TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 193, 1974

CHARACTERIZATION OF PRIVILEGED POLYDOMAINS

BY

YUM-TONG SIU[1]

ABSTRACT. This paper gives a number of equivalent conditions for a bounded polydomain to be privileged with respect to a coherent analytic sheaf in the sense of Douady. One of the equivalent conditions is in terms of the homological codimensions of the sheaf at the boundary of the polydomain. In the case of a polydisc, this condition about homological codimensions coincides with a conjecture of Douady. The other equivalent conditions concern some weaker concepts of privilegedness and the existence of privileged sets at the boundary.

0. In this paper we give a number of equivalent conditions for a bounded polydomain (with reasonable boundary) to be privileged with respect to a coherent analytic sheaf in the sense of Douady [1]. One of the equivalent conditions is in terms of the homological codimensions of the sheaf at the boundary of the polydomain. In the case of a polydisc, this condition about homological codimensions coincides with a conjecture of Douady. The other equivalent conditions concern some weaker concepts of privilegedness and the existence of privileged sets at the boundary.

To state the results, we have to introduce some notations and definitions. For a subset E of a topological space, E^- and ∂E denote respectively the topological closure and the topological boundary of E. ${}_n\mathcal{O}$ denotes the sheaf of germs of holomorphic functions on $\mathbf{C}^n$.

(0.1) For a bounded open subset G of $\mathbf{C}^n$, denote by $B(G)$ the set of all uniformly continuous holomorphic functions on G. $B(G)$ is a Banach space and is topologically isomorphic to the Banach space of all functions continuous on G^- and holomorphic on G.

(0.2) Suppose Ω is a Stein open subset of $\mathbf{C}^n$, G is a relatively compact open subset of Ω, and $\mathcal{F}$ is a coherent analytic sheaf on Ω. Suppose $\mathcal{F}$ admits a finite free resolution on Ω, i.e. there exists on Ω an exact sequence of sheaf-

Received by the editors August 2, 1971.

AMS (MOS) subject classifications (1970). Primary 32C35; Secondary 46B99.

Key words and phrases. Coherent analytic sheaves, homological codimensions, privileged sets, Banach spaces.

[1] Part of the research for this paper was done while the author was supported by a grant from the National Science Foundation.

homomorphisms of the form:

$$0 \to {}_n\mathcal{O}^{p_m} \to \cdots \to {}_n\mathcal{O}^{p_1} \to {}_n\mathcal{O}^{p_0} \to \mathcal{F} \to 0.$$

Let

$$(*) \qquad 0 \to B(G)^{p_m} \xrightarrow{\alpha_m} \cdots \xrightarrow{\alpha_2} B(G)^{p_1} \xrightarrow{\alpha_1} B(G)^{p_0}$$

be induced by the preceding sequence. We introduce the following definitions of privilegedness.

(i) G is said to be *weakly $\mathcal{F}$-privileged* if $(*)$ is exact and $\operatorname{Im} \alpha_1$ is closed in $B(G)^{p_0}$.

(ii) G is said to be *$\mathcal{F}$-privileged* if $(*)$ is direct-exact, i.e. $(*)$ is exact and, for $0 \leq i < m$, $\operatorname{Im} \alpha_{i+1}$ is closed in $B(G)^{p_i}$ and there exists a closed subspace E_i of $B(G)^{p_i}$ such that $B(G)^{p_i} = E_i \oplus \operatorname{Im} \alpha_{i+1}$.

(iii) G is said to be *strongly $\mathcal{F}$-privileged* if $(*)$ is direct-exact and the natural map $\operatorname{Coker} \alpha_1 \to \Gamma(G, \mathcal{F})$ is injective.

If G is weakly $\mathcal{F}$-privileged, define $B(G, \mathcal{F})$ to be $\operatorname{Coker} \alpha_1$.

These three definitions of privileged sets and the definition of $B(G, \mathcal{F})$ are independent of the choice of Ω and the choice of the finite free resolution of $\mathcal{F}$ on Ω, because, by using Theorem B of Cartan-Oka, we can easily prove that any two finite free resolutions of $\mathcal{F}$ on a given Stein open subset of Ω become isomorphic finite free resolutions after we apply to each of them a finite number of modifications [4, Definition VI.F.1], i.e. after we apply to each of them a finite number of times the process of replacing it by its direct sum with some finite free resolution of the zero-sheaf which has only two nonzero terms (cf. [4, p. 202, VI.F.3]).

Note that $B(G, {}_n\mathcal{O}^p) = B(G)^p$.

In the case where G is a polydisc, the definition of G being $\mathcal{F}$-privileged agrees with the definition given by Douady [1, p. 54, §7, Definition 1].

We introduce the following definitions of local privilegedness for bounded open subsets of $\mathbf{C}^n$:

(iv) G is said to be *locally weakly $\mathcal{F}$-privileged* if every point of ∂G admits a basis $\mathfrak{U}$ of open neighborhoods in $\mathbf{C}^n$ such that $G \cap U$ is weakly $\mathcal{F}$-privileged for every $U \in \mathfrak{U}$.

(v) G is said to be *locally $\mathcal{F}$-privileged* if every point of ∂G admits a basis $\mathfrak{U}$ of open neighborhood in $\mathbf{C}^n$ such that $G \cap U$ is $\mathcal{F}$-privileged for every $U \in \mathfrak{U}$.

(vi) G is said to be *locally strongly $\mathcal{F}$-privileged* if every point of ∂G admits a basis $\mathfrak{U}$ of open neighborhood in $\mathbf{C}^n$ such that $G \cap U$ is strongly $\mathcal{F}$-privileged.

(vii) G is said to be *semilocally weakly* $\mathfrak{F}$*-privileged* if every point of ∂G admits an open neighborhood U in $\mathbf{C}^n$ such that $G \cap U$ is weakly $\mathfrak{F}$-privileged.

(viii) G is said to be *semilocally* $\mathfrak{F}$*-privileged* if every point of ∂G admits an open neighborhood U in $\mathbf{C}^n$ such that $G \cap U$ is $\mathfrak{F}$-privileged.

(ix) G is said to be *semilocally strongly* $\mathfrak{F}$*-privileged* if every point of ∂G admits an open neighborhood U in $\mathbf{C}^n$ such that $G \cap U$ is strongly $\mathfrak{F}$-privileged.

(0.3) A bounded open subset G of $\mathbf{C}^n$ is called a *polydomain* if $G = G_1 \times \cdots \times G_n$ and each G_i is a connected open subset of $\mathbf{C}$ $(1 \le i \le n)$. For $0 \le k \le n$, the *boundary of order* k of G, denoted by $\partial_k G$, is defined as the set of all $(z_1, \cdots, z_n) \in G^-$ such that $z_i \in \partial G_i$ for at least k distinct values of i. Note that $\partial_0 G = G^-$, $\partial_1 G = \partial G$, and $\partial_n G$ is the distinguished boundary of G.

(0.4) For a bounded open subset D of $\mathbf{C}$, $x \in \partial D$ is said to be a *peak point* of $B(D)$ if there exists $f \in B(D)$ such that the continuous extension g of f to D^- satisfies $g(x) = 1$ and $|g(y)| < 1$ for $y \in D^- - \{x\}$.

If D equals the interior of D^- and there exists $d > 0$ such that every component of $\mathbf{C} - D^-$ has diameter $\ge d$, then every point of D is a peak point of $B(D)$ [3, p. 205, VIII.4.4].

(0.5) For a coherent analytic sheaf $\mathfrak{F}$ defined on an open subset Ω of $\mathbf{C}^n$, denote by $S_k(\mathfrak{F})$ the set of all points $x \in \Omega$ such that the homological codimension of $\mathfrak{F}_x$ over ${}_n\mathcal{O}_x$ does not exceed k. $S_k(\mathfrak{F})$ is always a subvariety of dimension $\le k$ in Ω (see e.g. [7, p. 31, (1.11)]).

Now we are ready to state the results.

(0.6) **Main Theorem.** *Suppose* Ω *is an open subset of* $\mathbf{C}^n$ *and* $\mathfrak{F}$ *is a coherent analytic sheaf on* Ω *admitting a finite free resolution on* Ω*. Suppose* $G = G_1 \times \cdots \times G_n$ *is a polydomain in* $\mathbf{C}^n$ *such that* $G \subset\subset \Omega$ *and every point of* G_i *is a peak point of* $B(G_i)$ *for* $1 \le i \le n$*. Then the following eleven statements are equivalent.*

(i) $S_k(\mathfrak{F}) \cap \partial_{k+1} G = \emptyset$ *for* $0 \le k < n$.

(ii) G *is* $\mathfrak{F}$*-privileged and there exist* $x_1, \cdots, x_k \in G$ *such that the natural map* $B(G, \mathfrak{F}) \to \bigoplus_{i=1}^k \mathfrak{F}_{x_i}$ *is injective.*

(iii) G *is strongly* $\mathfrak{F}$*-privileged.*

(iv) G *is* $\mathfrak{F}$*-privileged.*

(v) G *is weakly* $\mathfrak{F}$*-privileged.*

(vi) G *is locally strongly* $\mathfrak{F}$*-privileged.*

(vii) G *is locally* $\mathfrak{F}$*-privileged.*

(viii) G *is locally weakly* $\mathfrak{F}$*-privileged.*

(ix) G *is semilocally strongly* $\mathfrak{F}$*-privileged.*

(x) G *is semilocally* $\mathfrak{F}$*-privileged.*

(xi) G *is semilocally weakly* $\mathfrak{F}$*-privileged.*

(0.7) **Corollary.** *Suppose P is a bounded open polydisc in $\mathbf{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf defined on some open neighborhood of P^-. Then P is $\mathcal{F}$-privileged if and only if $S_k(\mathcal{F}) \cap \partial_{k+1} P = \emptyset$ for $0 \leq k < n$.*

The "if" part of Corollary (0.7) has also been proved by Pourcin [5]. Results similar to the "if" part of Corollary (0.7) have been proved by Douady-Frisch-Hirschowitz [2] for privileged sets which are defined by means of the Hilbert space of square-integrable holomorphic functions instead of the Banach space $B(G)$.

In this paper all topological spaces are assumed to be Hausdorff and have countable bases. $\mathbf{N}_*$ denotes the set of all nonnegative integers. Unless specified otherwise, the coordinates of $\mathbf{C}^n$ are denoted by $z_1, \cdots, z_n$. If $\mathfrak{U} = \{U_i\}$ is an open covering of a topological space X and $\mathcal{F}$ is a sheaf on X and $\xi \in C^p(\mathfrak{U}, \mathcal{F})$, then $\xi_{i_0 \cdots i_p} \in \Gamma(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F})$ denotes the value of ξ at the simplex $(i_0, \cdots, i_p)$ of the nerve of $\mathfrak{U}$. A continuous linear map f: $E \to F$ of Fréchet spaces is called *direct* if $\operatorname{Im} f$ is closed in F and there exist a closed subspace E' of E and a closed subspace F' of F such that $E' \oplus \operatorname{Ker} f = E$ and $F' \oplus \operatorname{Im} f = F$. A sequence of continuous linear maps of Fréchet spaces is called *direct-exact* if it is exact and if each map in it is direct.

1.

(1.1) Suppose R is a local ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module. A sequence $f_1, \cdots, f_k \in \mathfrak{m}$ is called an *M-sequence* if f_i is not a zero-divisor for $M/\sum_{j=1}^{i-1} f_j M$ for $1 \leq i \leq k$.

(1.2) Suppose S and Ω are respectively open subsets of $\mathbf{C}^k$ and $\mathbf{C}^n$. Let π: $\mathbf{C}^k \times \mathbf{C}^n \to \mathbf{C}^k$ be the natural projection. Suppose $\mathcal{F}$ is a coherent analytic sheaf on $S \times \Omega$. For $x \in S \times \Omega$, $\mathcal{F}$ is said to be *π-flat* at x if $\mathcal{F}_x$ as a ${}_k\mathcal{O}_{\pi(x)}$-module is flat over ${}_k\mathcal{O}_{\pi(x)}$.

Let $t_1, \cdots, t_k$ be the coordinates of $\mathbf{C}^k$ and $\pi(x) = (t_1^0, \cdots, t_k^0)$. Then $\mathcal{F}$ is π-flat at x if and only if $t_1 - t_1^0, \cdots, t_k - t_k^0$ form an $\mathcal{F}_x$-sequence.

$\mathcal{F}$ is said to be *π-flat* on a subset E of $S \times \Omega$ if $\mathcal{F}$ is π-flat at every point of E.

(1.3) **Lemma.** *Suppose G is a bounded polydomain in $\mathbf{C}^n$ and V is a subvariety of some open neighborhood of G^- in $\mathbf{C}^n$. If $\dim_x V \geq k$ for some $x \in G^-$, then $V \cap \partial_k G \neq \emptyset$.*

Proof. We can assume without loss of generality (w.l.o.g.) that V is pure-dimensional. We use induction on k. The case $k = 0$ is trivial. Suppose $k > 0$ and $V \cap \partial_k G = \emptyset$. Since by induction hypothesis $V \cap \partial_{k-1} G \neq \emptyset$, by reordering the coordinates of $\mathbf{C}^n$, we can assume that there exists $z^0 = (z_1^0, \cdots, z_n^0)$

$\in G^{-}$ V such that $z_i^0 \in \partial G_i$ for $1 \leq i \leq k-1$. Let $z' = (z_1^0, \cdots, z_{k-1}^0)$. It follows from $V \cap \partial_k G = \emptyset$ that

$$V \cap (\{z'\} \times \partial(G_{k+1} \times \cdots \times G_n)) = \emptyset.$$

By [4, p. 106, III.B.17], for some open neighborhood D of $(G_k \times \cdots \times G_n)^{-}$ in $\mathbf{C}^{n-k+1}$, $\dim V \cap (\{z'\} \times D) \leq 0$. Hence the dimension of the subvariety

$$\{(z_1, \cdots, z_n) \in V \mid z_i = z_i^0 \text{ for } 1 \leq i \leq k-1\}$$

at z^0 is 0. This contradicts $\dim_{z^0} V \geq k$ [4, p. 115, III.C.14]. Q.E.D.

(1.4) **Proposition.** *Suppose G is a bounded polydomain in $\mathbf{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf defined on some open neighborhood of G^{-}. Then the following two conditions are equivalent.*

(i) *$S_k(\mathcal{F}) \cap \partial_{k+1} G = \emptyset$ for $0 \leq k < n$.*

(ii) *If $z^0 = (z_1^0, \cdots, z_n^0) \in G^{-}$ and $i_1, \cdots, i_k$ are distinct elements of $\{1, \cdots, n\}$ such that $z_i^0 \in \partial G_i$ for $i = i_1, \cdots, i_k$, then $z_{i_1} - z_{i_1}^0, \cdots, z_{i_k} - z_{i_k}^0$ form an $\mathcal{F}_{z^0}$-sequence.*

Proof. (ii) $\Longrightarrow$ (i) is trivial. Suppose (i) holds. We use induction on k to prove (ii). We can assume w.l.o.g. that $i_\mu = \mu$ for $1 \leq \mu \leq k$.

Consider first the case where $k = 1$. Suppose $z_1 - z_1^0$ is a zero-divisor for $\mathcal{F}_{z^0}$. By [7, p. 40, (1.18)], for some $p \geq 0$ there exists a p-dimensional branch V of $S_p(\mathcal{F})$ such that $z^0 \in V \subset \{z_1^0\} \times \mathbf{C}^{n-1}$. By applying Lemma (1.3) to the subvariety V and the $(n-1)$-dimensional polydomain $\{z_1^0\} \times G_2 \times \cdots \times G_n$, we conclude that V intersects the boundary Z of order p of the $(n-1)$-dimensional polydomain $\{z_1^0\} \times G_2 \times \cdots \times G_n$. This contradicts $S_p(\mathcal{F}) \cap \partial_{p+1} G = \emptyset$, because $p < n$ and $Z \subset \partial_{p+1} G$.

The case where $k > 1$ is obtained by applying the induction hypothesis to the sheaf $(\mathcal{F}/(z_1 - z_1^0)\mathcal{F}) | \{z_1^0\} \times \mathbf{C}^{n-1}$ and the $(n-1)$-dimensional polydomain $\{z_1^0\} \times G_2 \times \cdots \times G_n$. Q.E.D.

2. In §2 we gather together a couple of simple facts we need about holomorphic Banach bundles and their applications. These facts are treated in much greater generalities and details by Douady in [1].

(2.1) For Banach spaces E_0, F_0, we denote by $L(E_0, F_0)$ the Banach space of all continuous linear maps from E_0 to F_0.

Suppose S is an open subset of $\mathbf{C}^n$ and E is a holomorphic Banach bundle on S with fiber E_0. For $s \in S$ we denote by E_s the fiber of E at s. For any open subset U of S, we denote by $E|U$ the restriction of E to U.

Suppose G is a relatively compact open subset of S. A continuous section f

of E over G is said to be *uniformly continuous* if, for every open subset U of S for which there exists a trivialization α: $E|U \xrightarrow{\approx} U \times E_0$ and for every relatively compact open subset W of U, the function $p\alpha f|W$ is a uniformly continuous function from W to E_0, where p: $U \times E_0 \longrightarrow E_0$ is the natural protection. A section of E over G is uniformly continuous if and only if it can be extended to a continuous section of E over G^-. Denote by $B(G, E)$ the Banach space of all holomorphic sections of E over G which are uniformly continuous.

Suppose F is a holomorphic Banach bundle on S with fiber F_0. A map γ: $E \longrightarrow F$ is called a *bundle-homomorphism* if, for every open subset U of S for which there are trivializations α: $E|U \xrightarrow{\approx} U \times E_0$ and β: $F|U \xrightarrow{\approx} U \times F_0$, there exists a holomorphic map $A(\cdot)$ from U to $L(E_0, F_0)$ such that $(\beta\gamma\alpha^{-1})(s, x) = (s, A(s)x)$ for $s \in U$ and $x \in E_0$. For any open subset U of S, we denote by $\gamma|U$ the bundle-homomorphism $E|U \longrightarrow F|U$ induced by γ. For $s \in S$, we denote by γ_s the map $E_s \longrightarrow F_s$ induced by γ.

A bundle-homomorphism γ: $E \longrightarrow F$ is called *direct* if both $\operatorname{Ker}\gamma$ and $\operatorname{Im}\gamma$ are holomorphic Banach bundles on S and there exist holomorphic Banach bundles E' and F' on S such that $E = E' \oplus \operatorname{Ker}\gamma$ and $F = F' \oplus \operatorname{Im}\gamma$. A sequence of bundle-homomorphisms is called *direct-exact* if it is exact and each bundle-homomorphism in it is direct.

(2.2) Suppose

$$0 \longrightarrow E^{(m)} \xrightarrow{\theta} E^{(m-1)} \longrightarrow \cdots \longrightarrow E^{(0)}$$

is a complex of bundle-homomorphisms of holomorphic Banach bundles on S. If for some $s_0 \in S$ the sequence

$$0 \longrightarrow E^{(m)}_{s_0} \xrightarrow{\theta_{s_0}} E^{(m-1)}_{s_0} \longrightarrow \cdots \longrightarrow E^{(0)}_{s_0}$$

is direct-exact, then there exists an open neighborhood U of s_0 in S such that the sequence

$$0 \longrightarrow E^{(m)}|U \longrightarrow E^{(m-1)}|U \longrightarrow \cdots \longrightarrow E^{(0)}|U$$

is direct-exact.

To prove this, it suffices to prove the case where $m = 1$ and $E^{(1)}$, $E^{(0)}$ are both trivial bundles. Let H be a closed subspace of $E^{(0)}_{s_0}$ which complements $\operatorname{Im}\theta_{s_0}$. Let σ: $E^{(1)} \oplus (S \times H) \longrightarrow E^{(0)}$ be the bundle-homomorphism induced by θ and the inclusion map

$$S \times H \hookrightarrow (S \times H) \oplus (S \times \operatorname{Im}\theta_{s_0}) = E^{(0)}.$$

The existence of U follows from the fact that, for some open neighborhood U of s_0 in S, $\sigma|U$ is a bundle-isomorphism (i.e. there exists a bundle-homomorphism which is the inverse of $\sigma|U$), which, in turn, follows from the fact that, if P is a Banach space and $A(\cdot)$ is a holomorphic map from S to $L(P, P)$ such that $A(s_0)$ admits an inverse $A(s_0)^{-1}$ in $L(P, P)$, then there exists an open neighborhood U of s_0 in S such that, for $s \in U$, $A(s)$ admits an inverse $A(s)^{-1}$ in $L(P, P)$ and the map $s \mapsto A(s)^{-1}$ is a holomorphic map from U to $L(P, P)$.

(2.3) Suppose S and Ω are respectively open subsets of $\mathbf{C}^k$ and $\mathbf{C}^n$, and $\mathcal{F}$ is a coherent analytic sheaf on $S \times \Omega$. For $s = (t_1^0, \cdots, t_k^0) \in S$, we denote by $\mathcal{F}(s)$ the sheaf $\mathcal{F}/\Sigma_{i=1}^k (t_i - t_i^0)\mathcal{F}$, where $t_1, \cdots, t_k$ are the coordinates of $\mathbf{C}^k$. $\mathcal{F}(s)$ can be regarded in a natural way as a sheaf on Ω.

For any positive integer p, we denote by $B(\Omega, {}_{k+n}\mathcal{O}^p)$ the trivial Banach bundle on S whose fiber is $B(\Omega, {}_n\mathcal{O}^p)$.

(2.4) Suppose S and Ω are respectively Stein open subsets of $\mathbf{C}^k$ and $\mathbf{C}^n$, and $\pi: S \times \Omega \to S$ is the natural projection. Suppose $\mathcal{F}$ is a π-flat coherent analytic sheaf on $S \times \Omega$ admitting a finite free resolution

$$(*) \qquad 0 \to {}_{k+n}\mathcal{O}^{p_m} \to \cdots \to {}_{k+n}\mathcal{O}^{p_1} \to {}_{k+n}\mathcal{O}^{p_0} \to \mathcal{F} \to 0$$

on $S \times \Omega$. Suppose $s \in S$ and G is a relatively compact open subset of Ω such that G is $\mathcal{F}(s)$-privileged. Then there exists an open neighborhood U of s in S such that $D \times G$ is $\mathcal{F}$-privileged for every relatively compact open subset D of U. Moreover, if G is strongly $\mathcal{F}(s)$-priviliged, then U can be chosen so the $D \times G$ is strongly $\mathcal{F}$-privileged for every relatively compact open subset D of U.

To prove this, consider the following sequence of bundle-homomorphisms induced by $(*)$:

$$(\#) \qquad 0 \to B(G, {}_{k+n}\mathcal{O}^{p_m}) \to \cdots \to B(G, {}_{k+n}\mathcal{O}^{p_1}) \to B(G, {}_{k+n}\mathcal{O}^{p_0}).$$

Since G is $\mathcal{F}(s)$-privileged and since by the π-flatness of $\mathcal{F}$ the sequence

$$0 \to {}_{k+n}\mathcal{O}^{p_m}(s) \to \cdots \to {}_{k+n}\mathcal{O}^{p_1}(s) \to {}_{k+n}\mathcal{O}^{p_0}(s) \to \mathcal{F}(s) \to 0$$

induced by $(*)$ is exact, we conclude that the sequence (#), when restricted to the singleton $\{s\}$, is direct-exact. By (2.2) there exists an open neighborhood U of s in S such that (#) is direct-exact on U. Our assertion follows from the fact that $B(D, B(G, {}_{k+n}\mathcal{O}^{p_i}))$ is topologically isomorphic to $B(D \times G, {}_{k+n}\mathcal{O}^{p_i})$ for any relatively compact open subset D of U $(0 \leq i \leq m)$.

3.

(3.1) Suppose X is a topological space. A sheaf $\mathcal{F}$ of $\mathbf{C}$-vector spaces on X is called a *Fréchet sheaf* if the following two conditions are satisfied.

(i) For any open subset W of X, $\Gamma(W, \mathcal{F})$ carries a Fréchet space structure.

(ii) For open subsets $W' \subset W$ of X the restriction map $\Gamma(W, \mathfrak{F}) \to \Gamma(W', \mathfrak{F})$ is continuous.

If $\mathfrak{U}$ is a countable open covering of X, then $C^p(\mathfrak{U}, \mathfrak{F})$ has a natural Fréchet space structure.

If $\mathfrak{G}$ is another Fréchet sheaf on X, then a sheaf-homomorphism $\phi: \mathfrak{G} \to \mathfrak{F}$ is called a *Fréchet-sheaf-homomorphism* if for every open subset W of X the map $\Gamma(W, \mathfrak{G}) \to \Gamma(W, \mathfrak{F})$ induced by ϕ is continuous.

The kernel of a Fréchet-sheaf-homomorphism is a Fréchet sheaf.

(3.2) Suppose X is a topological space. A collection $\mathfrak{E}$ of open subsets of X is said to be a *superbasis* for the topology of X if

(i) $\mathfrak{E}$ is a basis for the topology of X, and

(ii) for any $E \in \mathfrak{E}$ and any compact subset A of E there exists $E' \in \mathfrak{E}$ such that $A \subset E' \subset\subset E$.

A Fréchet sheaf $\mathfrak{F}$ on X is said to be *constructively fine relative to* $\mathfrak{E}$ if for every pair $E' \subset\subset E$ of members of $\mathfrak{E}$ and every finite covering $\mathfrak{U}$ of E by members of $\mathfrak{E}$ there exists a finite covering $\mathfrak{U}'$ of E' by members of $\mathfrak{E}$ which refines $\mathfrak{U}$ and there exists for $p \geq 1$ a continuous linear map

$$\alpha: Z^p(\mathfrak{U}, \mathfrak{F}) \to C^{p-1}(\mathfrak{U}', \mathfrak{F})$$

such that $\delta\alpha$ is the map $Z^p(\mathfrak{U}, \mathfrak{F}) \to Z^p(\mathfrak{U}', \mathfrak{F})$ induced by the refinement map $\mathfrak{U}' \to \mathfrak{U}$, where $\delta: C^{p-1}(\mathfrak{U}', \mathfrak{F}) \to Z^p(\mathfrak{U}', \mathfrak{F})$ is the coboundary map.

(3.3) Suppose X is a topological space and $\mathfrak{E}$ is a superbasis for the topology of X. Suppose

$$(*) \qquad 0 \to \mathfrak{L}_n \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} \mathfrak{L}_1 \xrightarrow{\phi^{(1)}} \mathfrak{L}_0$$

is an exact sequence of Fréchet-sheaf-homomorphisms on X.

Introduce the following notations: For every pair $E' \subset\subset E$ of members of $\mathfrak{E}$, denote by $\phi_E^{(\mu)}$ the map $\Gamma(E, \mathfrak{L}_\mu) \to \Gamma(E, \mathfrak{L}_{\mu-1})$ induced by $\phi^{(\mu)}$, and denote by $\rho_{E',E}^{(\mu)}$ the restriction map $\Gamma(E, \mathfrak{L}_\mu) \to \Gamma(E', \mathfrak{L}_\mu)$.

The sequence $(*)$ is said to be *locally direct relative to* $\mathfrak{E}$ if for $x \in E \in \mathfrak{E}$ there exists $E' \in \mathfrak{E}$ with $x \in E' \subset E$ satisfying the following condition:

(#) There exists a continuous linear map $\alpha_{E',E}^{(\mu)}: \Gamma(E, \mathfrak{L}_\mu) \to \Gamma(E', \mathfrak{L}_{\mu+1})$ for $0 \leq \mu < n$ such that $\phi_{E'}^{(\mu+1)}\alpha_{E',E}^{(\mu)}\phi_E^{(\mu+1)} = \rho_{E',E}^{(\mu)}\phi_E^{(\mu+1)}$.

The sequence $(*)$ is said to be *globally direct relative to* $\mathfrak{E}$ if for every pair $E' \subset\subset E$ of members of $\mathfrak{E}$ the condition (#) holds

(3.4) **Proposition.** *Suppose X is a topological space and*

$$0 \to \mathfrak{L}_n \xrightarrow{\phi^{(n)}} \cdots \xrightarrow{\phi^{(2)}} \mathfrak{L}_1 \xrightarrow{\phi^{(1)}} \mathfrak{L}_0$$

is an exact sequence of Fréchet-sheaf-homomorphisms on X. Suppose $\mathfrak{E}$ is a superbasis for the topology of X and $\mathfrak{L}_\mu$ is constructively fine relative to $\mathfrak{E}$ for $0 \le \mu \le n$. If the sequence is locally direct relative to $\mathfrak{E}$, then

(i) *the sequence is globally direct relative to $\mathfrak{E}$ and*

(ii) *for every pair $E' \subset\subset E$ of members of $\mathfrak{E}$ the image of the map $\Gamma(E', \mathfrak{L}_1) \to \Gamma(E', \mathfrak{L}_0)$ induced by $\phi^{(1)}$ contains the image of the restriction map $\Gamma(E, \operatorname{Im}\phi^{(1)}) \to \Gamma(E', \operatorname{Im}\phi^{(1)})$.*

Proof. By adding a zero term to the left of the sequence, we can assume w.l.o.g. that $\mathfrak{L}_n = 0$. In this proof we use the notations of (3.3) and the following notations: The letter E (with or without primes) denotes always an element of $\mathfrak{E}$. $\alpha^{(\mu)}_{E',E}$ always denotes a map satisfying (#) of (3.3). For a collection $\mathfrak{U}$ of open subsets of X and a sheaf $\mathfrak{F}$ on X, $\delta^p(\mathfrak{U}, \mathfrak{F})$ denotes the coboundary map $C^p(\mathfrak{U}, \mathfrak{F}) \to C^{p+1}(\mathfrak{U}, \mathfrak{F})$. If $\mathfrak{U}'$ is another collection of open subsets of X which refines $\mathfrak{U}$ by means of an index map τ, then $\tau^p(\mathfrak{U}', \mathfrak{U}, \mathfrak{F})$ denotes the map $Z^p(\mathfrak{U}, \mathfrak{F}) \to Z^p(\mathfrak{U}', \mathfrak{F})$ induced by τ.

Let $\mathfrak{G}_\mu = \operatorname{Im}\phi^{(\mu+1)}$ $(0 \le \mu < n)$ and, for $E' \subset E$, let $r^{(\mu)}_{E',E}$ denote the restriction map $\Gamma(E, \mathfrak{G}_\mu) \to \Gamma(E', \mathfrak{G}_\mu)$.

Since $\mathfrak{L}_\mu$ is constructively fine relative to $\mathfrak{E}$, the restriction map $H^\nu(E, \mathfrak{L}_\mu) \to H^\nu(E', \mathfrak{L}_\mu)$ is 0 for $E' \subset\subset E$, $\nu \ge 1$, and $0 \le \mu \le n$. From this we conclude that

(A) $\qquad \operatorname{Im} r^{(\mu)}_{E',E} \subset \operatorname{Im}\phi^{(\mu+1)}_{E'}$ for $E' \subset\subset E$ and $0 \le \mu < n$.

Conclusion (ii) of the proposition is proved.

We are going to prove that $\mathfrak{G}_0$ is a Fréchet sheaf by showing that $\Gamma(E, \mathfrak{G}_0)$ is closed in $\Gamma(E, \mathfrak{L}_0)$ for all E. Suppose $f \in \Gamma(E, \mathfrak{L}_0)$ belongs to the topological closure of $\Gamma(E, \mathfrak{G}_0)$. For $x \in E$ we can find $x \subset E' \subset E'' \subset\subset E$ such that $\alpha^{(0)}_{E',E''}$ exists. By (A), $\phi^{(1)}_{E'}$ maps $\alpha^{(0)}_{E',E''}\rho^{(0)}_{E'',E} f$ to $\rho^{(0)}_{E',E} f$. Hence $f_x \in (\mathfrak{G}_0)_x$. $\mathfrak{G}_0$ is a Fréchet sheaf.

We are going to prove the following:

For $x \in E$ there exist $x \in E' \subset E$ and a continuous linear map

(B) $\qquad \beta^{(\mu)}_{E',E}\colon \Gamma(E, \mathfrak{G}_\mu) \to \Gamma(E', \mathfrak{L}_{\mu+1})$ for $0 \le \mu < n$

such that $\phi^{(\mu+1)}_{E'}\beta^{(\mu)}_{E',E} = r^{(\mu)}_{E',E}$.

We can find $x \in E' \subset E'' \subset\subset E$ such that $\alpha^{(\mu)}_{E',E}$ exists. By (A), $\beta^{(\mu)}_{E',E} = \alpha^{(\mu)}_{E',E''}r^{(\mu)}_{E'',E}$ satisfies (B).

For the rest of this proof, $\beta^{(\mu)}_{E',E}$ denotes always a map satisfying (B).

We are going to prove the following two satements for $0 \leq \mu < n$ by descending induction on μ:

$(C)_\mu$ $\qquad\qquad \beta^{(\mu)}_{E',E}$ exists for $E' \subset\subset E$.

$(D)_\mu$ $\qquad\qquad \mathcal{G}_\mu$ is constructively fine relative to $\mathfrak{S}$.

Since $\mathcal{L}_n = 0$, $(C)_{n-1}$ and $(D)_{n-1}$ are trivial. Assume that $\mu < n-1$ and that $(C)_{\mu+1}$ and $(D)_{\mu+1}$ hold. We first prove $(C)_\mu$. Fix $E' \subset\subset E$. Take $E' \subset\subset E'' \subset\subset E$. We can find $U_i \in \mathfrak{S}$, $1 \leq i \leq k$, such that $E'' \subset \bigcup_{i=1}^k U_i \subset E$ and $\beta^{(\mu)}_{U_i,E}$ exists. By replacing U_i by $U_i \cap E''$ and replacing $\beta^{(\mu)}_{U_i,E}$ by $\rho^{(\mu+1)}_{U_i\cap E'',U_i}\beta^{(\mu)}_{U_i,E}$, we can assume that $E'' = \bigcup_{i=1}^k U_i$. Let

$$\mathfrak{U} = \{U_i\}_{i=1}^k \quad \text{and} \quad \beta_i = \beta^{(\mu)}_{U_i,E}.$$

Define

$$\xi\colon \Gamma(E, \mathcal{G}_\mu) \to Z^1(\mathfrak{U}, \mathcal{G}_{\mu+1})$$

by

$$\xi(f)_{ij} = (\beta_j(f) - \beta_i(f))| U_i \cap U_j$$

for $1 \leq i, j \leq k$. By $(D)_{\mu+1}$ there exists a finite covering $\mathfrak{U}' = \{U_i'\}_{i=1}^l$ of E' by members of $\mathfrak{S}$ which refines $\mathfrak{U}$ by means of an index map $\tau\colon \{1, \cdots, l\} \to \{1, \cdots, k\}$ and there exists a continuous linear map $\eta\colon Z^1(\mathfrak{U}, \mathcal{G}_{\mu+1}) \to C^0(\mathfrak{U}', \mathcal{G}_{\mu+1})$ such that $\delta^0(\mathfrak{U}', \mathcal{G}_{\mu+1})\eta = r^1(\mathfrak{U}', \mathfrak{U}, \mathcal{G}_{\mu+1})$. The map $\beta^{(\mu)}_{E',E}$ defined by

$$\beta^{(\mu)}_{E',E}(f)| U_i' = (\beta_{\tau(i)}(f)| U_i') - \eta\xi(f)_i$$

for $1 \leq i \leq l$ satisfies the requirement. $(C)_\mu$ is proved.

To prove $(D)_\mu$, fix $E' \subset\subset E$ and a finite covering $\mathfrak{U} = \{U_i\}_{i=1}^k$ of E by members of $\mathfrak{S}$. Take $E' \subset\subset E'' \subset\subset E''' \subset\subset E$. Choose $U_i''' \in \mathfrak{S}$ for $1 \leq i \leq k$ such that $U_i''' \subset\subset U_i$ and $E''' = \bigcup_{i=1}^k U_i'''$. Let $\mathfrak{U}''' = \{U_i'''\}_{i=1}^k$. By $(D)_{\mu+1}$ there exists a finite covering $\mathfrak{U}''$ of E'' by members of $\mathfrak{S}$ such that $\mathfrak{U}''$ refines $\mathfrak{U}'''$ by means of an index map τ_1 and such that for $p \geq 1$ there exists a continuous linear map γ_p: $Z^p(\mathfrak{U}''', \mathcal{G}_{\mu+1}) \to C^{p-1}(\mathfrak{U}'', \mathcal{G}_{\mu+1})$ satisfying $\delta^{p-1}(\mathfrak{U}'', \mathcal{G}_{\mu+1})\gamma_p = r_1^p(\mathfrak{U}'', \mathfrak{U}''', \mathcal{G}_{\mu+1})$. Since $\mathcal{L}_{\mu+1}$ is constructively fine relative to $\mathfrak{S}$, there exists a finite covering $\mathfrak{U}'$ of E' by members of $\mathfrak{S}$ such that $\mathfrak{U}'$ refines $\mathfrak{U}''$ by means of an index map τ_2 and such that for $p \geq 1$ there exists a continuous linear map

$$\nu_p\colon\ Z^p(\mathfrak{U}'', \mathfrak{L}_{\mu+1}) \to C^{p-1}(\mathfrak{U}', \mathfrak{L}_{\mu+1})$$

satisfying

$$\delta^{p-1}(\mathfrak{U}', \mathfrak{L}_{\mu+1})\nu_p = r_2^p(\mathfrak{U}', \mathfrak{U}'', \mathfrak{L}_{\mu+1}).$$

Fix $p \geq 1$. By $(C)_\mu$, for $1 \leq i_0 < \cdots < i_p \leq k$ there exists

$$\beta^{(\mu)}_{U'''_{i_0}\cap\cdots\cap U'''_{i_p}, U_{i_0}\cap\cdots\cap U_{i_p}}.$$

The collection of these maps defines a continuous linear map $\beta\colon Z^p(\mathfrak{U}, \mathfrak{G}_\mu) \to C^p(\mathfrak{U}'', \mathfrak{L}_{\mu+1})$.

Let $\psi\colon C^{p-1}(\mathfrak{U}', \mathfrak{L}_{\mu+1}) \to C^{p-1}(\mathfrak{U}', \mathfrak{G}_\mu)$ be induced by $\phi^{(\mu+1)}$. Define θ: $Z^p(\mathfrak{U}, \mathfrak{G}_\mu) \to C^{p-1}(\mathfrak{U}', \mathfrak{G}_\mu)$ by

$$\theta = \psi\nu_p(r^p(\mathfrak{U}', \mathfrak{U}'', \mathfrak{L}_{\mu+1})\beta - \gamma_{p+1}\delta^p(\mathfrak{U}'', \mathfrak{L}_{\mu+1})\beta).$$

Then $\delta^{p-1}(\mathfrak{U}', \mathfrak{G}_\mu)\theta = r^p(\mathfrak{U}', \mathfrak{U}, \mathfrak{G}_\mu)$. $(D)_\mu$ is proved.

Fix $E' \subset\subset E$ and $0 \leq \mu < n$. To finish the proof, we have to show that $\alpha^{(\mu)}_{E',E}$ exists. Take $E' \subset\subset E'' \subset\subset E''' \subset\subset E$. We can find $U_i \in \mathfrak{S}$, $1 \leq i \leq k$, such that $E''' = \bigcup_{i=1}^k U_i$ and such that $\alpha^{(\mu)}_{U_i,E}$ exists. Let $\mathfrak{U} = \{U_i\}_{i=1}^k$. Let $\alpha_i = \phi^{(\mu+1)}_{U_i}\alpha^{(\mu)}_{U_i,E}$. Define $\sigma\colon \Gamma(E, \mathfrak{L}_\mu) \to Z^1(\mathfrak{U}, \mathfrak{G}_\mu)$ by

$$\sigma(f)_{ij} = (\alpha_j(f) - \alpha_i(f))|U_i \cap U_j$$

for $1 \leq i, j \leq k$. By $(D)_\mu$ there exists a finite covering $\mathfrak{U}' = \{U'_i\}_{i=1}^l$ of E'' by members of $\mathfrak{S}$ which refines $\mathfrak{U}$ by means of an index map $\tau\colon \{1, \cdots, l\} \to \{1, \cdots, k\}$ and there exists a continuous linear map $\zeta\colon Z^1(\mathfrak{U}, \mathfrak{G}_\mu) \to C^0(\mathfrak{U}', \mathfrak{G}_\mu)$ such that $\delta^0(\mathfrak{U}', \mathfrak{G}_\mu)\zeta = r^1(\mathfrak{U}', \mathfrak{U}, \mathfrak{G}_\mu)$. Define $\lambda\colon \Gamma(E, \mathfrak{L}_\mu) \to \Gamma(E'', \mathfrak{G}_\mu)$ by

$$\lambda(f)|U'_i = (\alpha_{\tau(i)}(f)|U'_i) - \zeta\sigma(f)_i$$

for $1 \leq i \leq l$. By $(C)_\mu$, $\beta^{(\mu)}_{E',E''}$ exists. The map $\alpha^{(\mu)}_{E',E} = \beta^{(\mu)}_{E',E}\lambda$ satisfies the requirement. Q.E.D.

(3.5) **Corollary.** *If, in addition to all the assumptions of Proposition* (3.4), *X is compact and $X \in \mathfrak{S}$, then the sequence of Fréchet spaces*

$$0 \to \Gamma(X, \mathfrak{L}_n) \to \cdots \to \Gamma(X, \mathfrak{L}_1) \xrightarrow{\psi} \Gamma(X, \mathfrak{L}_0)$$

induced by the sequence in Proposition (3.4) *is direct-exact. Moreover,* $\operatorname{Im}\psi$ *is topologically isomorphic to* $\Gamma(X, \operatorname{Im}\phi^{(1)})$.

4.

(4.1) For $\alpha \in \mathbf{N}^n_*$ let $\alpha_1, \cdots, \alpha_n$ denote the components of α, let D^α denote $\partial^{\alpha_1+\cdots+\alpha_n}/\partial z_1^{\alpha_1}\cdots\partial z_n^{\alpha_n}$, and let $\bar{D}^\alpha$ denote $\partial^{\alpha_1+\cdots+\alpha_n}/\partial \bar{z}_1^{\alpha_1}\cdots\partial \bar{z}_n^{\alpha_n}$.

For a bounded polydomain $G = G_1 \times \cdots \times G_n$ in $\mathbf{C}^n$, denote by $\Lambda(G)$ the set of all infinitely differentiable (complex-valued) functions f on G such that, if $\alpha, \beta \in \mathbf{N}_*^n$, then $D^\alpha \bar{D}^\beta f$ is uniformly continuous on any polydomain of the form $G_1' \times \cdots \times G_n'$, where $G_i' = G_i$ for i satisfying $\alpha_i = 0$ and $G_j' \subset\subset G_j$ for j satisfying $\alpha_j > 0$. $\Lambda(G)$ is a Fréchet space.

Denote by $\Lambda_q^p(G)$ the set of all $(0, p)$-forms on G of the form

$$\sum_{1\le i_1<\cdots<i_p\le q} \xi_{i_1\cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p},$$

where $\xi_{i_1\cdots i_p}$ belongs to $\Lambda(G)$ and is holomorphic in $z_{q+1}, \cdots, z_n$ for fixed $z_1, \cdots, z_q$. $\Lambda_q^p(G)$ is a Fréchet space.

(4.2) **Lemma.** *If G is a bounded domain in $\mathbf{C}$, then there exists a continuous linear map I: $\Lambda(G) \to \Lambda(G)$ such that $(\partial/\partial\bar{z})I$ is the identity map of $\Lambda(G)$, where z is the coordinate of $\mathbf{C}$.*

Proof. For $f \in \Lambda(G)$ define $I(f)$ by

$$I(f)(z) = \frac{1}{2\pi i}\iint_G \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z} \qquad (z \in G).$$

We are going to prove that I satisfies the requirements.

Let R be a real number > 1 such that G is contained in the open disc of radius $(R-1)/2$ centered at 0. Let B be the open disc of radius R centered at 0. For $c \ge 0$ let $b(c)$ be the supremum of $\iint_E |d\zeta \wedge d\bar{\zeta}|/|\zeta|$ as E runs through all measurable sets of measure $\le c$ in B. Since $\iint_B |d\zeta \wedge d\bar{\zeta}|/|\zeta| < \infty$, the limit of $b(c)$ is 0 as c approaches 0.

First we show that $I(f)$ is infinitely differentiable. Let $\{G_m\}$ be an increasing sequence of relatively compact open subsets of G such that $\bigcup_m G_m = G$ and ∂G_m is the disjoint union of a finite number of simple rectifiable closed curves in G. Define $I_m(f)$ by

$$I_m(f)(z) = \frac{1}{2\pi i}\iint_{G_m} \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z} \qquad (z \in G).$$

Let μ_m be the measure of $G - G_m$. Then $\|I(f) - I_m(f)\|_G \le b(\mu_m)\|f\|_G$, where $\|\cdot\|_G$ is the sup norm on G. A trivial modification of [4, p. 25, I.D. 2] shows that $I_m(f)$ is infinitely differentiable on G_m and $(\partial/\partial\bar{z})I_m(f) = f$ on G_m. For $k \ge m$, $(\partial/\partial\bar{z})(I_k(f) - I_m(f)) = 0$ on G_m and hence $I_k(f) - I_m(f)$ is holomorphic on G_m.

Since $\mu_k \to 0$ as $k \to \infty$, $I_k(f)$ approaches $I(f)$ uniformly on G as $k \to \infty$. Hence for a fixed m, $I_k(f) - I_m(f)$ approaches $I(f) - I_m(f)$ uniformly on G as $k \to \infty$. It follows that $I(f) - I_m(f)$ is holomorphic on G_m for every m. Consequently $I(f)$ is infinitely differentiable and $(\partial/\partial\bar{z})I(f) = I(f)$ on G.

Next we have to show that $(\partial^k/\partial\bar{z}^k)I(f)$ is uniformly continuous on G for every fixed $k \geq 0$. When $k \geq 1$, this is clear, because $(\partial/\partial\bar{z})I(f) = f$ on G. So we need only show that $I(f)$ is uniformly continuous on G. Suppose $\epsilon > 0$. By the uniform continuity of f on G, there exists $0 < \delta < 1$ such that, if $a \in \mathbf{C}$ and $|a| < \delta$, then $|f(\zeta + a) - f(\zeta)| < \epsilon$ for $\zeta, \zeta + a \in G$. We can also assume that δ is so chosen that, for $|a| < \delta$, the measure of $(G(a) - G) \cup (G - G(a))$ is $\leq$ some nonnegative number c satisfying $h(c) < \epsilon$, where $G(a) = \{z \in \mathbf{C} | z = w - a \text{ for some } w \in G\}$. Then for $|a| < \delta$ and $z, z + a \in G$, we have

$$|I(f)(z + a) - I(f)(z)| < \frac{\epsilon}{2\pi}\left(\|f\|_G + \iint_B \frac{|d\zeta \wedge d\bar{\zeta}|}{|\zeta|}\right),$$

because

$$I(f)(z + a) - I(f)(z) = \frac{1}{2\pi i}\left(\iint_{G(a)\cap G} \frac{(f(\zeta + a) - f(\zeta))d\zeta \wedge d\bar{\zeta}}{\zeta - z}\right.$$

$$\left. + \iint_{G(a)-G} \frac{f(\zeta + a)d\zeta \wedge d\bar{\zeta}}{\zeta - z} + \iint_{G-G(a)} \frac{f(\zeta)d\zeta \wedge d\bar{\zeta}}{\zeta - z}\right).$$

Therefore $I(f)$ is uniformly continuous on G.

Finally we have to show that I is continuous. Suppose a sequence $f_k \to 0$ in $\Lambda(G)$ as $k \to \infty$. We have to prove the following two statements.

(i) Every partial derivative of $I(f_k)$ converges to 0 uniformly on every compact subset of G as $k \to \infty$

(ii) For every fixed $m \geq 0$, $\partial^m I(f_k)/\partial\bar{z}^m$ converges to 0 uniformly on G as $k \to \infty$.

Statement (ii) follows from $\partial I(f_k)/\partial\bar{z} = f_k$ and

$$\|I(f_k)\|_G \leq \|f_k\|_G \iint_B \frac{|d\zeta \wedge d\bar{\zeta}|}{|\zeta|}.$$

To prove statement (i), let F be the Fréchet space of all infinitely differentiable (complex-valued) functions on G. Since $\partial/\partial\bar{z}$ maps F continuously onto F, there exists a sequence $\{g_k\}$ in F such that $\partial g_k/\partial\bar{z} = f_k$ on G and $g_k \to 0$ in F as $k \to \infty$. By a trivial modification of [4, p. 24, I.D. 1],

$$g_k(z) = \frac{1}{2\pi i}\int_{\partial G_m} \frac{g_k(\zeta)\,d\zeta}{\zeta - z} + I_m(f_k)(z)$$

for $z \in G_m$. Hence for a fixed m, every partial derivative of $I_m(f_k)$ converges to 0 uniformly on every compact subset of G_m as $k \to \infty$. Since $\|I(f_k) - I_m(f_k)\|_G \leq b(\mu_m)\max_p \|f_p\|_G$, the sequence $I(f_k) - I_m(f_k)$ approaches 0 uniformly on G and *uniformly in* k as $m \to \infty$. Since $I(f_k) - I_m(f_k)$ is holomorphic on G_m, every partial derivative of $I(f_k) - I_m(f_k)$ approaches 0 uniformly on every compact subset of G_m and *uniformly in* k as $m \to \infty$. Statement (i) follows. Q.E.D.

(4.3) **Lemma.** *If G is a polydomain in $\mathbf{C}^n$ and $1 \leq q \leq n$, then there exists a continuous linear map J_q: $\Lambda(G) \to \Lambda(G)$ such that $(\partial/\partial\bar{z}_q)J_q$ is the identity map of $\Lambda(G)$ and J_q commutes with $\partial/\partial\bar{z}_i$ for $1 \leq i \leq n$ and $i \neq q$.*

Proof. We can assume w.l.o.g. that $q = 1$. Let $G = G_1 \times \cdots \times G_n$, where $G_i \subset \mathbf{C}$ $(1 \leq i \leq n)$. Let $G' = G_2 \times \cdots \times G_n$. For any function f on G and for any $z' \in G'$, denote by $f_{z'}$ the function on G_1 defined by $f_{z'}(z_1) = f(z_1, z')$ for $z_1 \in G_1$.

By Lemma (4.2) there exists a continuous linear map I: $\Lambda(G_1) \to \Lambda(G_1)$ such that $(\partial/\partial\bar{z}_1)I$ isthe identity map of $\Lambda(G_1)$. For $f \in \Lambda(G)$ define $J_1(f)$ by $J_1(f)_{z'} = I(f_{z'})$ for $z' \in G'$. It is straightforward to verify that J_1 satisfies the requirements. Q.E.D.

(4.4) **Proposition.** *If G is a bounded polydomain in $\mathbf{C}^n$ and $0 < p \leq q \leq n$, then there exists a continuous linear map I_q^p from the kernel Z_q^p of $\bar{\partial}$: $\Lambda_q^p(G) \to \Lambda_q^{p+1}(G)$ to Λ_q^{p-1} such that $\bar{\partial}I_q^p$ is the identity map of Z_q^p.*

Proof. We use induction on q for $0 \leq q \leq n$. The case $q = 0$ is vacuous. For $q > 0$, there exists J_q satisfying the conditions of Lemma (4.3). Define a map T: $Z_q^p \to \Lambda_q^{p-1}(G)$ as follows. For

$$\xi = \sum_{1 \leq i_1 < \cdots < i_p \leq q} \xi_{i_1 \cdots i_p} d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p} \in Z_q^p,$$

$$T(\xi) = \sum_{1 \leq i_1 < \cdots < i_{p-1} \leq q} (-1)^{p-1} J_q(\xi_{i_1 \cdots i_{p-1} q}) d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_{p-1}}.$$

The map I_q^p: $Z_q^p \to \Lambda_q^{p-1}(G)$ defined by

$$I_q^p(\xi) = \begin{cases} T(\xi) & \text{if } p = q \\ I_{q-1}^p(\xi - \bar{\partial}T(\xi)) + T(\xi) & \text{if } p < q \end{cases}$$

satisfies the requirement. Q.E.D.

(4.5) Suppose G is an open subset of $\mathbf{C}^n$. Denote by $\mathcal{O}_{G^-}$ the sheaf on G^- defined by the following presheaf:

(i) For every open subset U of G^-, $f \in \mathcal{O}_{G^-}(U)$ if and only if f is a continuous function on U and f is holomorphic on $G \cap U$.

(ii) For open subsets $U' \subset U$ of G^-, $\mathcal{O}_{G^-}(U) \to \mathcal{O}_{G^-}(U')$ is the restriction map.

The sheaf $\mathcal{O}_{G^-}$ is a Fréchet sheaf.

Denote by $\mathfrak{S}(G^-)$ the superbasis for the topology of G^- defined as follows. $E \in \mathfrak{S}(G^-)$ if and only if $E = G^- \cap \tilde{E}$ for some polydomain $\tilde{E}$ in $\mathbf{C}^n$.

(4.6) **Proposition.** *If G is a bounded polydomain in $\mathbf{C}^n$, then the Fréchet sheaf $\mathcal{O}_{G^-}$ on G^- is constructively fine relative to $\mathfrak{S}(G^-)$.*

Proof. Suppose $\mathfrak{U} = \{U_i\}_{i=1}^k$ and $\mathfrak{V} = \{V_i\}_{i=1}^k$ are coverings of G^- by members of $\mathfrak{S}(G^-)$ such that $V_i \subset\subset U_i$ for $1 \leq i \leq k$. Let $W_i = G \cap V_i$ $(1 \leq i \leq k)$ and $\mathfrak{W} = \{W_i\}_{i=1}^k$. Let $C_0^q(\mathfrak{W}, {}_n\mathcal{O})$ be the set of all $\xi \in C^q(\mathfrak{W}, {}_n\mathcal{O})$ such that $\xi_{i_0 \cdots i_q} \in B(W_{i_0} \cap \cdots \cap W_{i_q})$. Let $\mathfrak{A}^{(p)}$ be the sheaf of germs of infinitely differentiable $(0, p)$-forms on $\mathbf{C}^n$. Let $\Gamma_0(G, \mathfrak{A}^{(p)}) = \Lambda_n^p(G)$ and let $C_0^q(\mathfrak{W}, \mathfrak{A}^{(p)})$ be the set of $\eta \in C^q(\mathfrak{W}, \mathfrak{A}^{(p)})$ such that $\eta_{i_0 \cdots i_q} \in \Lambda_n^p(W_{i_0} \cap \cdots \cap W_{i_q})$. Consider the following commutative diagram of Fréchet spaces:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Gamma_0(G, \mathfrak{A}^{(0)}) & \to & \Gamma_0(G, \mathfrak{A}^{(1)}) & \to & \cdots \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & C_0^0(\mathfrak{W}, {}_n\mathcal{O}) & \hookrightarrow & C_0^0(\mathfrak{W}, \mathfrak{A}^{(0)}) & \to & C_0^0(\mathfrak{W}, \mathfrak{A}^{(1)}) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_0^1(\mathfrak{W}, {}_n\mathcal{O}) & \hookrightarrow & C_0^1(\mathfrak{W}, \mathfrak{A}^{(0)}) & \to & C_0^1(\mathfrak{W}, \mathfrak{A}^{(1)}) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where (except the inclusion maps) all the horizontal maps are induced by $\bar{\partial}$ and all the vertical maps are coboundary maps. By Proposition (4.4) all the rows are direct-exact. From the usual argument using a partition of unity, it follows that all the columns except the first are direct-exact. By diagram-chasing, we conclude that the first column is also direct-exact. Hence, for $r \geq 1$ there exists a continuous linear map ψ: $Z^r(\mathfrak{U}, \mathcal{O}_{G^-}) \to C^{r-1}(\mathfrak{V}, \mathcal{O}_{G^-})$ such that $\delta\psi$ is the restriction map $Z^r(\mathfrak{U}, \mathcal{O}_{G^-}) \to Z^r(\mathfrak{V}, \mathcal{O}_{G^-})$, where δ: $C^{r-1}(\mathfrak{V}, \mathcal{O}_{G^-}) \to Z^r(\mathfrak{V}, \mathcal{O}_{G^-})$ is the coboundary map. Q.E.D.

(4.7) **Corollary.** *If G is a bounded polydomain in* $\mathbf{C}^n$, *then* $H^p(G^-, \mathcal{O}_{G^-}) = 0$ *for* $p \geq 1$.

5.

(5.1) **Lemma.** *Suppose* $G_1 \subset\subset G_2$ *are Stein open subsets of* $\mathbf{C}^n$ *and* ϕ: ${}_n\mathcal{O}^p \to {}_n\mathcal{O}^q$ *is a sheaf-homomorphism on* G_2. *Then there exists a continuous linear map* α: $\Gamma(G_2, {}_n\mathcal{O}^q) \to \Gamma(G_1, {}_n\mathcal{O}^p)$ *such that* $\phi_1 \alpha \phi_2 = r\phi_2$, *where* ϕ_i: $\Gamma(G_i, {}_n\mathcal{O}^p) \to \Gamma(G_i, {}_n\mathcal{O}^q)$ *is induced by* ϕ $(i = 1, 2)$ *and* r: $\Gamma(G_2, {}_n\mathcal{O}^q) \to \Gamma(G_1, {}_n\mathcal{O}^q)$ *is the restriction map.*

Proof. Let G_3 be a Stein open subset of $\mathbf{C}^n$ satisfying $G_1 \subset\subset G_3 \subset\subset G_2$. Let $\Gamma_*(G_i, {}_n\mathcal{O}^k)$ be the set of all k-tuples of square-integrable holomorphic functions on G_i $(i = 1, 3;\ k = p, q)$. Let ϕ^*: $\Gamma_*(G_3, {}_n\mathcal{O}^q) \to \Gamma_*(G_3, {}_n\mathcal{O}^p)$ be induced by ϕ. Let $(\operatorname{Im} \phi^*)^-$ be the topological closure of $\operatorname{Im} \phi^*$ in $\Gamma_*(G_3, {}_n\mathcal{O}^q)$. By the closure-of-modules theorem [4, p. 85, II.D.3], $(\operatorname{Im} \phi^*)^-$ is a subset of $\Gamma(G_3, \operatorname{Im} \phi)$. Consider the following commutative diagram

$$
\begin{array}{ccccc}
\Gamma(G_2, {}_n\mathcal{O}^q) & \xrightarrow{\beta} & \Gamma_*(G_3, {}_n\mathcal{O}^q) & \xrightarrow{\gamma} & (\operatorname{Im} \phi^*)^- \\
 & & & & \downarrow \xi \\
 & & \Gamma(G_3, {}_n\mathcal{O}^p) & \xrightarrow{\sigma} & \Gamma(G_3, \operatorname{Im} \phi) \\
 & & \zeta \downarrow & & \downarrow \eta \\
(\operatorname{Ker} \tau)^\perp & \xhookrightarrow{\psi} & \Gamma_*(G_1, {}_n\mathcal{O}^p) & \xrightarrow{\tau} & \Gamma_*(G_1, {}_n\mathcal{O}^q) \\
 & \underset{\lambda}{\searrow} \approx & \mu \downarrow & \nearrow \nu & \\
 & & \Gamma_*(G_1, {}_n\mathcal{O}^p)/\operatorname{Ker} \tau, & &
\end{array}
$$

where γ is the orthogonal projection; β and ζ are restriction maps; η is induced by restriction and by the inclusion map $\operatorname{Im} \phi \hookrightarrow {}_n\mathcal{O}^q$; σ and τ are induced by ϕ; $(\operatorname{Ker} \tau)^\perp$ is the orthogonal complement of $\operatorname{Ker} \tau$ in $\Gamma_*(G_1, {}_n\mathcal{O}^p)$; μ is the quotient map; λ is induced by μ; and ν is induced by τ. Since σ is surjective, by the open mapping theorem for Fréchet spaces, $\nu^{-1}\eta$ is well defined and continuous. Let ρ: $\Gamma_*(G_1, {}_n\mathcal{O}^p) \to \Gamma(G_1, {}_n\mathcal{O}^p)$ be the inclusion map. Then $\alpha = \rho\psi\lambda^{-1}\eta^{-1}\nu\xi\gamma\beta$ satisfies the requirement. Q.E.D.

(5.2) **Lemma.** *Suppose* Ω *is a Stein open subset of* $\mathbf{C}^n$ *and*

$$0 \to {}_n\mathcal{O}^{p_m} \to \cdots \to {}_n\mathcal{O}^{p_1} \xrightarrow{\phi} {}_n\mathcal{O}^{p_0}$$

is an exact sequence on Ω. *Suppose G is a polydomain in* $\mathbf{C}^n$ *and* $G \subset\subset \Omega$. *If G*

is locally strongly (Im ϕ)*-privileged, then the sequence*

$$(*)\qquad 0 \to \mathcal{O}^{p_m}_{G^-} \to \cdots \to \mathcal{O}^{p_1}_{G^-} \to \mathcal{O}^{p_0}_{G^-}$$

is exact on G^-.

Proof. Let $\mathcal{G} = \mathrm{Im}\ \phi$. Take $x \in G^-$. Clearly (*) is exact at x if $x \in G$. Assume that $x \in \partial G$. There exists a basis $\mathfrak{U}$ of open neighborhoods of x in $\mathbf{C}^n$ such that $G \cap U$ is strongly $\mathcal{G}$-privileged for $U \in \mathfrak{U}$. The sequence

$$0 \to B(G \cap U, {}_n\mathcal{O}^{p_m}) \to \cdots \xrightarrow{\alpha_2} B(G \cap U, {}_n\mathcal{O}^{p_1}) \xrightarrow{\alpha_1} B(G \cap U, {}_n\mathcal{O}^{p_0})$$

is exact for $U \in \mathfrak{U}$ (Ker α_1 = Im α_2 follows from the injectivity of $B(G \cap U, \mathcal{G}) \to \Gamma(G \cap U, \mathcal{G})$). The lemma follows from the isomorphism $\mathrm{ind\,lim}_{U \in \mathfrak{U}} B(G \cap U, {}_n\mathcal{O}^{p_i}) \approx (\mathcal{O}^{p_i}_{G^-})_x$ for $0 \le i \le m$. Q.E.D.

(5.3) **Proposition.** *Suppose* Ω *is a Stein open subset of* $\mathbf{C}^n$ *and* $\mathcal{F}$ *is a coherent analytic sheaf on* Ω *admitting a finite free resolution on* Ω. *Suppose* G *is a polydomain in* $\mathbf{C}^n$ *and* $G \subset\subset \Omega$. *If* G *is locally strongly* $\mathcal{F}$*-privileged, then* G *is strongly* $\mathcal{F}$*-privileged.*

Proof. Let $0 \to {}_n\mathcal{O}^{p_m} \to \cdots \to {}_n\mathcal{O}^{p_1} \to {}_n\mathcal{O}^{p_0} \to \mathcal{F} \to 0$ be the finite free resolution of $\mathcal{F}$ on Ω and let

$$(*)\qquad 0 \to \mathcal{O}^{p_m}_{G^-} \xrightarrow{\phi^{(m)}} \cdots \xrightarrow{\phi^{(2)}} \mathcal{O}^{p_1}_{G^-} \xrightarrow{\phi^{(1)}} \mathcal{O}^{p_0}_{G^-}$$

be induced by it. By Lemma (5.2), (*) is exact.

Let $\mathfrak{S} = \mathfrak{S}(G^-)$. For $E \in \mathfrak{S}$ and $1 \le \mu \le m$, denote by $\phi^{(\mu)}_E$ the map $\Gamma(E, \mathcal{O}^{p_\mu}_{G^-}) \to \Gamma(E, \mathcal{O}^{p_{\mu-1}}_{G^-})$ induced by $\phi^{(\mu)}$. For $E' \subset E$ in $\mathfrak{S}$ and $0 \le \mu \le m$, denote the restriction map $\Gamma(E, \mathcal{O}^{p_\mu}_{G^-}) \to \Gamma(E', \mathcal{O}^{p_\mu}_{G^-})$ by $\rho^{(\mu)}_{E',E}$. We are going to prove that (*) is locally direct relative to $\mathfrak{S}$.

Suppose $x \in G^-$ and $E \in \mathfrak{S}$ such that $x \in E$. We have to show that there exists $E' \in \mathfrak{S}$ with $x \in E' \subset E$ and for $0 \le \mu < m$ there exists a continuous linear map $\alpha^{(\mu)}_{E',E}: \Gamma(E, \mathcal{O}^{p_\mu}_{G^-}) \to \Gamma(E', \mathcal{O}^{p_{\mu+1}}_{G^-})$ such that $\phi^{(\mu+1)}_{E'}\alpha^{(\mu)}_{E',E}\phi^{(\mu+1)}_E = \rho^{(\mu)}_{E',E}\phi^{(\mu+1)}_E$. We distinguish between two cases.

(i) $x \in G$. We can assume that $E \subset G$. Choose $E' \in \mathfrak{S}$ such that $x \in E' \subset\subset E$. Then the existence of $\alpha^{(\mu)}_{E',E}$ follows from Lemma (5.1).

(ii) $x \in \partial G$. There exists an open neighborhood U of x in $\mathbf{C}^n$ such that $(G \cap U)^- \subset E$ and $G \cap U$ is strongly $\mathcal{F}$-privileged. Choose $E' \in \mathfrak{S}$ such that $x \in E' \subset U$. We claim that $\alpha^{(\mu)}_{E',E}$ exists. Fix $0 \le \mu < m$. Let $W = G \cap U$. Let

β: $\Gamma(W^-, \mathcal{O}^{p_{\mu+1}}_{W^-}) \to \Gamma(W^-, \mathcal{O}^{p_\mu}_{W^-})$ be induced by $\phi^{(\mu+1)}$. Since $G \cap U$ is strongly $\mathcal{F}$-privileged, there exists a continuous linear map γ: $\Gamma(W^-, \mathcal{O}^{p_\mu}_{W^-}) \to \Gamma(W^-, \mathcal{O}^{p_{\mu+1}}_{W^-})$ such that $\beta\gamma\beta = \beta$. Let

$$\sigma: \Gamma(E, \mathcal{O}^{p_\mu}_{G^-}) \to \Gamma(W^-, \mathcal{O}^{p_\mu}_{W^-}) \quad \text{and} \quad \tau: \Gamma(W^-, \mathcal{O}^{p_{\mu+1}}_{W^-}) \to \Gamma(E', \mathcal{O}^{p_{\mu+1}}_{G^-})$$

be restriction maps. Then $\alpha^{(\mu)}_{E',E} = \tau\gamma\sigma$ satisfies the requirement.

By Corollary (3.5) and Proposition (4.6), the sequence of Banach spaces

$$0 \to \Gamma(G^-, \mathcal{O}^{p_m}_{G^-}) \to \cdots \to \Gamma(G^-, \mathcal{O}^{p_1}_{G^-}) \xrightarrow{\theta} \Gamma(G^-, \mathcal{O}^{p_0}_{G^-})$$

is direct-exact and $\operatorname{Im}\theta = \Gamma(G^-, \operatorname{Im}\phi^{(1)})$. Since $\Gamma(G^-, \mathcal{O}^{p_i}_{G^-})$ is topologically isomorphic to $B(G, {}_n\mathcal{O}^{p_i})$ for $0 \le i \le m$, G is $\mathcal{F}$-privileged.

Suppose $f \in \Gamma(G^-, \mathcal{O}^{p_0}_{G^-})$ and $f|G \in \Gamma(G, \operatorname{Im}\phi^{(1)})$. To prove that G is strongly $\mathcal{F}$-privileged, we have to show that $f \in \operatorname{Im}\theta$. Since $\operatorname{Im}\theta = \Gamma(G^-, \operatorname{Im}\phi^{(1)})$, we need only show that $f_x \in (\operatorname{Im}\phi^{(1)})_x$ for $x \in \partial G$. Take $x \in \partial G$. There exists an open neighborhood U of x in $\mathbf{C}^n$ such that $G \cap U$ is strongly $\mathcal{F}$-privileged. Let $W = G \cap U$. Since $f|W \in \Gamma(W, \operatorname{Im}\phi^{(1)})$, $f|W^-$ belongs to the image of the map $\Gamma(W^-, \mathcal{O}^{p_1}_{W^-}) \to \Gamma(W^-, \mathcal{O}^{p_0}_{W^-})$ induced by $\phi^{(1)}$. Hence $f_x \in (\operatorname{Im}\phi^{(1)})_x$. Q.E.D.

6.

(6.1) **Lemma.** *Suppose G is an open neighbhorhood of 0 in $\mathbf{C}^n$ and $V_0, \cdots, V_{n-1}$ are subvarieties in G such that* $\dim V_k \le k$ $(0 \le k < n)$. *Then there exists an open polydisc $P \ne \emptyset$ in $\mathbf{C}^n$ centered at 0 such that $P \subset\subset G$ and $V_k \cap \partial_{k+1} P = \emptyset$ for $0 \le k < n$.*

Proof. Use induction on n. The case $n = 0$ is trivial. Identify $\mathbf{C}^{n-1}$ with $\mathbf{C}^n \cap \{z_n = 0\}$. Let V'_{k-1} be the union of branches of $V_i \cap \{z_n = 0\}$ of dimension $\le k - 1$. By induction hypothesis, there exists an open $(n-1)$-dimension polydisc $P' \ne \emptyset$ centered at 0 such that $P' \subset\subset G \cap \{z_n = 0\}$ and $V'_k \cap \partial_{k+1} P' = \emptyset$ for $0 \le k < n - 1$. For $\delta > 0$ let D_δ be the open disc in $\mathbf{C}$ with center 0 and radius δ. There exists $\delta > 0$ such that $P' \times D_\delta \subset\subset G$ and $W \cap ((\partial_{k+1} P') \times D_\delta^-) = \emptyset$ for any branch W of V_k not contained in $\{z_n = 0\}$. Then $P = P' \times D_\delta$ satisfies the requirement. Q.E.D.

(6.2) **Proposition.** *Suppose Ω is a Stein open subset of $\mathbf{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf on Ω admitting a finite free resolution on Ω. Suppose G is a bounded polydomain in $\mathbf{C}^n$ and $G \subset\subset \Omega$. If $S_k(\mathcal{F}) \cap \partial_{k+1} G = \emptyset$ for $0 \le k < n$, then G is locally strongly $\mathcal{F}$-privileged and hence strongly $\mathcal{F}$-privileged.*

Proof. Use induction on n. The case $n = 0$ is trivial. Take $x \in G$ and take an open neighborhood U of x in $\mathbf{C}^n$. Let k be the largest integer such that $x \in \partial_k G$. We can assume w.l.o.g. that $x = (z_1^0, \cdots, z_n^0)$ with $z_i^0 \in \partial G_i$ for $1 \le i \le k$. Let $x' = (z_1^0, \cdots, z_k^0)$ and let

$$\mathcal{G} = \left(\mathcal{F} \Big/ \sum_{i=1}^{k} (z_i - z_i^0)\mathcal{F} \right) \Bigg| \{x'\} \times \mathbf{C}^{n-k}.$$

By induction hypothesis and Lemma (6.1), there exists a nonempty open $(n-1)$-dimensional polydisc P centered at $(z_{k+1}^0, \cdots, z_n^0)$ such that $P \subset G_{k+1} \times \cdots \times G_n$, $\{x'\} \times P \subset\subset U$, and $\{x'\} \times P$ is strongly $\mathcal{G}$-privileged. By Proposition (1.4) and (2.4), there exists an open neighborhood D of x' in $\mathbf{C}^k$ such that $D \times P \subset U$ and $((G_1 \times \cdots \times G_k) \cap D) \times P$ is strongly $\mathcal{F}$-privileged. Q.E.D.

(6.3) **Lemma.** *Suppose Ω is an open subset of $\mathbf{C}^n$ and V is a subvariety of pure dimension k in Ω. Suppose G is a polydomain in $\mathbf{C}^n$ such that $G \subset\subset \Omega$. If $V \cap \partial_{k+1} G = \emptyset$; then $V \cap G$ has only a finite number of branches.*

Proof. The cases where $k \le 0$ or $k \ge n$ are trivial. We can assume that $0 < k < n$. There exists $G_i' \subset\subset G_i$ $(1 \le i \le n)$ such that, for $1 \le i \le n$, $G_i - (G_i')^-$ has only a finite number of components and, for any distinct elements $i_1, \cdots, i_{k+1}$ of $\{1, \cdots, n\}$, the set

$$\{(z_1, \cdots, z_n) \in G^- \cap V | z_i \in G_i^- - G_i' \quad \text{for } i = i_1, \cdots, i_{k+1}\}$$

is empty. Let

$$D_{i_1 \cdots i_k} = \{(z_1, \cdots, z_n) \in G | z_i \in G_i - (G_i')^- \quad \text{for } i = i_1, \cdots, i_k\}.$$

Then, for any distinct elements $i_1, \cdots, i_k$ of $\{1, \cdots, n\}$, the projection $(z_1, \cdots, z_n) \mapsto (z_{i_1}, \cdots, z_{i_k})$ makes $D_{i_1 \cdots i_k} \cap V$ an analytic cover over $(G_{i_1} - (G_{i_1}')^-) \times \cdots \times (G_{i_k} - (G_{i_k}')^-)$. Hence, it suffices to show that every branch of $V \cap G$ intersects some $D_{i_1 \cdots i_k}$.

Suppose W is a branch of $V \cap G$. Let $G' = G_1' \times \cdots \times G_n'$. We can choose a polydomain G'' in $\mathbf{C}^n$ such that $G' \subset\subset G'' \subset\subset G$ and $W \cap (G'')^- \ne \emptyset$. By Lemma (1.3), $W \cap \partial_k G'' \ne 0$. Hence W intersects some $D_{i_1 \cdots i_k}$. Q.E.D.

(6.4) **Proposition.** *Suppose Ω is a Stein open subset of $\mathbf{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf on Ω admitting a finite free resolution on Ω. Suppose G is a polydomain in $\mathbf{C}^n$ and $G \subset\subset \Omega$. If $S_k(\mathcal{F}) \cap \partial_{k+1} G = \emptyset$ for $0 \le k < n$, then G is $\mathcal{F}$-privileged and there exist $x_1, \cdots, x_l \in G$ such that the natural map $B(G, \mathcal{F}) \to \bigoplus_{i=1}^{l} \mathcal{F}_{x_i}$ is injective.*

Proof. By Lemma (6.3), we can choose $x_1, \cdots, x_l \in G$ such that, for $0 \le$

$k < n$, every k-dimensional branch of $G \cap S_k(\mathfrak{F})$ contains some x_i. The proposition follows from Proposition (6.2) and [7, p. 49, (2.3)]. Q.E.D.

7. For a holomorphic Banach bundle E, denote by $\mathcal{O}(E)$ the sheaf of germs of holomorphic sections of E.

(7.1) **Lemma.** *Suppose Ω is an open subset of $\mathbf{C}^n$, $E^{(i)}$ is a globally trivial holomorphic Banach bundle on Ω $(0 \le i \le m)$, and*

$$(*) \qquad 0 \to E^{(m)} \xrightarrow{\phi^{(m)}} \cdots \to E^{(1)} \xrightarrow{\phi^{(1)}} E^{(0)}$$

is a complex of bundle-homomorphisms on Ω. Let

$$(\#) \qquad 0 \to \mathcal{O}(E^{(m)}) \xrightarrow{\psi^{(m)}} \cdots \to \mathcal{O}(E^{(1)}) \xrightarrow{\psi^{(1)}} \mathcal{O}(E^{(0)})$$

be induced by $(*)$. Suppose $G = G_1 \times \cdots \times G_n$ is a polydomain in $\mathbf{C}^n$ such that $G \subset\subset \Omega$ and every point of ∂G_i is a peak point of $B(G_i)$ for $1 \le i \le n$. Suppose the following three conditions are satisfied.

(i) *For $1 \le i \le m$ and $z^0 \in \partial_n G$, $\phi^{(i)}_{z^0}$ is direct.*

(ii) *For $1 \le i \le m$, the map $B(G, E^{(i)}) \to B(G, E^{(i-1)})$ induced by $\phi^{(i)}$ has a closed image.*

(iii) *There exists an open neighborhood D of $\partial_n G$ in Ω such that the complex $(\#)$ is exact on D.*

Then the complex $(*)$, when restricted to $\partial_n G$, is exact.

Proof. Use induction on m. The case $m = 0$ is trivial. Assume $m > 0$. Let $B^*(G^-, E^{(i)})$ be the Banach space of all sections of $E^{(i)}$ over G^- which are continuous in G^- and holomorphic on G $(0 \le i < m)$. Let $\tilde{\phi}^{(i)}: B^*(G^-, E^{(i-1)}) \to B^*(G^-, E^{(i-1)})$ be induced by $\phi^{(i)}$ $(0 < i \le m)$. Since $B^*(G^-, E^{(i)})$ is topologically isomorphic to $B(G, E^{(i)})$ $(0 \le i \le m)$, $\operatorname{Im} \tilde{\phi}^{(i)}$ is closed in $B^*(G^-, E^{(i-1)})$ $(0 < i \le m)$.

Fix $z^0 = (z^0_1, \cdots, z^0_n) \in \partial_n G$. By induction hypothesis and (2.2), for some open neighborhood U of z^0 in D, $\operatorname{Im} \phi^{(2)}|U$ is a holomorphic Banach bundle and there exists a holomorphic Banach bundle F on U such that

$$(\dagger) \qquad E^{(1)}|U = F \oplus (\operatorname{Im} \phi^{(2)}|U).$$

Let $\alpha: F \to E^{(0)}|U$ be the restriction of $\phi^{(1)}$. We need only show that $(\operatorname{Ker} \alpha)_{z^0} = 0$. Suppose the contrary. Then there exists $e \in F_{z^0}$ such that $e \neq 0$ and $\alpha_{z^0}(e) = 0$. Since $E^{(1)}$ is globally trivial, there exists $f \in B^*(G^-, E^{(1)})$ such that $f(z^0) = e$. We have $\tilde{\phi}^{(1)}(f)(z^0) = \alpha_{z^0}(e) = 0$.

Since z_i^0 is a peak point of $B(G_i)$, there exists a function λ_i on G_i^- which is continuous on G_i^- and is holomorphic on G_i such that $\lambda_i(z_i^0) = 1$ and $|\lambda_i(z_i)| < 1$ for $z_i \in G_i^- - \{z_i^0\}$ $(1 \leq i \leq n)$. Let $\lambda = \lambda_1 \cdots \lambda_n$. Let H be the fiber of the trivial bundle $E^{(0)}$ and let $\|\cdot\|_H$ be the norm of H. The sequence $\tilde{\phi}^{(1)}(\lambda^\mu f)$ approaches 0 in $B^*(G^-, E^{(0)})$ as $\mu \to \infty$, because, for any given $\epsilon > 0$, there exists an open neighborhood W of z^0 in $\mathbf{C}^n$ such that $\|\tilde{\phi}^{(1)}(f)(z)\|_H < \epsilon$ for $z \in G^- \cap W$ and there exists μ_0 such that

$$\left(\sup_{z \in G^- - W} |\lambda(z)|\right)^\mu \left(\sup_{z \in G^-} \|\tilde{\phi}^{(1)}(f)(z)\|_H\right) < \epsilon$$

for $\mu \geq \mu_0$. Since $\tilde{\phi}^{(1)}$ has a closed image, by applying the open mapping theorem to the map $B^*(G^-, E^{(1)}) \to \operatorname{Im} \tilde{\phi}^{(1)}$ induced by $\tilde{\phi}^{(1)}$, we conclude that there exists $g_\mu \in B^*(G^-, E^{(1)})$ $(1 \leq \mu < \infty)$ such that $g_\mu \to 0$ in $B^*(G^-, E^{(1)})$ as $\mu \to \infty$ and $\tilde{\phi}^{(1)}(g_\mu) = \tilde{\phi}^{(1)}(\lambda^\mu f)$ $(1 \leq \mu < \infty)$.

Let β: $E^{(1)}|U \to F$ be the projection obtained from the direct sum decomposition (†). Let $\tilde{\beta}$: $\mathcal{O}(E^{(1)})|U \to \mathcal{O}(F)$ be induced by β. Since $\tilde{\phi}^{(1)}(\lambda^\mu f - g_\mu) = 0$, we have

$$(\lambda^\mu f - g_\mu)|G \cap U \in \Gamma(G \cap U, \operatorname{Ker} \psi^{(1)})$$

for $1 \leq \mu < \infty$. Since the complex (#) is exact in D,

$$(\lambda^\mu f - g_\mu)|G \cap U \in \Gamma(G \cap U, \operatorname{Im} \psi^{(2)})$$

for $1 \leq \mu < \infty$. By (†), $\mathcal{O}(\operatorname{Im} \phi^{(2)}|U) = \operatorname{Im} \psi^{(2)}|U$. Hence $\tilde{\beta}((\lambda^\mu f - g_\mu)|G \cap U) = 0$ for $1 \leq \mu < \infty$. Since $\tilde{\beta}((\lambda^\mu f - g_\mu)|G^- \cap U)$ is a continuous section of F over $G^- \cap U$, we have $\tilde{\beta}((\lambda^\mu f - g_\mu)|G^- \cap U) = 0$ for $1 \leq \mu < \infty$. Hence

$$\beta_{z^0}(g_\mu(z^0)) = \lambda(z^0)^\mu \beta_{z^0}(f(z^0)) = e \qquad (1 \leq \mu < \infty).$$

Since $g_\mu \to 0$ in $B^*(G^-, E^{(1)})$ as $\mu \to \infty$, we have $\beta_{z^0}(g_\mu(z^0)) \to 0$ as $\mu \to \infty$, contradicting that $e \neq 0$. Q.E.D.

(7.2) **Lemma.** *Suppose $0 \to \mathfrak{G} \to \mathfrak{F} \to \mathfrak{R} \to 0$ is an exact sequence of coherent analytic sheaves and sheaf-homomorphisms on a Stein complex space $(X, \mathcal{O})$. Suppose $\mathfrak{G}$ and $\mathfrak{F}$ have finite free resolutions of length $\leq m$ on X. If Y is a relatively compact Stein open subset of X, then there exists a finite free resolution of $\mathfrak{R}$ on Y of length $\leq m + 1$.*

Proof. First we make the following observation. If

$$0 \to \mathcal{K} \to \mathcal{O}^{\mu_l} \to \dots \to \mathcal{O}^{\mu_0} \to \mathcal{F} \to 0,$$

$$0 \to \mathcal{K}' \to \mathcal{O}^{\mu'_l} \to \dots \to \mathcal{O}^{\mu'_0} \to \mathcal{F} \to 0$$

are two exact sequences of sheaf-homomorphisms on X, then there exist nonnegative integers μ and μ' such that $\mathcal{K} \oplus \mathcal{O}^{\mu} \approx \mathcal{K}' \oplus \mathcal{O}^{\mu'}$ on X, because, by a trivial modification of the proof of [4, p. 202, VI. F. 3], we can show, by using Theorem B of Cartan-Oka, that the two sequences become isomorphic sequences after we apply to each of them a finite number of modifications [4, p. 201, Definition VI. F. 1].

Construct the following commutative diagram on Y with exact rows and exact columns:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & \mathcal{N} & \to & \mathcal{O}^{r_{m-1}} & \to & \dots & \to & \mathcal{O}^{r_0} & \to & \mathcal{G} \to 0 \\
 & & \downarrow{\scriptstyle\sigma} & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & \mathcal{M} & \to & \mathcal{O}^{r_{m-1}+s_{m-1}} & \to & \dots & \to & \mathcal{O}^{r_0+s_0} & \to & \mathcal{F} \to 0 \\
 & & \downarrow{\scriptstyle\tau} & & \downarrow & & & & \downarrow & & \downarrow \\
0 & \to & \mathcal{S} & \xrightarrow{\phi_m} & \mathcal{O}^{s_{m-1}} & \xrightarrow{\phi_{m-1}} & \dots & \xrightarrow{\phi_1} & \mathcal{O}^{s_0} & \xrightarrow{\phi_0} & \mathcal{R} \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & & & 0 & & 0
\end{array}$$

By the preceding observation, for some nonnegative integer k, p, and q, there exist sheaf-isomorphisms

$$\alpha\colon \mathcal{N} \oplus \mathcal{O}^k \xrightarrow{\approx} \mathcal{O}^p \quad \text{and} \quad \beta\colon \mathcal{M} \oplus \mathcal{O}^k \xrightarrow{\approx} \mathcal{O}^q \quad \text{on } Y.$$

Define $\gamma\colon \mathcal{N} \oplus \mathcal{O}^k \to \mathcal{M} \oplus \mathcal{O}^k$ by $\gamma(f \oplus g) = \sigma(f) \oplus g$ for $f \oplus g \in \mathcal{N}_x \oplus \mathcal{O}^k_x$ and $x \in Y$.

We have the following finite free resolution of $\mathcal{R}$ on Y:

$$0 \to \mathcal{O}^p \xrightarrow{\tilde{\phi}_{m+1}} \mathcal{O}^q \xrightarrow{\tilde{\phi}_m} \mathcal{O}^{s_{m-1}} \oplus \mathcal{O}^k$$
$$\xrightarrow{\tilde{\phi}_{m-1}} \mathcal{O}^{s_{m-2}} \xrightarrow{\phi_{m-2}} \dots \xrightarrow{\phi_1} \mathcal{O}^{s_0} \xrightarrow{\phi_0} \mathcal{R} \to 0,$$

where $\tilde{\phi}_{m+1} = \beta\gamma\alpha^{-1}$; $\tilde{\phi}_{m-1}(a \oplus b) = \phi_{m-1}(a)$ for $a \oplus b \in \mathcal{O}^{s_{m-1}}_x \oplus \mathcal{O}^k_x$ and $x \in Y$; and $\tilde{\phi}_m$ is so defined that $\tilde{\phi}_m\beta(c \oplus d) = \phi_m\tau(c) \oplus d$ for $c \oplus d \in \mathcal{M}_x \oplus \mathcal{O}^k_x$ for $x \in Y$. Q.E.D.

(7.3) **Lemma.** *Suppose S and Ω are respectively open subsets of $\mathbf{C}^k$ and*

$\mathbf{C}^n$, *and* $\pi: S \times \Omega \to S$ *is the natural projection. Suppose*

$$0 \to {}_{k+n}\mathcal{O}^{p_m} \to \cdots \to {}_{k+n}\mathcal{O}^{p_1} \to {}_{k+n}\mathcal{O}^{p_0} \xrightarrow{\phi_0} \mathcal{F} \to 0$$

is an exact sequence of sheaf-homomorphisms on $S \times \Omega$. *Suppose* $s \in S$ *and* K *is a compact subset of* Ω *admitting a basis of Stein neighborhoods in* Ω. *If* $\operatorname{Ker} \phi_0$ *is* π*-flat on* $\{s\} \times K$ *and* $\mathcal{F}$ *is* π*-flat on* $\{s\} \times \partial K$, *then, for some open neighborhood* Ω' *of* K *in* Ω, $\mathcal{F}(s)$ *admits a finite free resolution on* Ω'.

Proof. Let $\mathcal{G} = \operatorname{Ker} \phi_0$. Since $\mathcal{G}$ is π-flat on $\{s\} \times K$ and K has a basis of Stein neighborhoods in Ω, there exists a Stein open neighborhood Ω'' of K in Ω such that

$$0 \to {}_{k+n}\mathcal{O}^{p_m}(s) \to \cdots \to {}_{k+n}\mathcal{O}^{p_1}(s) \to \mathcal{G}(s) \to 0$$

is exact on Ω''. Hence $\mathcal{G}(s)$ admits a finite free resolution on Ω''. Consider the sequence

$$\mathcal{G}(s) \xrightarrow{\alpha} {}_{k+n}\mathcal{O}^{p_0}(s) \to \mathcal{F}(s) \to 0.$$

Since $\mathcal{F}$ is π-flat on $\{s\} \times \partial K$, the support of $\operatorname{Ker} \alpha$ is disjoint from ∂K. For some relatively compact Stein open neighborhood Ω^* of K in Ω'', the support of $\operatorname{Ker} \alpha | \Omega^*$ is a finite set. $\operatorname{Ker} \alpha | \Omega^*$ can be trivially extended to a coherent analytic sheaf on $\mathbf{C}^n$. Hence $\operatorname{Ker} \alpha$ admits a finite free resolution on Ω^*. By applying Lemma (7.2) to the two sequences on Ω^*:

$$0 \to \operatorname{Ker} \alpha \to \mathcal{G}(s) \to \operatorname{Im} \alpha \to 0,$$

$$0 \to \operatorname{Im} \alpha \to {}_{k+n}\mathcal{O}^{p_0}(s) \to \mathcal{F}(s) \to 0,$$

we conclude that $\mathcal{F}(s)$ admits a finite free resolution on any relatively compact open neighborhood Ω' of K in Ω^*. Q.E.D.

(7.4) **Lemma.** *Suppose* Ω *is a Stein open subset of* $\mathbf{C}^n$ *and* $\mathcal{F}$ *is a coherent analytic sheaf on* Ω *admitting a finite free resolution on* Ω. *Suppose* G *is a polydomain in* $\mathbf{C}^n$ *such that* $G \subset\subset \Omega$ *and* G *is* $\mathcal{F}$*-privileged. If* ${}_n\mathcal{O}^{q_1} \xrightarrow{\psi} {}_n\mathcal{O}^{q_0} \to \mathcal{F}$ *is an exact sequence on* Ω, *then the map* $\tilde{\psi}: B(G, {}_n\mathcal{O}^{q_1}) \to B(G, {}_n\mathcal{O}^{q_0})$ *induced by* ψ *is direct.*

Proof. Let

$$0 \to {}_n\mathcal{O}^{p_m} \to \cdots \to {}_n\mathcal{O}^{p_1} \xrightarrow{\phi} {}_n\mathcal{O}^{p_0} \to \mathcal{F} \to 0$$

be the finite free resolution of $\mathcal{F}$ on Ω. By a trivial modification of the proof of [4, p. 202, VI. F. 3], we can construct the following commutative diagram on Ω:

$$\begin{array}{ccc} {}_n\mathcal{O}^{q_1} \oplus {}_n\mathcal{O}^{d} \oplus {}_n\mathcal{O}^{d'} & \xrightarrow{\sigma} & {}_n\mathcal{O}^{q_0} \oplus {}_n\mathcal{O}^{d} \\ \downarrow \wr & & \downarrow \wr \\ {}_n\mathcal{O}^{p_1} \oplus {}_n\mathcal{O}^{e} \oplus {}_n\mathcal{O}^{e'} & \xrightarrow{\tau} & {}_n\mathcal{O}^{p_0} \oplus {}_n\mathcal{O}^{e}, \end{array}$$

where $\sigma(a \oplus b \oplus c) = \psi(a) \oplus b$ for $a \oplus b \oplus c \in {}_n\mathcal{O}^{q_1}_n \oplus {}_n\mathcal{O}^{d}_x \oplus {}_n\mathcal{O}^{d'}_x$ and $x \in \Omega$; and $\tau(f \oplus g \oplus h) = \phi(f) \oplus g$ for $f \oplus g \oplus h \in {}_n\mathcal{O}^{p_1}_x \oplus {}_n\mathcal{O}^{e}_x \oplus {}_n\mathcal{O}^{e'}_x$ and $x \in \Omega$. Since G is $\mathcal{F}$-privileged, the map $B(G, {}_n\mathcal{O}^{p_1}) \to B(G, {}_n\mathcal{O}^{p_0})$ induced by ϕ is direct. Hence, from the commutative diagram above, we conclude that $\tilde{\psi}$ is direct. Q.E.D.

(7.5) **Lemma.** *Suppose S and Ω are respectively Stein open subsets of $\mathbf{C}^k$ and $\mathbf{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf on $S \times \Omega$ admitting a finite free resolution*

$$0 \to {}_{k+n}\mathcal{O}^{p_m} \to \cdots \to {}_{k+n}\mathcal{O}^{p_1} \to {}_{k+n}\mathcal{O}^{p_0} \to \mathcal{F} \to 0$$

on $S \times \Omega$. Suppose $s \in S$ and G is a relatively compact open subset of Ω. If the sequence

$$(*) \qquad 0 \to B(G, {}_{k+n}\mathcal{O}^{p_m}(s)) \to \cdots \to B(G, {}_{k+n}\mathcal{O}^{p_1}(s)) \to B(G, {}_{k+n}\mathcal{O}^{p_0}(s))$$

is exact, then $\mathcal{F}$ is π-flat on $\{s\} \times G$, where $\pi: S \times \Omega \to S$ is the natural projection.

Proof. We need only show that the sequence

$$0 \to {}_{k+n}\mathcal{O}^{p_m}(s) \xrightarrow{\phi_m} \cdots \xrightarrow{\phi_2} {}_{k+n}\mathcal{O}^{p_1}(s) \xrightarrow{\phi_1} {}_{k+n}\mathcal{O}^{p_0}(s)$$

is exact on G. This follows from the exactness of $(*)$, because, for $1 \leq i \leq m$, $\Gamma(\Omega, \operatorname{Ker} \phi_i)$ generates $\operatorname{Ker} \phi_i$ on Ω. Q.E.D.

(7.6) **Proposition.** *Suppose Ω is a Stein open subset of $\mathbf{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf on Ω admitting a finite free resolution on Ω. Suppose $G = G_1 \times \cdots \times G_n$ is a polydomain in $\mathbf{C}^n$ such that $G \subset\subset \Omega$ and every point of ∂G_i is a peak point of $B(G_i)$ for $1 \leq i \leq n$. If G is weakly $\mathcal{F}$-privileged, then $S_k(\mathcal{F}) \cap \partial_{k+1} G = \emptyset$ for $0 \leq k < n$.*

Proof. We can assume w.l.o.g. that Ω is a polydomain $\Omega_1 \times \cdots \times \Omega_n$. Let

$$(*)\qquad 0 \longrightarrow {}_n\mathcal{O}^{p_m} \xrightarrow{\alpha_m} \cdots \xrightarrow{\alpha_2} {}_n\mathcal{O}^{p_1} \xrightarrow{\alpha_1} {}_n\mathcal{O}^{p_0} \xrightarrow{\alpha_0} \mathcal{F} \longrightarrow 0$$

be the finite free resolution of $\mathcal{F}$ on Ω. We are going to prove the proposition by induction on m. The case $m = 0$ is trivial. Suppose $m > 0$. Let $\mathcal{G} = \operatorname{Ker} \alpha_0$. Since G is weakly $\mathcal{F}$-privileged, G is weakly $\mathcal{G}$-privileged. By induction hypothesis, $S_k(\mathcal{G}) \cap \partial_{k+1} G = \emptyset$ for $0 \le k < n$.

For distinct elements $i_1, \cdots, i_k$ of $\{1, \cdots, n\}$ let $\pi_{i_1 \cdots i_k}\colon \mathbf{C}^n \to \mathbf{C}^k$ be the projection defined by

$$\pi_{i_1 \cdots i_k}(z_1, \cdots, z_n) = (z_{i_1}, \cdots, z_{i_k}).$$

For $s = (z^0_{i_1}, \cdots, z^0_{i_k}) \in \Omega_{i_1} \times \cdots \times \Omega_{i_k}$, denote the sheaf

$$\left(\mathcal{F} \Big/ \sum_{\mu=1}^{k} (z_{i_\mu} - z^0_{i_\mu})\mathcal{F}\right) \Bigg| \Omega \cap \pi^{-1}_{i_1 \cdots i_k}(s)$$

by $\mathcal{F}_{i_1 \cdots i_k}(s)$, and denote the set

$$\{(z_1, \cdots, z_n) \in \mathbf{C}^n \mid z_i = z^0_i \text{ for } i = i_1, \cdots, i_k;$$
$$z_j \in G_j \text{ for } j \neq i_1, \cdots, i_k\}$$

by $G_{i_1 \cdots i_k}(s)$.

We are going to prove the following $(a)_k$ and $(b)_k$ for $1 \le k \le n+1$ by descending induction on k.

$(a)_k$ $\qquad \mathcal{F}$ is $\pi_{i_1 \cdots i_k}$-flat on $G^- \cap \pi^{-1}_{i_1 \cdots i_k}(\partial G_{i_1} \times \cdots \times \partial G_{i_k})$.

$(b)_k$ $\qquad$ For $s \in \partial G_{i_1} \times \cdots \times \partial G_{i_k}$,

$$S_l(\mathcal{F}_{i_1 \cdots i_k}(s)) \cap \partial_{l+1} G_{i_1 \cdots i_k}(s) = \emptyset \quad \text{for } 0 \le l < n-k.$$

Since $(a)_{n+1}$ and $(b)_{n+1}$ are vacuous statements, it suffices to prove the following statements.

(I) $(a)_{k+1}$ and $(b)_{k+1} \Rightarrow (b)_k$.

(II) $(a)_{k+1}$ and $(b)_k \Rightarrow (a)_k$.

To prove (I), we can assume that $i_\mu = \mu$ for $1 \le \mu \le k$. Take $1 \le l \le n-k$ and take $s^* \in \partial_l G_{1 \dots k}(s)$. We can assume w.l.o.g. that $s^* = (z^0_1, \cdots, z^0_n)$ with $z^0_i \in \partial G_i$ for $k+1 \le i \le k+l$. We have to show that $s^* \notin S_{l-1}(\mathcal{F}_{1 \dots k}(s))$. Suppose the contrary. Then the homological codimension of $\mathcal{F}_{1 \dots k}(s)$ at s^* is $\le l-1$. Since, by $(a)_{k+1}$, $\mathcal{F}$ is $\pi_{1 \dots (k+1)}$-flat on $\partial G_1 \times \cdots \times \partial G_{k+1} \times G^-_{k+2} \times \cdots \times G^-_n$, it follows that $z_{k+1} - z^0_{k+1}$ is not a zero-divisor for $\mathcal{F}_{1 \dots k}(s)$ at s^*. Let $s' =$

$(z_1^0, \cdots, z_{k+1}^0)$. Since

$$\mathcal{F}_{1\cdots(k+1)}(s') = \mathcal{F}_{1\cdots k}(s)/(z_{k+1} - z_{k+1}^0)\mathcal{F}_{1\cdots k}(s),$$

we conclude that the homological codimension of $\mathcal{F}_{1\cdots(k+1)}(s')$ at s^* is $\leq l-2$. However, since $s^* \in \partial_{l-1}G_{1\cdots(k+1)}(s')$, it follows from $(\mathrm{b})_{k+1}$ that $s^* \notin S_{l-2}(\mathcal{F}_{1\cdots(k+1)}(s'))$. We have a contradiction. (I) is proved.

To prove (II), we can again assume that $i_\mu = \mu$ for $1 \leq \mu \leq k$. Let $G' = G_1 \times \cdots \times G_k$ and $G'' = G_{k+1} \times \cdots \times G_n$. Let $E^{(i)} = B(G'', {}_n\mathcal{O}^{p_i})|\Omega_1 \times \cdots \times \Omega_k$ for $0 \leq i \leq m$. Let

$$(\#) \qquad 0 \to E^{(m)} \xrightarrow{\phi^{(m)}} \cdots \xrightarrow{\phi^{(2)}} E^{(1)} \xrightarrow{\phi^{(1)}} E^{(0)}$$

be induced by $(*)$. We will apply Lemma (7.1) to $(\#)$ to obtain (II).

In the rest of this proof, for $s \in \partial_k G'$, $\mathcal{F}(s)$ stands for $\mathcal{F}_{1\cdots k}(s)$. $\mathcal{F}(s)$ is regarded as a sheaf on $\Omega_{k+1} \times \cdots \times \Omega_n$. $\mathcal{G}(s)$ and ${}_n\mathcal{O}^{p_i}(s)$ have meanings similar to $\mathcal{F}(s)$.

Since $S_l(\mathcal{G}) \cap \partial_{l+1} G = \emptyset$ for $0 \leq l < n$, it follows from Proposition (1.4) that

$$(\dagger) \qquad \mathcal{G} \text{ is } \pi_{1\cdots k}\text{-flat on } (\partial_k G') \times (G'')^-$$

and

$$(\dagger\dagger) \qquad S_l(\mathcal{G}(s)) \cap \partial_{l+1}G'' = \emptyset \quad \text{for } 0 \leq l < n-k \text{ and } s \in \partial_k G'.$$

By Proposition (6.2) and $(\dagger\dagger)$,

$$(**) \qquad G'' \text{ is strongly } \mathcal{G}(s)\text{-privileged for } s \in \partial_k G'.$$

By (2.4) and $(\dagger)$, there exist Stein open subsets $U_1, \cdots, U_r$ of $\mathbf{C}^k$ such that $\partial_k G' \subset \bigcup_{j=1}^r U_j$ and $D \times G''$ is strongly $\mathcal{G}$-privileged for any open subset D of any U_j.

We claim that the complex

$$0 \to \mathcal{O}(E^{(m)}) \xrightarrow{\psi^{(m)}} \cdots \to \mathcal{O}(E^{(1)}) \xrightarrow{\psi^{(1)}} \mathcal{O}(E^{(0)})$$

induced by $(\#)$ is exact on $\bigcup_{j=1}^r U_j$. To verify the claim, take $1 \leq j \leq r$ and open subsets $D \subset\subset D' \subset U_j$. Take $f' \in \Gamma(D', \mathcal{O}(E^{(i)}))$ for some $0 < i \leq m$ such that $\psi^{(i)}(f') = 0$. $f'|D$ corresponds uniquely to an element $f \in B(D \times G'', {}_n\mathcal{O}^{p_i})$ and f satisfies $\alpha_i(f) = 0$. If $i = m$, from $\operatorname{Ker} \alpha_m = 0$ it follows that $f = 0$ and $f'|D = 0$. If $1 \leq i < m$, since $D \times G''$ is strongly $\mathcal{G}$-privileged, the sequence

$$B(D \times G'', {}_n\mathcal{O}^{p_{i+1}}) \to B(D \times G'', {}_n\mathcal{O}^{p_i}) \to B(D \times G'', {}_n\mathcal{O}^{p_{i-1}})$$

induced by $(*)$ is exact (the exactness for the case $i = 1$ comes from the injectivity of the natural map $B(D \times G'', \mathcal{G}) \to \Gamma(D \times G'', \mathcal{G}))$. Therefore there exists $g \in B(D \times G'', {}_n\mathcal{O}^{p_{i+1}})$ such that $\alpha_{i+1}(g) = f$. g corresponds uniquely to an element $g' \in \Gamma(D, \mathcal{O}(E^{(i+1)}))$. $\psi^{(i+1)}(g') = f'|D$. The claim is verified.

Next we claim that $\psi_s^{(i)}$ is direct for $1 \leq i \leq m$ and $s \in \partial_k G'$. By virtue of (**), we need only verify that $\psi_s^{(1)}$ is direct for $s \in \partial_k G'$. Fix $s \in \partial_k G'$. Since, by $(a)_{k+1}$, $\mathcal{F}$ is $\pi_{1\cdots ki}$-flat on $G^- \cap \pi_{1\cdots ki}^{-1}(\partial G_1 \times \cdots \times \partial G_k \times \partial G_i)$ for $k+1 \leq i \leq n$, it follows that

$$(\natural) \qquad \mathcal{F} \text{ is } \pi_{1\cdots k}\text{-flat on } \{s\} \times \partial G''.$$

By (†) and Lemma (7.3), $\mathcal{F}(s)$ admits a finite free resolution on an open neighborhood of $(G'')^-$ in $\Omega_{k+1} \times \cdots \times \Omega_n$. By applying Lemma (7.4) to the exact sequence

$${}_n\mathcal{O}^{p_1}(s) \to {}_n\mathcal{O}^{p_0}(s) \to \mathcal{F}(s) \to 0$$

on $\Omega_{k+1} \times \cdots \times \Omega_n$, we conclude from $(b)_k$ and Proposition (6.2) that $\psi_s^{(1)}$ is direct.

Since G is weakly $\mathcal{F}$-privileged, for $0 < i \leq m$ the map $B(G', E^{(i)}) \to B(G', E^{(i-1)})$ induced by $\phi^{(i)}$ has a closed image. By Lemma (7.1), the sequence (#), when restricted to any point of $\partial_k G'$, is exact. It follows from Lemma (7.5) and ($\natural$) that $(a)_k$ holds. (II) is proved.

The proposition follows from $(a)_1, \cdots, (a)_n$. Q.E.D.

8.

(8.1) **Proposition.** *Suppose Ω is an open subset of $\mathbb{C}^n$ and $\mathcal{F}$ is a coherent analytic sheaf on Ω admitting a finite free resolution on Ω. Suppose G is a polydomain in $\mathbb{C}^n$ such that $G \subset\subset \Omega$. If G is semilocally weakly $\mathcal{F}$-privileged, then G is weakly $\mathcal{F}$-privileged.*

Proof. Let

$$0 \to {}_n\mathcal{O}^{p_m} \to \cdots \to {}_n\mathcal{O}^{p_1} \xrightarrow{\phi} {}_n\mathcal{O}^{p_0} \to \mathcal{F} \to 0$$

be the finite free resolution of $\mathcal{F}$ on Ω. We prove by induction on m. Let $\mathcal{G} = \operatorname{Ker} \phi$. By induction hypothesis and Propositions (7.6) and (6.2), G is locally strongly $\mathcal{G}$-privileged. By Lemma (5.2), the sequence

$$(*) \qquad 0 \to \mathcal{O}^{p_m}_{G^-} \to \cdots \to \mathcal{O}^{p_1}_{G^-} \xrightarrow{\psi} \mathcal{O}^{p_0}_{G^-}$$

is exact on G^-. Since $B(G, {}_n\mathcal{O}^{p_i}) \approx \Gamma(G^-, \mathcal{O}^{p_i}_{G^-})$ for $i = 0, 1$, to finish the proof, we need only show that the map $\tilde{\psi}: \Gamma(G^-, \mathcal{O}^{p_1}_{G^-}) \to \Gamma(G^-, \mathcal{O}^{p_0}_{G^-})$ has a closed image.

Suppose $f \in (\operatorname{Im} \widetilde{\psi})^-$. Since G is semilocally weakly $\mathcal{F}$-privileged, there exist open subsets $U_1, \cdots, U_k$ of $\mathbf{C}^n$ such that $\partial G \subset \bigcup_{i=1}^k U_i$ and $G \cap U_i$ is weakly $\mathcal{F}$-privileged $(1 \leq i \leq k)$. For $1 \leq i \leq k$ let α_i: $B(G \cap U_i, {}_n\mathcal{O}^{p1}) \to B(G \cap U_i, {}_n\mathcal{O}^{p0})$ be induced by ϕ. $f|G \cap U_i \in \operatorname{Im} \alpha_i$ for $1 \leq i \leq k$, because $f|G \cap U_i \in (\operatorname{Im} \alpha_i)^-$ and $\operatorname{Im} \alpha_i$ is closed. It follows that $f|G^- \cap U_i \in \Gamma(G^- \cap U_i, \operatorname{Im} \psi)$ for $1 \leq i \leq k$. By [4, p. 85, II.D.3], $f|G \in \Gamma(G, \operatorname{Im} \psi)$. Hence $f \in \Gamma(G^-, \operatorname{Im} \psi)$. By Corollary (4.7) and the exactness of (*), $f \in \operatorname{Im} \widetilde{\psi}$. Q.E.D.

(8.2) **Proof of Main Theorem.** The Main Theorem follows from Propositions (6.2), (6.4), (7.6), and (8.1).

(8.3) **Remark.** The Main Theorem does not hold in its full strength if the following condition is not included in its hypotheses:

(*) Every point of ∂G_i is a peak point of $B(G_i)$ for $1 \leq i \leq n$.

A trivial counterexample is the following: Let Δ be the 1-dimensional open unit disc and let $\Delta' = \Delta - [0, 1)$. Let $G = \Delta' \times \Delta'$ and $\mathcal{F}$ be the maximum sheaf ideals on $\mathbf{C}^2$ whose zero-set is $\{0\}$. $S_1(\mathcal{F}) \cap \partial_2 G \neq \emptyset$, but G is strongly $\mathcal{F}$-privileged, because $B(\Delta' \times \Delta', {}_2\mathcal{O}) \approx B(\Delta \times \Delta, {}_2\mathcal{O})$.

However, without (*) some implications of the Main Theorem still hold, e.g. (i) $\Rightarrow$ (ii), and (i) $\Rightarrow$ (vi).

9. Finally we conclude this paper by a simple proposition concerning extending uniformly continuous holomorphic functions on a subvariety of a polydomain to the polydomain. This proposition follows easily from arguments analogous to those given in [6] and from the preceding results and techniques of this paper.

(9.1) **Proposition.** *Suppose G is a bounded polydomain in $\mathbf{C}^n$, Ω is an open neighborhood of G^- in $\mathbf{C}^n$, and $(X, \mathcal{O}_X)$ is a subvariety of pure division r in Ω endowed with the reduced complex structure. Suppose the following three conditions are satisfied.*

(i) *$X \cap \partial_{r+1} G = \emptyset$.*

(ii) *$S_k(\mathcal{O}_X) \cap \partial_k G = \emptyset$ for $1 \leq k < r$.*

(iii) *If $i_1, \cdots, i_r$ are distinct elements of $\{1, \cdots, n\}$ and $x = (z_1^0, \cdots, z_n^0) \in G^- \cap X$ such that $z_i \in \partial G_i$ for $i = i_1, \cdots, i_r$, then the map $\mathbf{C}^n \to \mathbf{C}^r$ defined by $(z_1, \cdots, z_n) \mapsto (z_{i_1}, \cdots, z_{i_r})$ maps some open neighborhood of x in X biholomorphically onto an open subset of $\mathbf{C}^r$.*

Then $B(G, \mathcal{O}_X)$ is canonically topologically isomorphic to the Banach space of all uniformly continuous holomorphic functions on $G \cap X$.

There is an analog of Proposition (9.1) for uniformly bounded holomorphic functions.

Added in proof. According to the Autorreferat written by G. Pourcin for the Zentralblatt für Mathematik (Vol. 211 (1971), p. 395), Douady has obtained, at the same time as I, a proof for Corollary (0.7).

REFERENCES

1. A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble) 16 (1966), fasc. 1, 1–95. MR 34 #2940.

2. A. Douady, J. Frisch and A. Hirschowitz, *Recouvrements privilégiés*, Ann. Inst. Fourier (Grenoble) 22 (1972), fasc. 4, 59–96.

3. T. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N. J., 1969.

4. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1965. MR 31 #4927.

5. G. Pourcin, *Polycylindres privilégiés*, C. R. Acad. Sci. Paris Sér. A 272 (1971), 795–798.

6. Y.-T. Siu, *Sheaf cohomology with bounds and bounded holomorphic functions*, Proc. Amer. Math. Soc. 21 (1969), 226–229. MR 38 #6108.

7. Y.-T. Siu and G. Trautmann, *Gap-sheaves and extension of coherent analytic subsheaves*, Lecture Notes in Math., vol. 172, Springer-Verlag, Berlin and New York, 1971. MR 44 #4240.

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520