ALMOST COMPLEX STRUCTURES ON
COMPLEX PROJECTIVE SPACES

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ABSTRACT. In this paper we classify the almost complex structures on a
complex projective space as roots of a certain polynomial equation.

Introduction. A real oriented vector bundle E is said to admit an almost com-
plex structure if there is a complex vector bundle F such that $E \cong F$ as oriented
vector bundles. An almost complex structure on E is a (complex) isomorphism
class of such bundles F. When X is a $2n$-dimensional oriented manifold and TX
is its tangent bundle, an almost complex structure on TX is called an almost com-
plex structure on X, and X together with an almost complex structure is called an
almost complex manifold. A special case of this is when X is a complex manifold
with complex tangent bundle $\mathfrak{I}X$. Since $(\mathfrak{I}X)_R = TX$, X always admits an almost
complex structure.

In this paper we study the generic complex manifolds, the complex projective
spaces, and classify their almost complex structures as roots of a certain polyno-
mial equation. In this context we give an answer to a question of Hirzebruch [7]
posed in 1954 concerning which elements of cohomology arise as Chern classes
of almost complex structures. The techniques used to solve this problem apply
equally well in other cases (for example to $U(n)$, S^{2n}, $G_2(\mathbf{C}^n)$ = Grassmannian of
2-planes in $\mathbf{C}^n$, $S^{2m+1} \times L^n(p)$—where $L^n(p)$ is the lens space and p is suffi-
ciently large, various products of these, etc.) and the details and calculations in
these cases will appear later. For example, one can easily show that $S^{2n+1} \times$
S^{2m+1} admits precisely one almost complex structure. (In fact this is the com-
plex tangent bundle of the complex structure defined by Calabi and Eckmann [9].)

1. Definitions and statement of main theorems. Let $\mathbf{Z}^m$ be the product of m
copies of the integers as an abelian group. We consider $\mathbf{Z}^m \subset \mathbf{Z}^{m+1}$ by the inclu-
sions $(a_1, \cdots, a_m) \mapsto (a_1, \cdots, a_m, 0)$, and put $\mathbf{Z}^\infty = \bigcup_{m \geq 1} \mathbf{Z}^m$. For each m, let
$p_m : \mathbf{Z}^\infty \to \mathbf{Z}$ be projection onto the mth coordinate.

Let $t \in H^2(CP^m)$ be the cohomology class corresponding to the submanifold CP^{m-1} via the natural orientation of CP^m. The abelian group $\widetilde{H}^{ev}(CP^m)$ is free with generators $t, t^2, \cdots, t^m$. For convenience of notation we identify this group with Z^m by means of the isomorphism $a_1 t + a_2 t^2 + \cdots + a_m t^m \mapsto (a_1, a_2, \cdots, a_m)$.

Definition. The kth Newton function $s_k \colon Z^\infty \to Z$ is defined inductively by

(i) $s_1 = p_1$,

(ii) $s_k(a) = p_1(a) s_{k-1}(a) - p_2(a) s_{k-2}(a) + \cdots + (-1)^{k-1} p_{k-1}(a) s_1(a) + (-1)^k k p_k(a)$.

We notice that $s_k(a)$ is a polynomial in $p_1(a), \cdots, p_k(a)$, the first k coordinates of a. Inductively we define a homomorphism $f_k \colon Z^k \to Z$ by

(i) $f_1 = $ identity,

(ii) $f_k(a_1, \cdots, a_k) = f_{k-1}(a_2, \cdots, a_k) - (k-1) f_{k-1}(a_1, \cdots, a_{k-1})$.

Theorem A. *The n-ple $a = (a_1, \cdots, a_n)$ is the Chern vector of a vector bundle over CP^n iff $f_k(s_1, \cdots, s_k) = 0 \bmod k!$ for $1 \le k \le n$ where $s_i = s_i(a)$.*

For n a positive integer let

$$r(n) = r = \text{largest integer} \le \tfrac{1}{2} n,$$
$$r'(n) = r' = \text{smallest integer} \ge \tfrac{1}{2} n,$$
$$u(n) = u = 0 \text{ if } n \text{ is even},$$
$$= 1 \text{ if } n \equiv 3 \bmod 4,$$
$$= 2 \text{ if } n \equiv 1 \bmod 4.$$

Define $f^n \colon Z^{r'} \to Z[[t]]$ (the ring of formal power series over the integers) by the equation

$$f^n(x_1, \cdots, x_{r'}) = (1 + t)^{n+1} (1 + (n-1)! t^n)^{u x_{r'}} \prod_{1 \le k \le r} \left(\frac{1 + kt}{1 - kt} \right)^{x_k}$$

and let $f^n_m(x_1, \cdots, x_{r'})$ be the coefficient of t^m, that is

$$f^n(x_1, \cdots, x_{r'}) = \sum_m f^n_m(x_1, \cdots, x_{r'}) t^m.$$

We note that $f^n_m(x_1, \cdots, x_{r'})$ is a polynomial of degree less than or equal to m.

Theorem B. *There is a 1-1 correspondence between the set of almost complex structures on CP^n and the set of solutions of the equation $f^n_n(x_1, \cdots, x_{r'}) = n + 1$. Moreover, the almost complex structure corresponding to the solution $(a_1, \cdots, a_{r'})$ has ith Chern class $f^n_i(a_1, \cdots, a_{r'})$ so that the 'total Chern class' is precisely $f^n(a_1, \cdots, a_{r'})$ truncated after the nth term.*

Inductively we define a homomorphism $g_k \colon Z^k \to Z$ by

(i) $g_1 = $ identity,

(ii) $g_k(a_1, \cdots, a_k) = g_{k-1}(a_2, \cdots, a_k) - (k-1)^2 g_{k-1}(a_1, \cdots, a_{k-1})$.

Theorem C. *The n-ple* $a = (a_1, \cdots, a_n)$ *is the Chern vector of an almost complex structure on* CP^n *iff*

(1) $a_1 = n + 1 \mod 2$;

(2) $a_n = n + 1$;

(3) $g_k(s_1, s_3, \cdots, s_{2k-1}) = 0 \mod 2((2k-1)!)$, $2 \le k \le r$;

(4) *if* n *is odd,* $g_r(s_1, \cdots, s_n) = 0 \mod u(n!)$;

(5) $s_{2k} = n + 1$ *for* $1 \le k \le r$;

where $s_i = s_i(a)$.

2. Preliminaries. We recall briefly from [2] the functors $\tilde{G}$ and $\tilde{H}$ and their relation with $\tilde{K}$-theory. The reader is referred to this paper for explicit and further details.

For X a finite connected based CW-complex, $\tilde{G}(X)$ is the abelian group whose elements are formal expressions

$$1 + a_1 + \cdots + a_i + \cdots \quad \text{where} \quad a_i \in H^{2i}(X)$$

with addition $\oplus$ defined by

$$(1 + a_1 + \cdots) \oplus (1 + b_1 + \cdots) = 1 + c_1 + \cdots$$

where $c_i = a_i + a_{i-1}b_1 + a_{i-2}b_2 + \cdots + b_i$.

If E is a C^n-bundle over X with ith Chern class $c_i(E) \in H^{2i}(X)$ then the 'total Chern class' of E

$$\tilde{c}(E) = 1 + c_1(E) + \cdots + c_n(E)$$

is an element of $G(X)$, and since if F is a C^m-bundle over X $\tilde{c}(E \oplus F) = \tilde{c}(E) \oplus \tilde{c}(F)$, we see that $\tilde{c}$ induces a natural homomorphism

$$\tilde{c} \colon \tilde{K}(X) \to \tilde{G}(X).$$

Then $\tilde{c}$ is an isomorphism modulo finite groups. (In fact a much stronger result is true [2].) The image of $\tilde{c}$ is the subgroup of 'vectors' which occur as total Chern class of a vector bundle.

Let $\tilde{H}(X) = \bigoplus_n \tilde{H}^{2n}(X)$ as an abelian group. We give $\tilde{H}(X)$ the structure of a graded ring with product denoted by juxtaposition defined in terms of cup product by the formula

$$x_m x_n = \frac{(m+n)!}{(m!)(n!)} x_m \cup x_n \quad \text{where} \quad x_i \in H^{2i}(X).$$

There is a natural homomorphism $\sigma \colon \tilde{G}(X) \to \tilde{H}(X)$ which is defined in terms of Newton polynomials with the property that if $a \in H^2(X)$ then $\sigma(1 + a) = (a, a^2, \cdots, a^n)$.

We define $\tilde{s}: \tilde{K}(X) \to \tilde{H}(X)$ to be the composite $\tilde{s} = \sigma\tilde{c}$, which is a ring homomorphism. If E is a complex vector bundle over X, then the ith term s_i of $\tilde{s}(E - \dim E)$ is related to the Chern classes c_i by the formula

$$s_i - c_1 s_{i-1} + \cdots + (-1)^k k c_k = 0.$$

Both σ and $\tilde{s}$ are isomorphisms modulo finite groups.

3. Proof of Theorem A. We now consider the case $X = \mathbf{CP}^n$. Let H be the line bundle over $\mathbf{CP}^n$ defined by

$$H = \{(V, v) \in \mathbf{CP}^n \times \mathbf{C}^{n+1} : v \in V\}$$

where we think of $\mathbf{CP}^n$ as the set of 1-dimensional subspaces of $\mathbf{C}^{n+1}$. Let $\eta = H - 1 \in \tilde{K}(\mathbf{CP}^n)$ and let η_R be the image of η under the 'realification' map

$$R: \tilde{K}(X) \to \widetilde{KO}(X).$$

The following theorem is well known. (See, for example, [3].)

3.1 Theorem. (1) $\tilde{K}(\mathbf{CP}^n)$ is generated by $\eta, \eta^2, \cdots, \eta^n$ subject to the single relation $\eta^{n+1} = 0$.

(2) $\widetilde{KO}(\mathbf{CP}^n)$ is generated by powers of η_R subject to the relations

(i)   if n is even, $(\eta_R)^{r+1} = 0$;

(ii) if $n \equiv 1 \mod 4$, $2(\eta_R)^{r+1} = 0$ and $(\eta_R)^{r'+1} = 0$;

(iii) if $n \equiv 3 \mod 4$, $(\eta_R)^{r+1} = 0$.

In particular, $\tilde{K}(\mathbf{CP}^n)$ is free abelian, and so is $\widetilde{KO}(\mathbf{CP}^n)$ provided $n \not\equiv 1 \mod 4$. If $n \equiv 1 \mod 4$, then $\widetilde{KO}(\mathbf{CP}^n)$ is the direct sum of a free abelian group and a copy of $\mathbf{Z}_2$, so has a unique element of order 2, namely $(\eta_R)^{r+1}$.

3.2 Lemma. $\tilde{G}(\mathbf{CP}^n)$ is a free abelian group of rank n.

Proof. Since $\tilde{G}(\mathbf{CP}^n)$ is isomorphic to $\tilde{K}(\mathbf{CP}^n)$ modulo finite groups, it is finitely generated, so it suffices to show it is torsion free. Suppose $a \in \tilde{G}(\mathbf{CP}^n)$ and $k \in \mathbf{Z}$ such that $ka = 0$. Writing $a = 1 + a_r +$ higher terms where $a_r \in H^{2r}(\mathbf{CP}^n)$ is nonzero, we see that

$$ka = 1 + ka_r + \text{higher terms}$$

so that $k = 0$ since $H^{2r}(\mathbf{CP}^n)$ is torsion free.

It follows that the maps

$$\tilde{c}: \tilde{K}(\mathbf{CP}^n) \to \tilde{G}(\mathbf{CP}^n), \quad \sigma: \tilde{G}(\mathbf{CP}^n) \to \tilde{H}(\mathbf{CP}^n), \quad \tilde{s}: \tilde{K}(\mathbf{CP}^n) \to \tilde{H}(\mathbf{CP}^n)$$

are all monomorphisms. We identify $\tilde{G}(\mathbf{CP}^n)$ with $\mathbf{Z}^n$ by means of the bijection

$$1 + a_1 t + \cdots + a_n t^n \mapsto (a_1, \cdots, a_n).$$

Let $C = \operatorname{Im} \tilde{c}$, and $S = \operatorname{Im} \tilde{s}$. Then C can be identified with the set of elements $\{a \in \mathbf{Z}^n$ such that $(s_1(a), s_2(a), \cdots, s_n(a)) \in S\}$.

For $k \in \mathbf{Z}$, let $v_k = H^k - 1 \in \tilde{K}(\mathbf{CP}^n)$. Then $\tilde{c}(v_k) = 1 + kt \in \tilde{G}(\mathbf{CP}^n)$ and $\tilde{s}(v_k) = (k, k^2, \cdots, k^n)$. Since $v_k \cdot v_l = v_{k+l} - v_k - v_l$, and $\eta = v_1$, one can show inductively that

$$\eta^k = v_k - \binom{k}{1} v_{k-1} + \binom{k}{2} v_{k-2} - \cdots + (-1)^{k-1} \binom{k}{k-1} v_1$$

and hence $\tilde{K}(\mathbf{CP}^n)$ is generated by $v_1, \cdots, v_n$. So S consists of all elements of the form

$$a_1 s(v_1) + a_2 s(v_2) + \cdots + a_n s(v_n) \quad \text{where } a_i \in \mathbf{Z},$$

i.e. those elements $(b_1, \cdots, b_n) \in \mathbf{Z}^n$ such that the system of simultaneous equations

$$a_1 + 2a_2 + 3a_3 + \cdots + na_n = b_1,$$
$$a_1 + 2^2 a_2 + 3^2 a_3 + \cdots + n^2 a_n = b_2,$$
$$\vdots$$
$$a_1 + 2^n a_2 + 3^n a_3 + \cdots + n^n a_n = b_n$$

has a solution with a_i integers.

3.3 **Lemma.** (i) $f_i(j, j^2, j^3, \cdots, j^i) = j!/(j-i)!$.

(ii) *The augmented matrix of the above system of equations is row equivalent over the integers to the matrix M defined by*

$$M_{ij} = 0 \quad \text{if } i > j,$$
$$= f_i(j, j^2, \cdots, j^i) \quad \text{for } i \le j \le n,$$
$$= f_i(b_1, \cdots, b_i) \quad \text{for } j = n+1.$$

Proof. Part (i) is proved by induction on i, starting with $f_1(j) = j = j!/(j-1)!$. Then

$$f_{i+1}(j, \cdots, j^{i+1}) = f_i(j^2, \cdots, j^{i+1}) - if_i(j, \cdots, j^i) = (j-i)f_i(j, \cdots, j^i)$$

using the fact that f_i is a homomorphism, and the result follows.

Part (ii) is proved using (i) and by applying to the augmented matrix those row operations which inductively define f_i.

Since $M_{ii} = i!$ by the lemma, we see that

$$S = \{(b_1, \cdots, b_n): f_i(b_1, \cdots, b_i) = 0 \mod i! \text{ for } 1 \le i \le n\}$$

which completes the proof of Theorem A.

For X a CW-complex, let $\mathrm{Vect}_n(X)$ be the set of (isomorphism classes of) C^n-bundles over X. If $\dim X = 2n$, then $\mathrm{Vect}_n(X)$ is an abelian group. If $E, F \in \mathrm{Vect}_n(X)$, then by stability properties there is a C^n-bundle, say $E \cdot F$ determined uniquely up to isomorphism such that $E \oplus F = (E \cdot F) \oplus n$, where n denotes the trivial C^n-bundle. The function $(E, F) \mapsto E \cdot F$ defines the group structure, and the natural map

$$\mathrm{Vect}_n(X) \to K(X), \qquad E \mapsto E - n$$

is an isomorphism.

If $E \in \mathrm{Vect}_n(CP^n)$, then $\tilde{c}(E) \in C$, and this defines a function $f: \mathrm{Vect}_n(CP^n) \to C$. Conversely, if $a \in C$ then there is a C^n-bundle E with $f(E) = a$. If E' is a second such C^n-bundle, then $\tilde{c}(E - E') = 0$ so $E - E' = 0$ in $\tilde{K}(CP^n)$ and by stability, $E \cong E'$ and the isomorphism class of E is well defined. This defines a function $g: C \to \mathrm{Vect}_n(CP^n)$ such that $fg = 1$.

3.4 Lemma. *The maps* $f: \mathrm{Vect}_n(CP^n) \to C$ *and* $g: C \to \mathrm{Vect}_n(CP^n)$ *are inverse isomorphisms (of abelian groups).*

This lemma together with Theorem A classifies C^n-bundles over CP^n. If E is a C^r-bundle over CP^n we say that E extends over CP^{n+s} if there is a C^r-bundle E' over CP^{n+s} which restricts to E. It is an interesting and unsolved problem to determine necessary and sufficient conditions for a C^r-bundle over CP^n to extend over CP^{n+s}. The following is a solution in the simplest case ($r = n$, $s = 1$).

3.5 Proposition. *A* C^n*-bundle over* CP^n *extends over* CP^{n+1} *iff*

$$f_{n+1}(s_1, \cdots, s_{n+1}) = 0 \mod (n + 1)!$$

where $s_i = s_i(c)$, *and* $c = (c_1(E), c_2(E), \cdots, c_n(E), 0)$.

Proof. Necessity is an easy corollary of Theorem A. Let F be the C^{n+1}-bundle over CP^{n+1} with $c_i(F) = c_i(E)$ for $1 \le i \le n$, and $c_{n+1}(F) = 0$. By hypothesis and Lemma 3.4, there is such a bundle. Over CP^n we can write $F \cong G \oplus 1$ by stability, and then by Lemma 3.4 $G \cong E$ by examining Chern classes. So F has a nonzero section over CP^n and since the Euler class of F is zero this section extends to a nonzero section over CP^{n+1}. So $F \cong E' \oplus 1$ for some E' an n-dimensional bundle over CP^{n+1}, and E' restricted to CP^n is E.

4. **Almost complex structures.** If X is a $2n$-dimensional oriented manifold, let

$$e: \tilde{K}(X) = \mathrm{Vect}_n(X) \to H^{2n}(X) = \mathbf{Z}$$

be the function which associates to an n-dimensional complex vector bundle its nth Chern class, that is, its Euler class.

4.1 Proposition. *If X is a $2n$-dimensional oriented manifold, there is a 1-1 correspondence between the set of almost complex structures on X and the set of elements $x \in \widetilde{K}(X)$ satisfying*
(1) $e(x) = e(X) = $ *Euler number of X*;
(2) $R(x) = TX - 2n$ *in* $\widetilde{KO}(X)$, *where $R: \widetilde{K}(X) \to \widetilde{KO}(X)$ is the realification homomorphism, and TX is the tangent bundle of X.*

Proof. If A is an almost complex structure then $R(A - n) = TX - 2n$ and $e(A - n)$ $= c_n(A) = $ Euler class of $TX = $ Euler class of X. Conversely if $x \in \widetilde{K}(X)$, let $A \in$ $\mathrm{Vect}_n(X)$ be the corresponding element. Then $R(A - n) = TX - 2n$, so $A_R \cong_{\text{stably}}$ TX, whence $A_R \oplus 1 \cong TX \oplus 1$ so $A_R \cong TX$ since they have the same Euler class, which is the obstruction to removing the trivial line bundle.

4.2 Corollary. *If X is a $2n$-dimensional oriented manifold such that*
(1) $H^{2j}(X)$ *has only torsion relatively prime to $(j - 1)!$,*
(2) $\widetilde{KO}(X)$ *has no 2-torsion,*
then there is a 1-1 correspondence between almost complex structures on X and elements $E \in \mathrm{Vect}_n(x)$ such that
(a) $c_{2i}(E \oplus \bar{E}) = p_i(X) = $ *ith Pontrjagin class,* $1 \leq i \leq n$,
(b) $e(E) = e(X)$.

Proof. If $x \in \widetilde{K}(X)$, then $R(X) = TX - 2n$ iff $CR(x) = C(TX) - 2n$ (where $C: \widetilde{KO}(X) \to \widetilde{K}(X)$ is the complexification homomorphism) since $\widetilde{KO}(X)$ has no 2-torsion and RC is multiplication by 2. So $R(E - n) = TX - 2n$ iff $E \oplus \bar{E} - 2n$ $= C(TX) - 2n$, that is iff $E \oplus \bar{E}$ and $C(TX)$ are stably isomorphic. By a result of Peterson [4] under the conditions of the hypothesis, 2 vector bundles are stably isomorphic iff they have the same Chern classes, and the result follows.

A particular example of Corollary 4.2 is $X = \mathbf{CP}^n$ for $n \not\equiv 1 \mod 4$.

4.3 Proposition. *The kernel of $R: \widetilde{K}(\mathbf{CP}^n) \to \widetilde{KO}(\mathbf{CP}^n)$ is freely generated by*
(1) $w_1, \cdots, w_r$, n *even*;
(2) $w_1, \cdots, w_r, \eta^n$, $n \equiv 3 \mod 4$;
(3) $w_1, \cdots, w_r, 2\eta^n$, $n \equiv 1 \mod 4$ *where* $w_k = H^k - H^{-k}$.

Proof. This is an easy consequence of the theorems of Fujii on the structure of $\widetilde{KO}(\mathbf{CP}^n)$ ([5], [6]) and the fact that

$$(\eta - \bar{\eta})(\eta + \bar{\eta})^k = w_{k+1} + (\text{linear combination of } w_1, \cdots, w_k)$$

which is easily proved by induction starting with $k = 0$.

Let $\mathcal{T}$ denote the complex tangent bundle of the complex manifold $\mathbf{CP}^n$ and let T denote the real tangent bundle of the differentiable manifold $\mathbf{CP}^n$. Then $\mathcal{T}_R \cong T$, and so if $x \in \widetilde{K}(\mathbf{CP}^n)$, $R(x) = T - 2n$ iff

$$x = (\mathcal{T} - n) + a_1 w_1 + \cdots + a_r w_r + ua_{r'} \eta^n$$

for some integers $a_1, \cdots, a_r, a_{r'}$. For such an x,

$$\widetilde{c}(x) = (1 + t)^{n+1}(1 + (n-1)!t^n)^{ua_{r'}} \prod_{1 \leq k \leq r} \left(\frac{1 + kt}{1 - kt} \right)^{a_k}$$

$$= f^n(a_1, \cdots, a_{r'})$$

and so $c(x) = n$th Chern class $= f_n^n(a_1, \cdots, a_{r'})$. So using Proposition 4.1 we see that any almost complex structure determines a solution of the equation

$$f_n^n(x_1, \cdots, x_{r'}) = n + 1.$$

Conversely, if $(a_1, \cdots, a_{r'})$ is a solution, let

$$x = (\mathcal{T} - n) + a_1 w_1 + \cdots + a_r w_r + ua_{r'} \eta^n$$

so that $R(x) = T$ and $e(x) = n + 1$ so by Proposition 4.1 x determines an almost complex structure. This completes the proof of Theorem B.

The proof of Theorem C now follows from Proposition 4.3 by methods similar to §3. For an element $x \in \widetilde{K}(\mathbf{CP}^n)$ is of the form

$$x = (\mathcal{T} - n) + a_1 w_1 + \cdots + a_r w_r + ua_{r'} \eta^n \quad \text{for some } a_i$$

iff $s(x) = s(\mathcal{T} - n) + a_1 s(w_1) + \cdots + a_r s(w_r) + ua_{r'} s(\eta^n)$ for some a_i iff the n equations $(1 \leq i \leq n)$

$$a_1 s_i(w_1) + \cdots + a_r s_i(w_r) + a_{r'} us_i(\eta^n) = s_i(x) - s_i(\mathcal{T} - n)$$

have a solution with a_i integers. Since

$$s(w_k) = 2(k, 0, k^3, 0, k^5, 0, \cdots)$$

we see that $s_{2i}(x) = s_{2i}(\mathcal{T} - n) = n + 1$. The equations for i odd are

$$2(a_1 + 2a_2 + \cdots + ra_r) = s_1(x) - (n + 1),$$

$$2(a_1 + 2^3 a_2 + \cdots + r^3 a_r) = s_3(x) - (n + 1),$$

$$\vdots$$

$$2(a_1 + 2^{2r-1} a_2 + \cdots + r^{2r-1} a_r) = s_{2r-1}(x) - (n + 1)$$

and if n is odd

$$2(a_1 + 2^n a_2 + \cdots + r^n a_r) - u(n!)a_{r'} = s_n(x) - (n+1).$$

Theorem C now follows from the following lemma.

4.4 Lemma. (i) $g_i(j, j^3, \cdots, j^{2i-1}) = (j + i - 1)! / (j - i)!.$

(ii) *The augmented matrix of the above system of equations is row equivalent over the integers to the matrix N defined by*

$$
\begin{aligned}
N_{ij} &= 0 \quad \text{if } i > j, \\
&= 2g_i(j, j^3, \cdots, j^{2i-1}) \quad \text{for } i \leq j \leq r, \\
&= g_i(s_1, s_3, \cdots, s_{2i-1}) \quad \text{for } 2 \leq i, \text{ and } j = r' + 1 \\
&= s_1 - (n+1) \quad \text{for } i = 1, \; j = r' + 1
\end{aligned}
\right\} \text{ where } s_i = s_i(x),
$$

and if n is odd

$$
\begin{aligned}
N_{ij} &= 0 \quad \text{if } 1 \leq i \leq r, \; j = r', \\
&= -2u(n!), \quad i = j = r'.
\end{aligned}
$$

The proof of this lemma is similar to the proof of Lemma 3 with the additional observation that

$$
\begin{aligned}
g_i(s_1 - (n+1), s_3 &- (n+1), \cdots, s_{2i-1} - (n+1)) \\
&= g_i(s_1, \cdots, s_{2i-1}) - g_i(n+1, n+1, \cdots, n+1) \\
&= g_i(s_1, \cdots, s_{2i-1}) \quad \text{for } i \geq 2, \\
&= s_1 - (n+1) \quad \text{for } i = 1.
\end{aligned}
$$

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