If $\lambda_j > \lambda_k \geq 0$ for $k = 0, 1, 2, \cdots, j-1, j+1, \cdots, n$ and $P_n(z) = \sum_{k=0}^{n} a_k z^k$ vanishes at $z = \rho e^{i\theta_0}$ then*

$$\sum_{k=0}^{n} \lambda_k |a_k|^2 \leq (\lambda_j - \lambda^{(\rho)}) \sum_{k=0}^{n} |a_k|^2 \tag{1.26}$$

where $\lambda^{(\rho)}$ is the unique root of the equation

$$\sum_{k=0}^{n} \frac{\rho^{2k}}{\lambda_j - \lambda_k - x} = 0 \tag{1.27}$$

in $(0, \Lambda = \min_{0 \leq k \leq n; k \neq j} (\lambda_j - \lambda_k))$.

Equality holds in (1.26) for

$$P_n(z) = a \sum_{k=0}^{n} \frac{\rho^k}{\lambda_j - \lambda_k - \lambda^{(\rho)}} (e^{-i\theta_0} z)^k.$$

Since $f(x) = \prod_{k=0}^{n} (\lambda_j - \lambda_k - x)^{\rho^{2k}}$ vanishes at $x = 0$, $x = \Lambda$ the equation

$$\sum_{k=0}^{n} \frac{\rho^{2k}}{\lambda_j - \lambda_k - x} \equiv -\frac{f'(x)}{f(x)} = 0$$

must have at least one root in $(0, \Lambda)$ by the mean value theorem. That it has only one becomes obvious on writing it as

$$\frac{\rho^{2j}}{x} = \sum_{k=0; k \neq j}^{n} \frac{\rho^{2k}}{\lambda_j - \lambda_k - x}$$

and noting that the left-hand side decreases whereas the right-hand side increases as x increases from 0 to Λ.

On setting $\lambda_k = k^2$ $(0 \leq k \leq n)$, $\lambda_k = R^{2k}$ $(0 \leq k \leq n)$ in Theorem 7″ we obtain sharp estimates for

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n'(e^{i\theta})|^2 \, d\theta\right) \Big/ \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta\right),$$

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(Re^{i\theta})|^2 \, d\theta\right) \Big/ \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2 \, d\theta\right)$$

under the hypothesis that $P_n(z)$ has a zero on $|z| = \rho$. These estimates constitute generalizations of Corollaries 1, 2 respectively which deal with the case $\rho = 1$.

The following corollary is obtained on setting $\lambda_j = 1$ and $\lambda_k = 0$ for $k \neq j$ in Theorem 7″.

Corollary 3. *If* $P_n(z) = \sum_{k=0}^{n} a_k z^k$ *has a zero on* $|z| = \rho$ *then for* $0 \leq j \leq n$

$$|a_j| \leq \left\{\left(\sum_{k=0;k\neq j}^{n} \rho^{2k}\right)\Big/\left(\sum_{k=0}^{n} \rho^{2k}\right)\right\}^{1/2} \left(\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2\, d\theta\right)^{1/2}.$$

If $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ is a trigonometric polynomial of degree n with $S_n(\theta_0) = 0$ for some θ_0 in $[0, 2\pi)$ then $e^{in\theta}S_n(\theta) = P_{2n}(e^{i\theta})$ where $P_{2n}(z) = \sum_{k=0}^{2n} a_k z^k = \sum_{k=0}^{2n} c_{k-n} z^k$ is a polynomial of degree $2n$ with $P_{2n}(e^{i\theta_0}) = 0$. Corollary 3 is applicable with $2n$ in place of n giving sharp upper bound for $|a_{k+n}| \equiv |c_k|$ in terms of $((1/2\pi)\int_0^{2\pi} |S_n(\theta)|^2\, d\theta)^{1/2}$. That is how we get

Theorem 8. *If* $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ *is a trigonometric polynomial of degree* n *with* $S_n(\theta_0) = 0$ *for some* θ_0 *in* $[0, 2\pi)$ *then for* $-n \leq k \leq n$

$$|c_k| \leq \left(\frac{2n}{2n+1}\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi} |S_n(\theta)|^2\, d\theta\right)^{1/2} \tag{1.28}$$

where equality holds if and only if $S_n(\theta)$ *is a constant multiple of*

$$\sum_{l=-n;l\neq k}^{n} (e^{-i\theta_0}e^{i\theta})^l - 2n(e^{-i\theta_0}e^{i\theta})^k.$$

Theorem 8 is the L^2 analogue of a theorem of Boas [6] according to which

$$|c_0| \leq \frac{n}{n+1} \max_{0\leq\theta<2\pi} |S_n(\theta)|. \tag{1.29}$$

He, in fact, showed that the maximum on the right-hand side of (1.29) may be taken only over the points $\theta = \theta_0 + 2k\pi/(n+1)$, $k = 1, 2, \cdots, n$. It follows from his argument that

$$|c_0| \leq \frac{1}{n+1}\left\{a + n \max_{1\leq k\leq n}\left|S_n\left(\theta_0 + \frac{2k\pi}{n+1}\right)\right|\right\} \tag{1.30}$$

if $|S_n(\theta_0)| = a$ where a is *any* nonnegative number. Applying (1.30) to the trigonometric polynomial $|P_n(e^{i\theta_0})|^2$ as Boas [6] did we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |P_n(e^{i\theta})|^2\, d\theta \leq \frac{1}{n+1}\left\{a^2 + n \max_{1\leq k\leq n}\left|P_n\left(e^{i(\theta_0+2k\pi/(n+1))}\right)\right|^2\right\} \tag{1.31}$$

if $P_n(z)$ is a polynomial of degree n with $|P_n(e^{i\theta_0})| = a$. In (1.31) equality holds for all polynomials $P_n(z)$ for which

$$|P_n(e^{i\theta})|^2 = a^2 + (\mu^2 - a^2)\left\{1 - \left(\frac{1}{n+1}\ \frac{\sin(n+1)(\theta-\theta_0)/2}{\sin \frac{1}{2}(\theta-\theta_0)}\right)^2\right\}$$

for $0 \le \theta < 2\pi$. For all such polynomials $|P_n(e^{i\theta_0})| = a$,

$$\max_{1\le k\le n} |P_n(e^{i(\theta_0+2k\pi/(n+1))})| = \mu^2 = \max_{|z|=1} |P_n(z)| \quad \text{if } a \le \mu.$$

Hence if $a \le \mu$ we will not get any improvement in (1.31) if maximum of $|P_n(z)|$ is taken on $|z| = 1$ rather than at the points

$$z = e^{i(\theta+2k\pi/(n+1))} \qquad (k = 1, 2, \cdots, n).$$

In general

(1.32) $$\|P_n(e^{i\theta})\|_{p_1} \le \|P_n(e^{i\theta})\|_{p_2}$$

if $0 < p_1 < p_2 \le \infty$. We know from above that

$$\|P_n(e^{i\theta})\|_2 \le \{(a^2 + n\|P_n(e^{i\theta})\|_\infty^2)/(n+1)\}^{1/2}$$

if $|P_n(e^{i\theta_0})| = a$ for some θ_0 in $[0, 2\pi)$. It is natural to ask what improvement results in (1.32) if $|P_n(e^{i\theta_0})| = a < \|P_n(e^{i\theta})\|_\infty$ for some θ_0 in $[0, 2\pi)$. Although we cannot answer this general question we note that by applying Theorem 8 to the trigonometric polynomial $|P_n(e^{i\theta})|^2$ we get

Corollary 4. *If* $P_n(e^{i\theta_0}) = 0$ *for some* θ_0 *in* $[0, 2\pi)$, *then*

(1.33) $$\|P_n(e^{i\theta})\|_2 \le (2n/(2n+1))^{1/4}\,\|P_n(e^{i\theta})\|_4.$$

In (1.33) equality holds for all polynomials $P_n(z)$ for which

$$|P_n(e^{i\theta})|^2 = n - \sum_{k=1}^{n} \cos k(\theta - \theta_0)$$

for $0 \le \theta < 2\pi$.

Entire functions of exponential type. If $P_n(z)$ is a polynomial of degree n such that $|P_n(z)| \le 1$ for $|z| = 1$ then $f(z) = P_n(e^{iz})$ is an entire function of exponential type n and $|f(x)| \le 1$ for $-\infty < x < \infty$. From (1.1) we know that $|f'(x)| \le n$ for $-\infty < x < \infty$ whereas according to (1.2) $|f(x+iy)| \le e^{n|y|}$ for $y < 0$. It was shown by S. Bernstein (see [3, Chapter 11]) that for every entire function $f(z)$ of exponential type τ satisfying $|f(x)| \le 1$ for $-\infty < x < \infty$ we have $|f'(x)| \le \tau$ for $-\infty < x < \infty$. Besides, it is a simple consequence of the Phragmén-Lindelöf principle (for references see [3, p. 82]) that for all y: $|f(x+iy)| \le e^{\tau|y|}$ for an arbitrary entire function $f(z)$ of exponential type τ satisfying $|f(x)| \le 1$ for $-\infty < x < \infty$.

If $P_n(z) \neq 0$ in $|z| < 1$ then $P_n(e^{iz})$ is an entire function $f(z)$ of exponential type of a special kind: if $h(\theta)$ is its indicator, we have $h(-\pi/2) = n$ and (since $P_n(0) \neq 0$) $h(\pi/2) = 0$. Thus Boas [5] extended (1.3), (1.4) to entire functions of exponential type by proving that for all real x,

$$|f'(x)| \leq \tau/2 \tag{1.34}$$

and for $y < 0$,

$$|f(x+iy)| \leq \tfrac{1}{2}(e^{\tau|y|} + 1) \tag{1.35}$$

if $f(z)$ is an entire function of exponential type τ with $|f(x)| \leq 1$ for real x, $h(\pi/2) = 0$ and $f(z) \neq 0$ for $\operatorname{Im} z > 0$.

Here we shall prove

Theorem 9. *Let $f(z)$ be an entire function of exponential type τ with $|f(x)| \leq 1$ for real x,*

$$h(\pi/2) = \limsup_{y\to\infty} y^{-1} \log |f(iy)| \leq 0$$

and $f(0) = 0$. Then for all real x we have

$$|f'(x)| \leq \tau\{1 - (4-\pi)/2(\tau|x| + 2)^2\} \tag{1.36}$$

and for $y < 0$ we have

$$|f(x+iy)| \leq e^{\tau|y|}\left\{1 - \frac{1}{2}\tau|y| \frac{(1 - e^{-\tau|y|})(4-\pi)}{(\tau y)^2 + (\tau|x| + 2)^2}\right\}. \tag{1.37}$$

We shall show that (1.36) is "essentially" best possible.

Theorem 6 can be reformulated as follows:

Theorem 6′. *If the entire function $f(z)$ of exponential type n is periodic on the real axis with period 2π (and hence bounded for real x), such that $h(\pi/2) \leq 0$, $f(0) = 0$ and $|f(j\pi/n)| \leq 1$ for $j = 1, 2, \cdots, 2n-1$, then, for $0 < \omega < n$,*

$$|f(-i\log(1 - \omega/n))| \leq 1 - (1/\omega - 1/2n) + (1/\omega - 1/2n)(1 - \omega/n)^n. \tag{1.14′}$$

Indeed, an entire function $f(z)$ of exponential type τ is periodic on the real axis with period 2π if and only if $f(z) = \sum_{k=-n}^{n} a_k e^{ikz}$ $(n \leq \tau)$. If, in addition, $h(\pi/2) \leq 0$, then

$$f(z) = \sum_{k=0}^{n} a_k e^{ikz} = P_n(e^{iz})$$

where $P_n(z)$ satisfies the hypotheses of Theorem 6. Hence (1.14′) holds.

We observe that the requirement of periodicity in Theorem 6′ can be dropped with little change in the conclusion if $f(z)$ is bounded on the real axis.

Theorem 10. *If $f(z)$ is an entire function of exponential type τ such that $|f(x)|$ is bounded on the real axis, $h(\pi/2) \le 0$, $f(0) = 0$ and $|f(j\pi/\tau)| \le 1$ for $j = \pm 1, \pm 2, \cdots$, then, for $0 < \omega < \tau$, we have*

$$|f(-i\log(1-\omega/\tau))| \le 1 + \frac{1-(1-\omega/\tau)^\tau}{\log(1-\omega/\tau)^\tau} \tag{1.38}$$
$$< 1 - (1/\omega - 1/2\tau) + (1/\omega - 1/2\tau)(1-\omega/\tau)^\tau + \omega\{1-(1-\omega/\tau)^\tau\}/2\tau^2$$

2. Lemmas. We now prove or simply quote certain results which we shall need later. First, an interpolation formula:

Lemma 1. *If $P_n(z)$ is a polynomial of degree n then for $R > 1$*

$$P_n(Re^{i\phi}) = P_n(e^{i\phi}) + \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k A_k P_n(e^{i(\phi+k\pi/n)}) \tag{2.1}$$

where

$$A_k = (R^n - 1) + 2\sum_{j=1}^{n-1}(R^{n-j}-1)\cos j\frac{k\pi}{n}.$$

The coefficients A_k are positive and

$$\frac{1}{2n}\sum_{k=1}^{2n} A_k = R^n - 1. \tag{2.2}$$

Proof. Let $t(\theta) = P_n(e^{i\theta}) = \Sigma_{\nu=0}^n(a_\nu\cos\nu\theta + b_\nu\sin\nu\theta)$. As

$$a_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\cos\nu\theta\,d\theta, \qquad b_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta)\sin\nu\theta\,d\theta$$

for $\nu = 1, 2, \cdots, n$, we have

$$P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \sum_{\nu=1}^{n}(R^\nu - 1)(a_\nu\cos\nu\phi + b_\nu\sin\nu\phi)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\phi)\left(\sum_{\nu=1}^{n}(R^\nu-1)\cos\nu\theta\right)d\theta.$$

Since $t(\theta+\phi)$ is a trigonometric polynomial of degree n in θ we may add to the sum terms in $e^{i\nu\theta}$ $(|\nu| > n)$ without changing the value of the integral. Thus, we can write

$$P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\phi)\left[(R^n-1)\cos n\theta + \sum_{\nu=1}^{n-1}(R^\nu-1)\{\cos\nu\theta - \cos(2n-\nu)\theta\}\right]d\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t(\theta+\phi)(\cos n\theta)\left\{R^n - 1 + 2\sum_{j=1}^{n-1}(R^{n-j}-1)\cos j\theta\right\}d\theta.$$

Once again, we can replace $\cos n\theta$ by $h(n\theta)$, where $h(\theta)$ is simply required to be continuous, periodic (with period 2π) and have a Fourier series of the form $b(\theta) \sim \cos\theta + c_2 \cos 2\theta + d_2 \sin 2\theta + \cdots$. If $0 < \rho < 1$, we may choose

$$b(\theta) = \frac{1}{4\rho}\left(\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2} - \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2}\right)$$

$$= \cos\theta + \rho^2\cos 3\theta + \rho^4\cos 5\theta + \cdots,$$

the series being uniformly convergent. We will then have

$$P_n(Re^{i\phi}) - P_n(e^{i\phi}) = \lim_{\rho\to 1}\frac{1}{4\pi\rho}\int_{-\pi}^{\pi} t(\theta+\phi)\left\{R^n - 1 + 2\sum_{j=1}^{n-1}(R^{n-j}-1)\cos j\theta\right\}$$

$$\times\left(\frac{1-\rho^2}{1-2\rho\cos n\theta+\rho^2} - \frac{1-\rho^2}{1+2\rho\cos n\theta+\rho^2}\right)d\theta.$$

Now, by a well-known property of Poisson's kernel, we have, for every continuous periodic function $F(\theta)$ with period 2π,

$$\lim_{\rho\to 1}\frac{1}{4\pi\rho}\int_{-\pi}^{\pi} F(\theta)\left(\frac{1-\rho^2}{1-2\rho\cos n\theta+\rho^2} - \frac{1-\rho^2}{1+2\rho\cos n\theta+\rho^2}\right)d\theta$$

$$= \frac{1}{2n}\sum_{k=1}^{2n}(-1)^k F(k\pi/n).$$

Applying this to the function

$$F(\theta) = t(\theta+\phi)\left\{R^n - 1 + 2\sum_{j=1}^{n-1}(R^{n-j}-1)\cos j\theta\right\}$$

we get the interpolation formula (2.1). The relation (2.2) follows from (2.1) if we set $P_n(z) = z^n$ and $\phi = 0$.

The idea of the above proof comes from [19].

The fact that the coefficients A_k are positive is clear from Lemma 2 below.

Lemma 2. *For* $R > 1$

$$A_k = (R^n - 1) + 2\sum_{j=1}^{n-1}(R^{n-j}-1)\cos j\frac{k\pi}{n} \geq R^{n-2}(R-1)^2. \tag{2.3}$$

Proof. If $\lambda_n \geq 0$, $\lambda_{n-1} - 2\lambda_n \geq 0$ and $\lambda_{j-1} - 2\lambda_j + \lambda_{j+1} \geq 0$ for $j = 1, 2, \cdots, n-1$ then (see [17, p. 75])

$$\lambda_0 + 2\sum_{j=1}^{n}\lambda_j\cos j\theta \geq 0$$

for all real θ. This result may be applied with $\lambda_j = R^{n-j} - 1$ for $1 \leq j \leq n$ and $\lambda_0 = 2R^{n-1} - R^{n-2} - 1$. Thus

$$\chi(\theta) = (2R^{n-1} - R^{n-2} - 1) + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \cos j\theta \geq 0$$

for all real θ and $A_k \equiv R^{n-2}(R-1)^2 + \chi(k\pi/n) \geq R^{n-2}(R-1)^2$.

For the proof of Theorems 1 and 3 we shall need the following

Lemma 3. *Let $P_n(z)$ be a polynomial of degree n. If* $\max_{|z|=1} |P_n(z)| = 1$ *and* $|P_n(1)| = a$, *then, for* $|\theta| \leq (1-a)/n$,

(2.4) $$|P_n(e^{i\theta})| \leq (1+a)/2.$$

Proof. Without loss of generality we may suppose that $P_n(z) \neq 0$ in $|z| < 1$. In fact, if $z_1, z_2, \cdots, z_k$ are the zeros of $P_n(z)$ in $|z| < 1$ then

$$P_n^*(z) = P_n(z) \prod_{\nu=1}^{k} \frac{1 - \bar{z}_\nu z}{z - z_\nu}$$

is a polynomial of degree n which does not vanish in $|z| < 1$ and $|P_n^*(e^{i\theta})| = |P_n(e^{i\theta})|$ for $0 \leq \theta < 2\pi$. Since

$$P_n(e^{i\theta}) = P_n(1) + \int_1^{e^{i\theta}} P_n'(z)\, dz$$

we have

$$|P_n(e^{i\theta})| \leq a + |e^{i\theta} - 1| \max_{|z|=1} |P_n'(z)|.$$

Hence, if $\max_{|z|=1} |P_n(z)| \leq 1$ and $P_n(z) \neq 0$ in $|z| < 1$ (as we may assume), (1.3) implies

$$|P_n(e^{i\theta})| \leq a + n|\sin(\theta/2)| \leq a + n|\theta|/2 \leq (1+a)/2$$

provided that $|\theta| \leq (1-a)/n$. This concludes the proof of Lemma 3.

Whereas the bound in (2.4) is not attained for any $\theta \neq 0$ (unless $a = 1$) the following lemma gives sharp estimate for $|P_n(e^{i\theta})|$ for every θ in $[-\pi/n, \pi/n]$.

Lemma 4. *If $P_n(z)$ is a polynomial of degree n such that* $\max_{0 \leq k \leq n-1} |P_n(e^{i((2k+1)/n)\pi})| \leq 1$ *and* $P_n(1) = 0$, *then for* $|\theta| \leq \pi/n$

(2.5) $$|P_n(e^{i\theta})| \leq |\sin(n\theta/2)|.$$

Proof. This result is, essentially, due to Boas. Consider the trigonometric polynomial of degree n

$$S_n(\theta) = e^{-in\theta} P_n(e^{2i\theta}).$$

Since $\max_{0 \leq k \leq 2n-1} |S_n[(2k+1)\pi/(2n)]| \leq 1$ and $S_n(0) = 0$ it follows from a theorem of Boas (see [6, p. 43]) that $|S_n(\theta)| \leq |\sin n\theta|$ for $|\theta| \leq \pi/(2n)$. In terms of $P_n(z)$ this says that, for $|\theta| \leq \pi/n$, $|P_n(e^{i\theta})| \leq |\sin(n\theta/2)|$ which is the desired estimate.

Lemma 5 below will be used in the proofs of Theorems 2 and 4.

Lemma 5. *The integral*

$$I = \int_0^{\pi/2} - n \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2 (n+1)\theta}{\sin^2 \theta}\right\} d\theta$$

remains bounded as n tends to infinity over the positive integers.

Proof. Break the range of integration $[0, \pi/2]$ into two parts, namely $[0, \pi/(2n)]$, $[\pi/(2n), \pi/2]$. Let the integral over $[0, \pi/(2n)]$ be denoted by I_1 and that over $[\pi/(2n), \pi/2]$ by I_2.

In the range $\pi/(2n) \leq t \leq \pi/2$ we have

$$0 < |\sin (n+1)t|/\{(n+1) \sin t\} \leq \pi/\{2(n+1)t\} < 1$$

so that

$$0 \leq - n \int_{\pi/(2n)}^{\pi/2} \log \left(1 - \frac{1}{(n+1)^2} \frac{\sin^2 (n+1)t}{\sin^2 t}\right) dt$$

$$\leq - n \int_{\pi/(2n)}^{\pi/2} \log \left(1 - \frac{\pi^2}{2(n+1)^2 t^2}\right) dt$$

$$= n \int_{\pi/(2n)}^{\pi/2} \left\{\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\pi}{2(n+1)}\right)^{2k} t^{-2k}\right\} dt$$

$$= n \frac{\pi}{2} \sum_{k=1}^{\infty} \left(\frac{1}{n+1}\right)^{2k} \frac{1}{k(1-2k)} (1 - n^{2k-1}) < \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k(2k-1)} \left(\frac{n}{n+1}\right)^{2k},$$

i.e., for all positive n, $|I_2| < (\pi/2)\sum_{k=1}^{\infty} (1/k(2k-1))$.

On the other hand,

$$\frac{\sin (n+1)t}{\sin t} = e^{-int} + e^{-i(n-2)t} + \cdots + e^{i(n-2)t} + e^{int}$$

so that

$$\frac{\sin (n+1)t}{\sin t} = 1 + 2 \sum_{k=1}^{n/2} \cos 2kt$$

if n is even and

$$\frac{\sin (n+1)t}{\sin t} = 2 \sum_{k=0}^{(n-1)/2} \cos (2k+1)t$$

in case n is odd. For $0 \leq x \leq 1$,

$$\cos x = 1 - x^2/2 + x^4/24 - x^6/720 + \cdots \leq 1 - x^2/2 + x^4/24 \leq 1 - 11\, x^2/24.$$

Hence, for $0 \leq t \leq \pi/(2n)$, we have

$$\cos 2kt \leq 1 - 11\, k^2t^2/6 \qquad (1 \leq k \leq j < n/\pi)$$

and

$$\cos (2k+1)t \leq \; 1 - 11\,(2k+1)^2t^2/24 \qquad (0 \leq k \leq j < (2n-\pi)/2\pi).$$

Thus, for even n and $j < n/\pi$,

$$\frac{\sin (n+1)t}{\sin t} \leq 1 + 2\left(j - \frac{11}{6}t^2 \sum_{k=1}^{j} k^2\right) + 2(n/2 - j) = (n+1) - \frac{11}{8}t^2 j(j+1)(2j+1),$$

whereas, for odd n and $j < (2n-\pi)/2\pi$,

$$\frac{\sin (n+1)t}{\sin t} \leq 2\left\{j + 1 - \frac{11}{24}t^2 \sum_{k=0}^{j} (2k+1)^2\right\} + 2((n-1)/2 - j)$$

$$= (n+1) - \frac{11}{12}t^2(2j/3 + 1)(j+1)(2j+1).$$

In case n is even we may choose $j = [n/\pi] > (n/\pi) - 1$ to conclude that for $0 < t \leq \pi/(2n)$:

$$(2.6) \qquad 0 < \frac{\sin (n+1)t}{(n+1)\sin t} \leq 1 - \frac{11}{9}t^2 \frac{(n-\pi)n(n-\pi/2)}{(n+1)\pi^3} < 1 \quad \text{if } n \geq 4.$$

Therefore for even $n \geq 4$

$$0 \leq I_1 \leq -n\int_0^{\pi/(2n)} \log\left\{1 - \left(1 - \frac{11}{9}t^2\frac{(n-\pi)n(n-\pi/2)}{(n+1)\pi^3}\right)^2\right\} dt$$

$$= -\int_0^{\pi/2} \log\left\{1 - \left(\frac{11}{9}s^2 \frac{(n-\pi)(n-\pi/2)}{n(n+1)\pi^3}\right)^2\right\} ds = O(1).$$

For odd n, we may choose $j = [(n-\pi)/\pi]$ to obtain, corresponding to (2.6),

$$(2.7) \quad 0 < \frac{\sin (n+1)t}{(n+1)\sin t} \leq 1 - \frac{11}{36}t^2 \frac{(2n-\pi)(n-\pi)(2n-3\pi)}{(n+1)\pi^3} < 1 \quad \text{if } n \geq 5,$$

and then argue as before.

For the proof of Theorem 2 we shall also need

Lemma 6. *The integral*

$$-n\int_{\pi/4}^{\pi/2} \log\left(1 - \frac{1}{(n+1)^2}\,\frac{\sin^2(n+1)t}{\sin^2 t}\right)\frac{dt}{1+\cos 2t}$$

remains bounded as n *tends to infinity over the odd positive integers.*

Proof. Write the integral as

$$\left(\int_{\pi/4}^{\pi/2-\pi/(2n)} + \int_{\pi/2-\pi/(2n)}^{\pi/2}\right) - n \log\left(1 - \frac{1}{(n+1)^2}\frac{\sin^2(n+1)t}{\sin^2 t}\right)\frac{dt}{1+\cos 2t}.$$

For the second integral, we have, since n is odd

$$0 \le \int_{\pi/2-\pi/(2n)}^{\pi/2} - n \log\left(1 - \frac{1}{(n+1)^2}\frac{\sin^2(n+1)t}{\sin^2 t}\right)\frac{dt}{1+\cos 2t}$$

$$= \int_0^{\pi/(2n)} - n \log\left(1 - \frac{1}{(n+1)^2}\frac{\sin^2(n+1)t}{\cos^2 t}\right)\frac{dt}{2\sin^2 t}$$

$$\le \frac{\pi^2}{8}\int_0^{\pi/(2n)} - n \log\left(1 - \frac{t^2}{\cos^2 t}\right)\frac{dt}{t^2}$$

$$= \frac{\pi^2}{8} n \int_0^{\pi/(2n)} \sum_{k=1}^{\infty} \frac{1}{k}\frac{t^{2(k-1)}}{\cos^{2k} t}\, dt = O(1).$$

The first integral can be estimated as follows.

$$0 \le \int_{\pi/4}^{\pi/2-\pi/(2n)} - n \log\left(1 - \frac{1}{(n+1)^2}\frac{\sin^2(n+1)t}{\sin^2 t}\right)\frac{dt}{1+\cos 2t}$$

$$\le \int_{\pi/4}^{\pi/2-\pi/(2n)} - n \log\left(1 - \frac{\pi^2}{(n+1)^2 4t^2}\right)\frac{dt}{2\sin^2(t-\pi/2)}$$

$$\le \frac{\pi^2}{8}\int_{-\pi/4}^{-\pi/(2n)} - n \log\left(1 - \frac{\pi^2}{(n+1)^2 4(t+\pi/2)^2}\right)\frac{dt}{t^2}$$

$$= \frac{\pi^2}{8}\int_{-\pi/4}^{-\pi/(2n)} n\left\{\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\pi}{2(n+1)}\right)^{2k}\frac{1}{t^2(t+\pi/2)^{2k}}\right\}dt$$

$$\le \frac{\pi^2}{8} n \sum_{k=1}^{\infty}\left\{\frac{1}{k}\left(\frac{\pi}{2(n+1)}\right)^{2k}\int_{-\pi/4}^{-\pi/(2n)}\frac{dt}{(\pi/4)^{2k}t^2}\right\}$$

$$< \frac{\pi n^2}{(n+1)^2}\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left(\frac{2}{n+1}\right)^{2k} = O(1).$$

The next lemma is, in fact, the well-known interpolation formula of M. Riesz [16] expressing the derivative of a trigonometric polynomial in terms of the values of the polynomial at $2n$ different points.

Lemma 7. *If $S(\theta)$ is a trigonometric polynomial of degree n then*

$$S'(\theta) = \frac{1}{2n}\sum_{k=1}^{2n}\frac{(-1)^k}{1-\cos\{(2k+1)/2n\}\pi}\, S\left(\theta + \frac{2k+1}{2n}\pi\right). \tag{2.8}$$

Besides,

(2.9) $$\frac{1}{2n}\sum_{k=1}^{2n}\frac{1}{1-\cos\{(2k+1)/2n\}\pi}=n.$$

There is a corresponding interpolation formula for entire functions of exponential type. We shall need it for the proof of the first inequality appearing in the statement of Theorem 9 and it reads as follows (see [3, p. 210]).

Lemma 8. *If $f(z)$ is an entire function of exponential type τ bounded on the real axis, then for all real x*

(2.10) $$f'(x)=\tau(4/\pi^2)\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(2n+1)^2}f\left(x+\frac{2n+1}{2\tau}\pi\right).$$

Further,

(2.11) $$(4/\pi^2)\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2}=1.$$

The next interpolation formula will be needed for the proof of the second inequality appearing in the statement of Theorem 9 as well as for the proof of Theorem 10.

Lemma 9. *If $f(z)$ is an entire function of exponential type τ bounded on the real axis, such that $h(\pi/2)\le 0$, then*

(2.12) $$f(x+iy)=\tau y\sum_{k=-\infty}^{\infty}\frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2+(k\pi)^2}f\left(x+\frac{k\pi}{\tau}\right).$$

Proof. Suppose first that $f(z)=\int_0^\tau e^{izt}\phi(t)dt$ with $\phi\in L^2(0,\tau)$. Setting

$$g(t)=\begin{cases}e^{-yt} & \text{if } 0\le t\le\tau,\\ e^{yt} & \text{if } -\tau\le t\le 0,\end{cases}\quad\text{and}\quad \psi(t)=\begin{cases}e^{ixt}\phi(t) & \text{if } 0\le t\le\tau,\\ 0 & \text{if } -\tau\le t\le 0,\end{cases}$$

we can write

(2.13) $$f(x+iy)=\int_{-\tau}^{\tau}g(t)\psi(t)\,dt.$$

Now let $\sum_{k=-\infty}^{\infty}c_k\,e^{ik(\pi/\tau)t}$ be the Fourier series of the continuous function of bounded variation $g(t)$ on the interval $[-\tau,\tau]$. Then

$$c_k=\frac{1}{2\pi}\int_{-\tau}^{\tau}g(t)e^{-ik(\pi/\tau)t}\,dt=\tau y\,\frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2+(k\pi)^2}$$

and

$$g(t)=\tau y\sum_{-\infty}^{\infty}\frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2+(k\pi)^2}\,e^{ik(\pi/\tau)t}$$

for $-\tau\le t\le\tau$. Since the series is uniformly convergent, we may substitute it for $g(t)$ in (2.13) and integrate term by term to obtain

$$f(x+iy)=\tau y\sum_{k=-\infty}^{\infty}\frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2+(k\pi)^2}\int_0^{\tau} e^{i(x+k\pi/\tau)t}\phi(t)\,dt$$

$$=\tau y\sum_{k=-\infty}^{\infty}\frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2+(k\pi)^2}\,f(x+k\pi/\tau),$$

which proves the desired interpolation formula in the special case under consideration. In the general case, consider the function

$$f_\delta(z)=f(z)e^{i\delta z}\,\frac{\sin\delta z}{\delta z}$$

with $\delta>0$. It is clear that $f_\delta(z)$ is an entire function of exponential type $\tau+2\delta$ such that

$$\limsup_{y\to\infty} y^{-1}\log|f_\delta(iy)|\le 0.$$

Besides, it belongs to L^2 on the real axis. Hence by the Paley-Wiener theorem (see [3, p. 103]), it has the form $\int_0^{\tau+2\delta}e^{izt}\phi_\delta(t)dt$ with $\phi_\delta\in L^2[0,\tau+2\delta]$. By the above argument,

$$f_\delta(x+iy)=(\tau+2\delta)y\sum_{k=-\infty}^{\infty}\frac{1-(-1)^k e^{-(\tau+2\delta)y}}{((\tau+2\delta)y)^2+(k\pi)^2}\,f_\delta\left(x+k\frac{\pi}{\tau+2\delta}\right)$$

and we obtain (2.12) on letting δ tend to zero.

This kind of reasoning has been used before for proving certain other interpolation formulas (see for instance [4], [13]).

For the proof of Theorem 9 we shall also need:

Lemma 10. *Let $f(z)$ be an entire function of exponential type τ. If $f(0)=0$,*

$$\sup_{-\infty<x<\infty}|f(x)|\le 1 \quad\textit{and}\quad h_f(\pi/2)=\limsup_{y\to\infty}y^{-1}\log|f(iy)|\le 0$$

then for all real x

$$|f(x)|\le(\tau/2)|x|. \tag{2.14}$$

Proof. An entire function of order less than 1 which is bounded on the real axis is necessarily a constant. If $f(0)=0$ then it is identically zero and (2.14) is trivially true.

So let $f(z)$ be an entire function of order 1 type τ. If $h_f(\pi/2)=c$ let

$$F(z)=e^{-i((\tau-c)/2)z}z^{-1}f(z).$$

The function $F(z)$ is an entire function of order 1 type $\frac{1}{2}(\tau+c)$ and $h_F(\pi/2)=h_F(-\pi/2)=\frac{1}{2}(\tau+c)$. Let x_0 be a point of the real axis where $|F(x_0)|=\max_{-\infty<x<\infty}|F(x)|$. Such a point exists since $|F(x)|$ tends to zero as $|x|\to\infty$.

Choose γ such that $e^{i\gamma}F(x_0)$ is positive. If $e^{i\gamma}F(z) = \sum_{n=0}^{\infty} a_n z^n$ then the function

$$G(z) = \sum_{n=0}^{\infty} (\operatorname{Re} a_n) z^n$$

is an entire function of exponential type $\frac{1}{2}(\tau + c)$. Besides it is real for real x and

$$|G(x)| = |\operatorname{Re}\{e^{i\gamma}F(x)\}| \le |x|^{-1}.$$

The function $H(z) = z\,G(z)$ is therefore an entire function of exponential type $\frac{1}{2}(\tau + c)$ which is real for real x and whose modulus is bounded by 1 on the real axis. According to a theorem of Duffin and Schaeffer (see [9, p. 555]) we have for all real x

$$((\tau + c)/2)^2 H^2(x) + \{H'(x)\}^2 \le ((\tau + c)/2)^2,$$

i.e.

$$((\tau + c)/2)^2 x^2 G^2(x) + \{G(x) + xG'(x)\}^2 \le ((\tau + c)/2)^2.$$

At the point x_0 where $G(x)$ attains its maximum we have $G'(x_0) = 0$ and therefore

$$((\tau + c)/2)^2 x_0^2 G^2(x_0) + G^2(x_0) \le ((\tau + c)/2)^2$$

or

$$G(x_0) \le ((\tau + c)/2)/\{1 + ((\tau + c)/2)^2 x_0^2\}^{1/2} \le (\tau + c)/2 \le \tau/2.$$

Since

$$G(x_0) = \max_{-\infty<x<\infty} |G(x)| = \max_{-\infty<x<\infty} |e^{i\gamma}F(x)| = \max_{-\infty<x<\infty} |x|^{-1}|f(x)|$$

we get $|f(x)| \le (\tau/2)|x|$ for all real x. This proves Lemma 10.

3. Proofs of results announced in §1.

Proof of Theorem 1. Let $P_n(z) \in \mathcal{P}_{n,a}$. There is no loss of generality in supposing that $|P_n(1)| = a$. If $P_n(z) = a_0 + a_1 z + \cdots + a_n z^n$ then

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(e^{i\phi})e^{ik\phi}\,d\phi \qquad (0 \le k \le n).$$

Hence

$$e^{i\theta}P_n'(e^{i\theta}) = \sum_{k=1}^{n} k a_k e^{ik\theta} = \sum_{k=1}^{n} \frac{k}{2\pi}\int_{-\pi}^{\pi} P_n(e^{i\phi})e^{ik(\theta-\phi)}\,d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(e^{i(\theta+t)})e^{-it}\sum_{k=1}^{n} k e^{-i(k-1)t}\,dt.$$

Since $P_n(e^{i(\theta+t)})e^{-it} = a_0 e^{-it} + a_1 e^{i\theta} + a_2 e^{2i\theta}e^{it} + \cdots + a_n e^{in\theta}e^{i(n-1)t}$, we may add to the sum $\Sigma_{k=1}^{n} k\, e^{-i(k-1)t}$ terms in e^{-ikt} with $k \geq n$ without changing the value of the integral *provided* $n > 1$. In particular, noting the identity

$$\left(\sum_{k=1}^{n} z^{k-1}\right)^2 = \sum_{k=1}^{n} kz^{k-1} + \text{higher powers of } z$$

we can write (also if $n = 1$)

$$e^{i\theta}P_n'(e^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(e^{i(\theta+t)})e^{-it}\{1 + e^{-it} + e^{-i2t} + \cdots + e^{-i(n-1)t}\}^2\, dt.$$

Consequently,

$$|P_n'(e^{i\theta})| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |P_n(e^{i(\theta+t)})|\{(\sin nt/2)/(\sin t/2)\}^2\, dt.$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt = n$$

we find, using (2.4) together with $|P_n(z)| \leq 1$ $(|z| \leq 1)$ that

$$\begin{aligned}
|P_n'(e^{i\theta})| &\leq \frac{1}{2\pi}\left(\int_{(1-a)/n\leq|\theta+t|\leq\pi} + \int_{|\theta+t|\leq(1-a)/n}\right)|P_n(e^{i(\theta+t)})|\left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt \\
&\leq n - \frac{1}{2\pi}\int_{|\theta+t|\leq(1-a)/n}\{1 - |P_n(e^{i(\theta+t)})|\}\left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt \\
&\leq n - \frac{1-a}{4\pi}\int_{|\theta+t|\leq(1-a)/n}\left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt \\
&\leq n - \frac{1-a}{4\pi}\int_{-\theta-(1-a)/n}^{-\theta+(1-a)/n} \sin^2\frac{nt}{2}\, dt \\
&= n - \frac{1-a}{4\pi n}\{(1-a) - \cos n\theta \sin(1-a)\} \\
&\leq n - \frac{1-a}{4\pi n}\{(1-a) - \sin(1-a)\}.
\end{aligned}$$

This proves Theorem 1.

Proof of Theorem 2. Since the trigonometric polynomial

$$1 - \frac{1}{(n+1)^2}\left(\sum_{k=0}^{n} e^{ik\theta}\right)\left(\sum_{l=0}^{n} e^{-il\theta}\right) = 1 - \frac{1}{(n+1)^2}\,\frac{\sin^2(n+1)\theta/2}{\sin^2\theta/2}$$

is nonnegative for all real θ, it follows from a classical result of Fejér and Riesz (see [1, p. 152]) that the equation

$$|P_n(e^{i\theta})|^2 + \frac{1}{(n+1)^2}\,\frac{\sin^2(n+1)\theta/2}{\sin^2\theta/2} = 1$$

defines a family of polynomials belonging to $\mathcal{P}_{n,0}$. Let $\overset{+}{P}_n(z)$ be the one which does not vanish in $|z| < 1$ and assumes a positive value at the origin. For $|z| < 1$

$$\log \overset{+}{P}_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} \log |\overset{+}{P}_n(e^{it})|\, dt. \tag{3.1}$$

In fact, given z_0 $(|z_0| < 1)$

$$\log \overset{+}{P}_n(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho e^{it}+z}{\rho e^{it}-z} \log |\overset{+}{P}_n(\rho e^{it})|\, dt$$

for $|z_0| < \rho < 1$ by Poisson-Schwarz formula. Since

$$1 \geq |\rho e^{it} - 1| \geq 2\sqrt{\rho}|\sin t/2| \geq 2\sqrt{\rho}|t| \geq \sqrt{2}|t|/\pi$$

for $\rho \geq ½$ and hence

$$|\log|\rho e^{it} - 1|| \leq \log(\pi/\sqrt{2}) + |\log|t||,$$

Lebesgue's dominated convergence theorem allows us to let ρ tend to 1 under the integral sign giving (3.1). Besides, the function $\log |\overset{+}{P}_n(e^{it})|$ being integrable over $[0, 2\pi]$, we can differentiate (3.1) under the integral sign and get

$$\overset{+}{P}{}_n'(z) = \overset{+}{P}_n(z) \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2} \log |\overset{+}{P}_n(e^{it})|^2\, dt \tag{3.2}$$

for $|z| < 1$. But we are interested in $\overset{+}{P}{}_n'(-1)$ and will like (3.2) to hold for $z = -1$ as well. We observe that for *odd* n it is indeed the case. For odd n, the function

$$\log \{\overset{+}{P}_n(e^{iz})\}^2 = \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2 (n+1)z/2}{\sin^2 z/2}\right\}$$

is analytic in $|z - \pi| < \rho_0$ for some $\rho_0 > 0$ and has a double zero at $z = \pi$. Let $\delta = ½ \min(\rho_0, 1)$. Then for $|t - \pi| \leq \delta$:

$$\log |\overset{+}{P}_n(e^{it})|^2 = (t-\pi)^2 \alpha(t)$$

where $\alpha(t)$ is continuous on $|t - \pi| \leq \delta$. Hence, if

$$f_\rho(t) = \frac{e^{it}}{(e^{it}+\rho)^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2 (n+1)t/2}{\sin^2 t/2}\right\}$$

then, for $0 < \rho < 1$, $|t - \pi| \leq \delta$,

$$|f_\rho(t)| = \frac{(t-\pi)^2\alpha(t)}{|e^{it}+\rho|^2} \leq \frac{(t-\pi)^2\alpha(t)}{4\rho \sin^2((t-\pi)/2)} = \frac{1}{\rho}\beta(t)$$

where the function $\beta(t)$ is integrable over $|t - \pi| \leq \delta$. For $0 < \rho < 1$, $\delta < |t - \pi| \leq \pi$, we have

$$|f_\rho(t)| \leq \frac{-1}{1 + 2\rho \cos(\pi - \delta) + \rho^2} \log \left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2 (n+1)t/2}{\sin^2 t/2}\right\};$$

Once again, Lebesgue's dominated convergence theorem shows that

$$\lim_{\rho \to 1} \frac{1}{2\pi} \int_0^{2\pi} f_\rho(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f_1(t)\, dt,$$

i.e.

$$\text{(3.3)} \qquad \overset{+}{P}{}'_n(-1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it}+1)^2} \log\left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t/2}{\sin^2 t/2}\right\} dt.$$

Formula (3.3) can be rewritten as

$$|\overset{+}{P}{}'_n(-1)| = -\overset{+}{P}{}'_n(-1) = -\frac{1}{\pi} \int_0^{\pi/2} \log\left\{1 - \frac{1}{(n+1)^2} \frac{\sin^2(n+1)t}{\sin^2 t}\right\} \frac{dt}{1+\cos 2t}.$$

Therefore, Lemmas 5 and 6 show that there exists a positive constant c such that $|\overset{+}{P}{}'_n(-1)| \le c/n$. If n is even and $m = n/2$ is odd, $g_n(z) = (\overset{+}{P}_m(z))^2$ is a polynomial of degree n such that $g_n(1) = 0$ and $\max_{|z|=1} |g_n(z)| = g_n(-1) = 1$. Besides,

$$|g'_n(-1)| = 2|\overset{+}{P}{}'_m(-1)| \le 4c/n.$$

If n is even and $m = n/2$ is also even, the polynomial $h_n(z) = \overset{+}{P}_{m-1}(z)\overset{+}{P}_{m+1}(z) \in \mathcal{P}_{n,0}$ (with $h_n(-1) = 1$) and

$$|h'_n(-1)| = -h'_n(-1) \le 4cn/(n^2-4).$$

Thus, there exist a positive constant c_1 and, for each n, a polynomial $P_n(z) \in \mathcal{P}_{n,0}$ with $P_n(1) = 0$, $P_n(-1) = 1$, $P'_n(-1) < 0$ and

$$\text{(3.4)} \qquad |P'_n(-1)| \le c_1/n.$$

Now, let $G_n(z) = (-1)^n z^n P_n(1/z)$ which belongs to $\mathcal{P}_{n,0}$ with $G_n(1) = 0$, $G_n(-1) = 1$. Besides, using (3.4) we see that

$$|G'_n(-1)| = -G'_n(-1) \ge n - c_1/n.$$

Finally, given $a \in [0, 1]$, set $H_n(z) = (-1)^n a\, z^n + (1-a)G_n(z)$. Then $H_n(z) \in \mathcal{P}_{n,a}$ (with $|H_n(1)| = a$ and $H_n(-1) = 1$) and

$$|H'_n(-1)| = |na - (1-a)G'_n(-1)| = na + (1-a)|G'_n(-1)| \ge n - c_1(1-a)/n.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Let $P_n(z) \in \mathcal{P}_{n,a}$. Without loss of generality we may suppose $P_n(1) = a$. Let $m = [\pi/(1-a)] + 1$; and consider the polynomial $P^*(z) = (P_n(z))^m$. By Lemma 1, we have for $R > 1$

$$P^*(Re^{i\phi}) = P^*(e^{i\phi}) + \frac{1}{2mn} \sum_{k=1}^{2mn} (-1)^k A_k P^*(e^{i(\phi + k\pi/mn)})$$

where

$$\frac{1}{2mn} \sum_{k=1}^{2mn} A_k = R^{mn} - 1,$$

and according to Lemma 2 $A_k \geq R^{mn}(1 - R^{-1})^2$. By Lemma 3

$$|P^*(e^{i\theta})| \leq ((1 + a)/2)^m < ((1 + a)/2)^{\pi/(1-a)} < e^{-\pi/2} \quad \text{for } |\theta| < (1 - a)/n.$$

At least two of the points $\phi_k = \phi + k\pi/mn$, $k = 1, 2, \cdots, 2mn$ (say ϕ_{j_1}, ϕ_{j_2}) lie in the angle $|\theta| \leq \pi/mn < (1 - a)/n$. Hence from above

$$|P^*(Re^{i\phi})| \leq 1 + \frac{1}{2mn} \sum_{k=1}^{2mn} A_k |P^*(e^{i(\phi + k\pi/mn)})|$$

$$\leq 1 + \frac{1}{2mn} \sum_{k=1}^{2mn} A_k - \frac{1}{2mn} (1 - e^{-\pi/2})(A_{j_1} + A_{j_2})$$

$$\leq R^{mn}\{1 - (mn)^{-1}(1 - e^{-\pi/2})(1 - R^1)^2\}$$

and

$$|P_n(Re^{i\phi})| = |P^*(Re^{i\phi})|^{1/m} \leq R^n\{1 - (mn)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2\}^{1/m}$$
$$\leq R^n\{1 - (m^2n)^{-1}(1 - e^{-\pi/2})(1 - R^{-1})^2\}$$

by virtue of the inequality $(1 - x)^a < 1 - ax$ valid for $0 < a < 1$ and $0 < x < 1$. Since $m = [\pi/(1 - a)] + 1$ we have, in fact, proved (1.7).

Proof of Theorem 4. We again consider the polynomial

$$\overset{+}{P}_n(z) = \sum_{k=0}^{n} \overset{+}{a}_k z^k \in \mathcal{P}_{n,0}$$

defined by the relation

$$|P_n(e^{i\theta})|^2 = 1 - \frac{1}{(n + 1)^2} \frac{\sin^2(n + 1)\theta/2}{\sin^2\theta/2}$$

not vanishing in $|z| < 1$ and assuming a positive value at the origin. From (3.1)

$$0 < -\log \overset{+}{a}_0 = \frac{1}{\pi}\int_0^{\pi/2} -\log\left\{1 - \frac{1}{(n + 1)^2} \frac{\sin^2(n + 1)\theta}{\sin^2\theta}\right\} d\theta.$$

Hence according to Lemma 5 there exists a constant c_2 such that $-\log \overset{+}{a}_0 < (c_2 - 1)/n$, i.e. $\overset{+}{a}_0 > \exp\{-(c_2 - 1)/n\} > 1 - (c_2 - 1)/n$. Now if

$$\overset{+}{Q}_n(z) = z^n \overset{+}{P}_n(1/z) = \sum_{k=0}^{n} \overset{+}{a}_k z^{n-k}$$

then

$$|\overset{+}{Q}_n(-R)| \geq \overset{+}{a}_0 R^n - \sum_{k=1}^{n} |a_k| R^{n-k}$$

$$\geq \overset{+}{a}_0 R^n - nR^{n-1} \quad (\text{since } |\overset{+}{Q}_n(z)| \leq 1 \text{ for } |z| = 1)$$

$$> R^n\{1 - (c_2 - 1)/n - n/R\} > R^n(1 - c_2/n)$$

if $R > n^2$. The polynomial $\overset{+}{Q}_n(z)$ is real on the real axis and for odd n, $\overset{+}{Q}_n(-R) < 0$ if $R > 1$ (note that $\overset{+}{Q}_n(z)$ does not vanish for $|z| > 1$ and $Q_n(-R)$ tends to $-\infty$ as $R \to \infty$). Hence, if n is odd and

$$V_n(z) = (1-a)\{[a/(1-a)]z^n + \overset{+}{Q}_n(z)\}$$

then $|V_n(z)| \leq 1$ for $|z| \leq 1$, $V_n(-1) = -1$ and $V_n(1) = a$. Thus $V_n(z) \in \mathcal{P}_{n,a}$ and for $R > n^2$:

$$\begin{aligned}|V_n(-R)| &= (1-a)\left\{\frac{a}{1-a}R^n + |\overset{+}{Q}_n(-R)|\right\}\\ &> aR^n + (1-a)R^n(1-c_2/n) = R^n\{1 - c_2(1-a)/n\}.\end{aligned}$$

If n is even, we may consider

$$W_n(z) = (1-a)\{(a/(1-a))z^n + z\overset{+}{Q}_{n-1}(z)\}$$

instead of $V_n(z)$. Clearly $W_n(z) \in \mathcal{P}_{n,a}$ and for $R > n^2$

$$|W_n(-R)| > R^n\{1 - c_2(1-a)/n\}.$$

Proof of (1.5″). Let $P_n(z) \in \mathcal{P}_{n,0}$, where we may suppose $P_n(1) = 0$. Since $P_n(e^{i\theta})$ is a trigonometric polynomial of degree n we have by Lemma 7

$$|P_n'(e^{i\theta})| \leq \frac{1}{2n}\sum_{k=1}^{2n} \frac{1}{1-\cos(2k+1)\pi/2n}\,|P_n(e^{i(\theta+(2k+1)\pi/2n)})|.$$

One of the points $e^{i(\theta(2k+1)\pi/2n)}$ $(1 \leq k \leq 2n)$ is 1 if θ is an odd multiple of $\pi/(2n)$ and hence for such values of θ

$$\begin{aligned}|P_n'(e^{i\theta})| &\leq n - \frac{1}{2n}\,\frac{1}{1-\cos(2j+1)\pi/2n} \qquad \text{(for some integer } j)\\ &\leq n - \frac{1}{2n}\,\frac{1}{1+\cos(\pi/2n)} < n - \frac{1}{4n}.\end{aligned}$$

If θ is not an odd multiple of $\pi/(2n)$ then precisely two of the points $\theta + (2k+1)\pi/2n$ (say $\theta + \pi(2j_1+1)/2n$, $\theta + \pi(2j_2+1)/2n$) lie in the interval $(-\pi/n, \pi/n)$ and by Lemma 4

$$\begin{aligned}|P_n'(e^{i\theta})| &\leq n - \frac{1}{2n}\left\{\frac{1-|\sin n(\theta+(2j_1+1)\pi/2n)/2|}{1-\cos(2j_1+1)\pi/2n} + \frac{1-|\sin n(\theta+(2j_2+1)\pi/2n)/2|}{1-\cos(2j_2+1)\pi/2n}\right\}\\ &= n - \frac{1}{2n}\left\{\frac{1-|\sin n(\theta+(2j_1+1)\pi/2n)/2|}{1-\cos(2j_1+1)\pi/2n} + \frac{1-|\cos n(\theta+(2j_1+1)\pi/2n)/2|}{1-\cos(2j_2+1)\pi/2n}\right\}\\ &\leq n - \frac{1}{2n}\left\{\frac{2-|\sin n(\theta+(2j_1+1)\pi/2n)/2| - |\cos n(\theta+(2j_1+1)\pi/2n)/2|}{1+\cos(\pi/2n)}\right\}\\ &\leq n - \frac{2-\sqrt{2}}{4n}\end{aligned}$$

which completes the proof of (1.5″).

Proof of (1,7″). Let $P_n(z) \in \mathcal{P}_{n,0}$. Clearly, we may suppose $P_n(1) = 0$. In this case two and possibly three of the interpolation points in the formula

$$P_n(Re^{i\phi}) = P_n(e^{i\phi}) + \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k A_k P_n e^{i(\phi + k\pi/n)}$$

lie in $|\theta| \leq \pi/n$ where by Lemma 4 $|P_n(e^{i\theta})| \leq |\sin(n\theta/2)|$. If three of the points (say $\phi + j_1\pi/n$, $\phi + j_2\pi/n$, $\phi + j_3\pi/n$) fall in the interval $[-\pi/n, \pi/n]$ then they have to be $-\pi/n$, 0, π/n and

$$|P_n(Re^{i\phi})| \leq R^n - \frac{1}{2n}\{A_{j_1}(1 - |\sin -\pi/2|) + A_{j_2}(1 - |\sin 0|) + A_{j_3}(1 - |\sin \pi/2)|\}$$

$$\leq R^n\{1 - (1 - R^{-1})^2/2n\}.$$

If there are only two points (say $\phi + j_1\pi/n$, $\phi + j_2\pi/n$) in $[-\pi/n, \pi/n]$, then

$$|\sin n(\phi + j_2\pi/n)/2| \equiv |\cos n(\phi + j_1\pi/n)/2|$$

and hence

$$|P_n(Re^{i\phi})| \leq R^n - \frac{1}{2n}\{A_{j_1}(1 - |\sin n(\phi + j_1\pi/n)/2| + A_{j_2}(1 - |\cos n(\phi + j_1\pi/n)/2|)\}$$

$$\leq R^n\{1 - (2 - \sqrt{2})(1 - R^{-1})^2/2n\}$$

which completes the proof of (1.7″).

Proof of Theorem 5. In the proof of Theorem 2 it was shown that for each positive integer n there exists a polynomial $G_n(z) \in \mathcal{P}_{n,0}$ with $G_n(1) = 0$, $G_n(-1) = 1$ and $|G_n'(-1)| = -G_n'(-1) \geq n - c_1/n$ where c_1 is an absolute constant. Given $\epsilon > 0$, and a positive integer n let $m = [\sqrt{n}]$, $k = [\epsilon\sqrt{n}/2c_1]$, $j = n - [\sqrt{n}][\epsilon\sqrt{n}/2c_1]$. Then $P_n(z) = \{G_m(z)\}^k G_j(z)$ has a zero of multiplicity $k + 1 > \epsilon\sqrt{n}/2c_1 = \delta\sqrt{n}$ at $z = 1$. Moreover

$$|P_n'(-1)| = |kG_m'(-1) + G_j'(-1)|$$

$$\geq k(m - c_1/m) + (j - c_1/j) = n - c_1(k/m + 1/j) > n - \epsilon$$

if n is large enough, say $n > n_0$.

Proof of Theorem 6. If $Q_n(z) = z^n \overline{P_n(1/\bar{z})}$ then by the hypothesis of Theorem 6

$$Q_n(1) = P_n(1) = 0, \qquad |Q_n(\exp ij\pi/n)| = |P_n(\exp ij\pi/n)| \leq 1$$

$$\text{for } j = 1, 2, \cdots, 2n - 1.$$

Hence applying Lemma 1 to $Q_n(z)$ we get for $R > 1$

$$|Q_n(R)| \leq \frac{1}{2n} \sum_{k=1}^{2n} A_k - \frac{1}{2n} A_{2n} = R^n - 1 - \frac{1}{2n} A_{2n}$$

$$= R^n - 1 - \frac{1}{2n} \left\{ R^n - 1 + 2 \sum_{j=1}^{n-1} (R^{n-j} - 1) \right\}$$

$$= R^n - 1 - \frac{1}{2n} \left\{ (R^n - 1) \frac{R+1}{R-1} - 2n \right\}.$$

Thus

$$|P_n(1/R)| \leq \left\{ (1 - R^{-n}) \left(1 - \frac{1}{2n} \frac{1 + R^{-1}}{1 - R^{-1}} \right) + R^{-n} \right\}.$$

If $0 < \omega \leq n$ then $(1 - \omega/n)^{-1} > 1$ and hence, in particular,

$$|P_n(1 - \omega/n)| \leq 1 - (1/\omega - 1/2n) + (1/\omega - 1/2n)(1 - \omega/n)^n.$$

The above reasoning also shows that *if* $|P_n(1)| = a < 1$ *and* $|P_n(\exp ij\pi/n)| \leq 1$ *for* $j = 1, 2, \cdots, 2n - 1$ *then for* $0 < \omega \leq n$

(3.5) $\quad |P_n(1 - \omega/n)| \leq 1 - (1 - a)(1/\omega - 1/2n) + (1 - a)(1/\omega - 1/2n)(1 - \omega/n)^n.$

Besides, the formula obtained by equating the real parts on the two sides of (2.1) may be applied to the polynomial $Q_n(z) = z^n \overline{P_n(1/\bar{z})}$ to prove in exactly the same way as above that:

(i) *if* $P_n(z)$ *is a polynomial of degree* n *such that* $|\text{Re } P_n(1)| = a < 1$, $|\text{Re } P_n(\exp ij\pi/n)| \leq 1$ *for* $j = 1, 2, \cdots, 2n - 1$ *then for* $0 < \omega \leq n$

(3.6) $\quad |\text{Re } P_n(1 - \omega/n)| \leq 1 - (1 - a)(1/\omega - 1/2n) + (1 - a)(1/\omega - 1/2n)(1 - \omega/n)^n;$

(ii) *if* $P_n(z)$ *is a polynomial of degree* n *such that* $\text{Re } P_n(1) = a < 1$, $\text{Re } P_n(\exp ij\pi/n) \leq 1$ *for* $j = 1, 2, \cdots, 2n - 1$ *then for* $0 < \omega \leq n$

(3.7) $\quad \text{Re } P_n(1 - \omega/n) \leq 1 - (1 - a)(1/\omega - 1/2n) + (1 - a)(1/\omega - 1/2n)(1 - \omega/n)^n.$

Proof of Theorem 7. There is no loss of generality in supposing that the point of the unit circle where the polynomial vanishes is 1, i.e.

(3.8)
$$\sum_{k=0}^{n} a_k = 0.$$

We write the left-hand side of (1.19) as

$$\sum_{k=0}^{n} \lambda_k |a_k|^2 = \lambda_j \sum_{k=0}^{n} |a_k|^2 - \sum_{k=0;k\neq j}^{n} (\lambda_j - \lambda_k)|a_k|^2$$

$$= \lambda_j \sum_{k=0}^{n} |a_k|^2 - \sum_{k=0;k\neq j}^{n} (\lambda_j - \lambda_k - \lambda)|a_k|^2 - \lambda \sum_{k=0;k\neq j}^{n} |a_k|^2,$$

where for the moment λ is a constant such that

$$0 < \lambda < \Lambda = \min_{0 \le k \le n; k \ne j} (\lambda_j - \lambda_k).$$

From (3.8) and Schwarz's inequality we obtain

$$|a_j|^2 = \left| \sum_{k=0; k\ne j}^{n} a_k \right|^2 \le \left(\sum_{k=0; k \ne j}^{n} |a_k| \right)^2$$

$$= \left\{ \sum_{k=0; k\ne j}^{n} (\lambda_j - \lambda_k - \lambda)^{1/2} |a_k| (\lambda_j - \lambda_k - \lambda)^{-1/2} \right\}^2$$

$$\le \sum_{k=0; k\ne j}^{n} (\lambda_j - \lambda_k - \lambda)|a_k|^2 \sum_{k=0; k\ne j}^{n} (\lambda_j - \lambda_k - \lambda)^{-1},$$

so that

$$- \sum_{k=0; k\ne j}^{n} (\lambda_j - \lambda_k - \lambda)|a_k|^2 \le - |a_j|^2 \left\{ \sum_{k=0; k\ne j}^{n} (\lambda_j - \lambda_k - \lambda)^{-1} \right\}^{-1}.$$

Now if λ happens to be the root of the equation (1.20) lying in $(0, \Lambda)$, then

$$\left\{ \sum_{k=0; k\ne j}^{n} (\lambda_j - \lambda_k - \lambda)^{-1} \right\}^{-1} = \lambda$$

and we get

$$\sum_{k=0}^{n} \lambda_k |a_k|^2 \le \lambda_j \sum_{k=0}^{n} |a_k|^2 - \lambda |a_j|^2 - \lambda \sum_{k=0; k\ne j}^{n} |a_k|^2 = (\lambda_j - \lambda) \sum_{k=0}^{n} |a_k|^2$$

which is the desired inequality.

Observe that in (1.19) equality is possible only if

$$\left| \sum_{k=0; k\ne j}^{n} a_k \right| = \sum_{k=0; k\ne j}^{n} |a_k|$$

and equality holds in Schwarz's inequality, i.e. $a_k = a(\lambda_j - \lambda_k - \lambda)^{-1}$ if $k \ne j$ whereas $a_j = - a/\lambda$. Thus the extremal polynomial has the form

$$P_n(z) = a \sum_{k=0}^{n} \frac{1}{\lambda_j - \lambda_k - \lambda} (e^{-i\theta_0} z)^k.$$

The proof of Theorem 7″ is analogous and we omit it.

Proof of Theorem 9. For an entire function $f(z)$ satisfying the hypotheses of the theorem we have by virtue of Lemma 8

$$|f'(x)| \le \tau(4/\pi^2) \sum_{k=-\infty}^{\infty} \frac{1}{(2n+1)^2} |f(x+(2n+1)\pi/2\tau)|$$

for all real x. According to (2.14) $|f(x)| < (\tau/2)|x|$ in $(-2/\tau, 2/\tau)$. If only one of the interpolation points (say $x + (2m+1)\pi/2\tau$) falls in this interval it must, in fact, lie in the subinterval $[-(\pi-2)/\tau, (\pi-2)/\tau]$ and hence by (2.14) in conjunction with (2.11)

$$|f'(x)| \le \tau \left\{1 - \frac{4}{\pi^2} \frac{1}{(2m+1)^2} \left(1 - \frac{\tau}{2} \left| x + \frac{2m+1}{2\tau}\pi \right| \right)\right\} \le \tau \left\{1 - \frac{4}{\pi^2} \frac{2-\pi/2}{(2m+1)^2}\right\}.$$

Since

$$|(2m+1)\pi/2\tau| - |x| \le |x + (2m+1)\pi/2\tau| \le (\pi-2)/\tau$$

we have

$$1/(2m+1)^2 \ge \pi^2/4(\tau|x| + \pi - 2)^2$$

and hence

$$|f'(x)| \le \tau\{1 - (4-\pi)/2(\tau|x| + \pi - 2)^2\}.$$

If, on the other hand, two points $x + (2m+1)\pi/2\tau$, $x + (2m+3)\pi/2\tau$ lie in $(-2/\tau, 2/\tau)$, then

$$|f'(x)| \le \tau \left\{1 - \frac{4}{\pi^2} \left(\frac{1 - \tau|x + (2m+1)\pi/2\tau|/2}{(2m+1)^2} + \frac{1 - \tau|x + (2m+3)\pi/2\tau|/2}{(2m+3)^2}\right)\right\}.$$

Since $|x + (2m+1)\pi/2\tau| < 2/\tau$ and $|x + (2m+3)\pi/2\tau| < 2/\tau$, we have

$$1/(2m+1)^2 > \pi^2/4(\tau|x| + 2)^2, \quad 1/(2m+3)^2 > \pi^2/4(\tau|x| + 2)^2$$

and hence

$$|f'(x)| < \tau\{1 - (4-\pi)/2(\tau|x| + 2)^2\}$$

which completes the proof of (1.36).

To prove inequality (1.37), we use formula (2.12) which we may apply to the constant function 1 to first conclude that

$$\tau y \sum_{k=-\infty}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(\tau y)^2 + (k\pi)^2} = 1. \tag{3.9}$$

Now let $y > 0$. If only one of the interpolation points, say $x + n\pi/\tau$, belongs to $[-2/\tau, 2/\tau]$, we have in fact $|x + n\pi/\tau| \le (\pi-2)/\tau$ so that $(n\pi)^2 \le (\pi - 2 + \tau|x|)^2$ and

$$\begin{aligned} |f(x+iy)| &\le \tau y \sum_{k=-\infty; k\ne n}^{\infty} \frac{1 - (-1)^k e^{-\tau y}}{(\tau y)^2 + (k\pi)^2} + \tau y \frac{1 - (-1)^n e^{-\tau y}}{(\tau y)^2 + (n\pi)^2} \cdot \tau|x + n\pi/\tau|/2 \\ &= 1 - \frac{\tau y(1 - (-1)^n e^{-\tau y})}{(\tau y)^2 + (n\pi)^2}(1 - \tau|x + n\pi/\tau|/2) \\ &\le 1 - \frac{\tau y(1 - (-1)^n e^{-\tau y})}{(\tau y)^2 + (n\pi)^2}(2 - \pi/2) \le 1 - \tau y \frac{(2-\pi/2)(1 - e^{-\tau y})}{(\tau y)^2 + (\pi - 2 + \tau|x|)^2}. \end{aligned}$$

If two interpolation points $x + n\pi/\tau$ and $x + (n+1)\pi/\tau$ are in $[-2/\tau, 2/\tau]$, we similarly obtain

$$|f(x+iy)| \leq 1 - \tau y \frac{(2-\pi/2)(1-e^{-\tau y})}{(\tau y)^2 + (2+\tau|x|)^2}$$

for $y > 0$. Thus, in any case

$$|f(x+iy)| \leq 1 - \tau y \frac{(2-\pi/2)(1-e^{-\tau y})}{(\tau y)^2 + (2+\tau|x|)^2}$$

if $y > 0$.

Finally, applying this inequality to the function $g(z) = \overline{f(\overline{z})}e^{i\tau z}$ which satisfies the same hypothesis as the function $f(z)$ we get

$$|f(x+iy)| \leq e^{\tau|y|}\left\{1 - \tau|y| \frac{(1-e^{-\tau|y|})(2-\pi/2)}{(\tau y)^2 + (2+\tau|x|)^2}\right\}$$

for $y < 0$.

We observe that inequality (1.36) is best possible in the sense that there exists an absolute constant A such that given τ and a positive number B, we can find a function f satisfying the hypothesis of Theorem 9 and a point $x > B$ where

$$|f'(x)| > \tau - A/\tau x^2. \tag{3.10}$$

In the proof of Theorem 2 it was shown that for each positive integer n there exists a polynomial $P_n(z) \in \mathcal{P}_{n,0}$ such that $P_n(1) = 0$, $P_n(-1) = 1$, $P_n'(-1) < 0$ and $|P_n'(-1)| > n - c_1/n$ where c_1 is an absolute constant. If $P_m(z)$ is such a polynomial of degree $m = [\tau B/\pi] + 1$ then

$$f(z) = P_m(\exp(i\tau z/m))$$

satisfies all the hypotheses of Theorem 9 and

$$|f'(m\pi/\tau)| = \frac{\tau}{m}|P_m'(-1)| > \tau - \frac{c_1\pi^2}{\tau(m\pi/\tau)^2} = \tau - \frac{A}{\tau(m\pi/\tau)^2}$$

where, clearly, $x = m\pi/\tau > B$.

We do not make such a remark concerning inequality (1.37).

Proof of Theorem 10. By Lemma 9 we have

$$f(iy) = \tau y \sum_{k=-\infty}^{\infty} \frac{1-(-1)^k e^{-\tau y}}{(\tau y)^2 + (k\pi)^2} f(k\pi/\tau),$$

which in conjunction with (3.9) gives us $|f(iy)| \leq 1 - (1-e^{-\tau y})/\tau y$. If $0 < \omega < \tau$ we may set $y = -\log(1-\omega/\tau)$ to get the desired inequality

$$|f(-i\log(1-\omega/\tau))| \leq 1 + \frac{1-(1-\omega/\tau)^\tau}{\log(1-\omega/\tau)^\tau}.$$

In very much the same way, inequalities (3.5), (3.6) and (3.7) can also be extended to entire functions of exponential type.

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