

## A RELATION BETWEEN $K$ -THEORY AND COHOMOLOGY

BY

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**ABSTRACT.** It is well known that for  $X$  a CW-complex,  $K(X)$  and  $H^{\text{ev}}(X)$  are isomorphic modulo finite groups, although the "isomorphism" is not natural. The purpose of this paper is to improve this result for  $X$  a finite CW-complex.

**1. Preliminaries.** For the basic definitions and theory<sup>(1)</sup> of  $\lambda$ -rings, we refer to [2], [12]. The ring  $Z$  of integers has a  $\lambda$ -ring structure with  $\lambda^m: Z \rightarrow Z$  the function

$$\lambda^m(n) = \binom{n}{m} = \frac{n(n-1) \cdots (n-m+1)}{m!}.$$

**Definition.** An augmented  $\lambda$ -ring is a  $\lambda$ -ring  $R$  together with  $\lambda$ -ring homomorphisms  $i: Z \rightarrow R$  and  $\epsilon: R \rightarrow Z$  such that  $\epsilon i = 1$ .

Since the Definition implies that  $i: Z \rightarrow R$  is a monomorphism, we think of  $Z \subset R$  as the multiples of the identity. Let  $\mathcal{B}$  denote the category of augmented  $\lambda$ -rings and  $\lambda$ -ring homomorphisms which commute with the augmentation. If  $B \in \text{Ob } \mathcal{B}$ , we write  $B_n^\gamma$  for the  $n$ th term of the  $\gamma$ -filtration on  $B$ , and  $\Gamma(B)$  for the associated graded ring. We note that  $\Gamma(B)_0 = Z$ . Let  $\mathcal{A}$  be the category of commutative graded rings  $A$  with  $A_0 = Z$ . We define  $\tilde{\Lambda}: \mathcal{A} \rightarrow \mathcal{B}$  as follows: If  $A \in \text{Ob } \mathcal{A}$ , then as a set  $\tilde{\Lambda}(A) = \prod_{n \geq 0} A_n$ . If  $a \in \tilde{\Lambda}(A)$ , we denote the component in  $A_n$  by  $a_n$  and for convenience of notation define  $a_0 = 1 \in A_0$  and write  $a$  as either of the formal expressions

$$1 + a_1 + a_2 + \cdots + a_n + \cdots \quad \text{or} \quad \sum_{i \geq 0} a_i.$$

If  $a, b \in \tilde{\Lambda}(A)$ , we define their sum,  $a \oplus b$ , componentwise:

$$(a \oplus b)_n = \sum_{i+j=n} a_i b_j$$

so that ' $\oplus$ ' is analogous with multiplication of formal power series. This operation makes  $\tilde{\Lambda}(A)$  into an Abelian group, and we denote the inverse of  $a$  by  $\ominus a$ . In particular if  $a$  is an element with  $a_n = 0$  for  $n \geq 2$ , then

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(1) We are indebted to Atiyah and Tall [2] for explicit proofs, in particular those based on 'formal algebra'.

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$$\ominus a = 1 - a_1 + a_1^2 - a_1^3 + \dots + (-a_1)^n + \dots$$

A map  $f: A \rightarrow B$  in  $\mathcal{A}$  induces a function  $\tilde{\Lambda}(f): \tilde{\Lambda}(A) \rightarrow \tilde{\Lambda}(B)$  componentwise:

$$(\tilde{\Lambda}(f)(a))_n = f(a_n).$$

Since ' $\oplus$ ' is defined in terms of the ring structure and  $f$  preserves the ring structure, clearly  $\tilde{\Lambda}(f)$  is a homomorphism.

Define  $\Lambda(A) = Z \oplus \tilde{\Lambda}(A)$  as an abelian group, and write  $[m, a]$  for the element  $m \oplus a$ .

**Proposition 1 (Grothendieck).** (a) *There is a unique multiplication on  $\tilde{\Lambda}(A)$ , denoted by  $\otimes$ , which is associative, commutative and distributive over addition such that*

(i) *for each integer  $n$  there is a polynomial  $P_n(X_1, \dots, X_n, Y_1, \dots, Y_n)$  with integer coefficients such that*

$$(a \otimes b)_n = P_n(a_1, \dots, a_n, b_1, \dots, b_n);$$

(ii) *if  $a_1, b_1 \in A_1$ , then  $(1 + a_1) \otimes (1 + b_1) = (1 + a_1 + b_1) \ominus (1 + a_1) \ominus (1 + b_1)$ .*

(b) *If we give  $\Lambda(A)$  the ring structure obtained from  $A$  by adjoining a unit (i.e. define  $[m, a] \otimes [n, b] = [mn, mb + na + a \otimes b]$ ), then  $\Lambda(A)$  admits a unique  $\lambda$ -ring structure such that*

(i) *for each pair of integers  $m, n$ , there is a polynomial  $Q_{m,n}(X_1, \dots, X_n)$  with integer coefficients such that*

$$\lambda^m[0, a] = [0, b] \quad \text{where } b_n = Q_{m,n}(a_1, \dots, a_n);$$

(ii)  $\lambda^m[0, 1 + a_1] = (-1)^{m-1}[0, 1 + a_1]$  if  $a_1 \in A_1$  and  $m \geq 1$ .

The existence and uniqueness depend on formal algebra.

We define a functor  $D: \mathcal{A} \rightarrow \mathcal{A}$  as follows:

If  $A \in \text{Ob } \mathcal{A}$  let  $\times$  denote its multiplication. Define  $D(A) = A$  as a graded abelian group but with multiplication (denoted by  $\cdot$ ) defined as follows: if  $x_m \in A_m$  and  $x_n \in A_n$  then

$$x_m \cdot x_n = \frac{(m+n)!}{m!n!} x_m \times x_n.$$

If  $f: A \rightarrow B$  in  $\mathcal{A}$  define  $D(f) = f: D(A) \rightarrow D(B)$ .

Let  $N_n = N_n(\sigma_1, \dots, \sigma_r)$  be the polynomial defined inductively for  $n \geq r$  by  $N_1(\sigma_1, \dots, \sigma_r) = \sigma_1$  and the formula

$$N_n - \sigma_1 N_{n-1} + \sigma_2 N_{n-2} - \cdots + (-1)^n n \sigma_n = 0.$$

Define  $\sigma: \Lambda(A) \rightarrow D(A)$  by

$$(\sigma[m, a])_0 = m, \quad (\sigma[m, a])_n = N_n(a_1, \dots, a_n) \quad \text{for } n \geq 1,$$

where  $N_n$  is evaluated in  $A$ , not in  $D(A)$ .

**Proposition 2.**  $\sigma: \Lambda(A) \rightarrow D(A)$  is a ring homomorphism.

**Proof.** That  $\sigma$  is additive depends on certain identities satisfied by the polynomials  $N_n$ , and is omitted.

To show  $\sigma$  preserves multiplication, by virtue of the universality of the definition of multiplication, it suffices to examine  $\sigma(x \otimes y)$  where  $x = [1, 1 + a]$  and  $y = [1, 1 + b]$  with  $a, b \in b_1$ .

In this case  $x \otimes y = [1, 1 + a + b]$  so that  $(\sigma(x \otimes y))_n = (a + b)^n$ . But

$$(\sigma(x)\sigma(y))_n = \sum_{r+s=n} \frac{(r+s)!}{r!s!} a^r b^s = (a+b)^n.$$

So  $\sigma(x \otimes y) = \sigma(x) \cdot \sigma(y)$ .

**2. The main theorem.** Let  $\mathcal{V}_*$  be the category of finite based connected CW-complexes and based maps. If  $X \in \mathcal{V}_*$  then  $H^{ev}(X) = \bigoplus_{n \geq 0} H^{2n}(X, \mathbb{Z})$  is a graded commutative ring and, since  $H^0(X, \mathbb{Z}) = \mathbb{Z}$ , belongs to  $\mathcal{Q}$ . Thus  $H^{ev}: \mathcal{V}_* \rightarrow \mathcal{Q}$  is a functor, and we define

$$\mathcal{Z} = \mathcal{Z}_{H^{ev}}: \mathcal{V}_* \rightarrow \text{rings},$$

$$G = \Lambda H^{ev}: \mathcal{V}_* \rightarrow \mathcal{B},$$

$$H = D H^{ev}: \mathcal{V}_* \rightarrow \mathcal{Q}.$$

The internal multiplication  $G(X) \otimes G(X) \rightarrow G(X)$  induces an external multiplication  $G(X) \otimes G(Y) \rightarrow G(X \times Y)$  in the usual way. This in turn induces a multiplication  $\mathcal{Z}(X) \otimes \mathcal{Z}(Y) \rightarrow \mathcal{Z}(X \wedge Y)$ .

If  $E \rightarrow X$  is a complex vector bundle, let  $c_i(E) \in H^{2i}(X, \mathbb{Z})$  denote its  $i$ th Chern class and define

$$\tilde{c}(E) = 1 + c_1(E) + c_2(E) + \cdots \in \mathcal{Z}(X),$$

$$c(E) = [\text{rank } E, \tilde{c}(E)] \in G(X).$$

**Lemma 1.** If  $E, F$  are vector bundles over  $X$  and  $G$  over  $Y$  then

$$(1) \quad \tilde{c}(E \oplus F) = \tilde{c}(E) \otimes \tilde{c}(F) \quad \text{and} \quad c(E \oplus F) = c(E) \otimes c(F).$$

(2)  $c(E \hat{\otimes} G) = c(E) \otimes c(G)$  where  $E \hat{\otimes} G$  is the exterior tensor product bundle over  $X \times Y$ .

(3)  $c(\lambda^i E) = \lambda^i c(E)$ .

**Proof.** The formulae are the standard ones—see Hirzebruch [3].

Hence  $\tilde{c}$  defines a ring homomorphism  $\tilde{c}: \tilde{K}(X) \rightarrow \tilde{G}(X)$  and  $c$  defines a  $\lambda$ -ring homomorphism  $c: K(X) \rightarrow G(X)$ . Let  $s: K(X) \rightarrow H(X)$  be the composite  $K(X) \xrightarrow{\tilde{c}} G(X) \xrightarrow{\sigma} H(X)$ .

The purpose of this section is to prove the following theorem. If  $n$  is a positive integer let  $l_n$  be the set of primes less than  $n$ .

**Theorem 1.** *If  $X$  is a finite CW-complex of dimension  $\leq 2n + 1$ , then*

- (i)  $\tilde{c}: \tilde{K}(X) \rightarrow \tilde{G}(X)$  and  $c: K(X) \rightarrow G(X)$  are isomorphisms modulo  $l_n$ -torsion.
- (ii)  $s: K(X) \rightarrow H(X)$  is an isomorphism modulo  $l_{n+1}$ -torsion.

The proof of Theorem 1 is an easy consequence of the following mod  $\mathcal{C}$  version of a well-known theorem on half-exact functors.

**Proposition 4.** *Let  $\mathcal{C}$  be a Serre class of abelian groups, and let  $\rho: t_1 \rightarrow t_2$  be a map of half-exact functors, where  $t_i: \mathcal{U}_* \rightarrow \text{Abelian groups}$ , such that  $\rho: t_1(S^n) \rightarrow t_2(S^n)$  is an isomorphism mod  $\mathcal{C}$  for  $n \leq m$ . Then  $\rho: t_1(X) \rightarrow t_2(X)$  is an isomorphism mod  $\mathcal{C}$  when  $X$  is finite and  $\dim X \leq m$ .*

Thus in order to prove the theorem, we examine the maps on spheres.

**Lemma 2.** (i)  $\tilde{K}(S^{2n+1}) \xrightarrow{\tilde{c}} \tilde{G}(S^{2n+1})$  is an isomorphism.

(ii)  $\tilde{K}(S^{2n}) \xrightarrow{\tilde{c}} \tilde{G}(S^{2n})$  is a monomorphism with cokernel  $\mathbb{Z}_{(n-1)!}$ .

(iii)  $G(S^{2n}) \xrightarrow{\sigma} H(S^{2n})$  is a monomorphism with cokernel  $\mathbb{Z}_n$ .

**Proof.** (i) Trivial, since both groups are zero.

(ii) The map  $K(S^{2n}) \rightarrow G(S^{2n}) \cong H^{2n}(S^{2n})$  is given by the  $n$ th Chern class, and by theorems of Borel-Hirzebruch the  $n$ th Chern class of a complex vector bundle on  $S^{2n}$  is a multiple of  $(n-1)!$  times the generator, and every such multiple arises.

(iii) The map  $G(S^{2n}) \xrightarrow{\sigma} H(S^{2n}) \cong H^{2n}(S^{2n})$  is easily seen by calculation to be multiplication by  $n$ .

Let  $\mathcal{C}_n$  be the class of abelian groups whose order is a product of primes in  $l_n$ . By Lemma 2

$\tilde{c}: \tilde{K}(S^m) \rightarrow \tilde{G}(S^m)$  is an isomorphism mod  $\mathcal{C}_n$  for  $m \leq 2n + 1$ ,

$s: K(S^m) \rightarrow H(S^m)$  is an isomorphism mod  $\mathcal{C}_{n+1}$  for  $m \leq 2n + 1$ ,

whence the theorem follows.

3. Finite CW-complexes of dimension  $\leq 5$ . In the case  $n = 2$ , Theorem 1 says that if  $\dim X \leq 5$ ,  $c: K(X) \rightarrow G(X)$  is an isomorphism of  $\lambda$ -rings, i.e.  $K(X) \cong \Lambda H(X)$  [in particular, if  $\dim X \leq 4$ , we see that  $K^1(X) \cong H^1(X) \oplus H^3(X)$ ], so that the graded ring structure of  $H^{ev}(X)$  determines the  $\lambda$ -ring structure of  $K(X)$ . In this section we show that in these low dimensions the converse is true, namely  $H^{ev}(X) \cong \Gamma K(X)$ . Since we already know that  $K(X) \cong G(X)$  as  $\lambda$ -rings, it suffices to show that  $\Gamma G(X) \cong H^{ev}(X)$ .

Let  $\alpha: \tilde{G}(X) \rightarrow H^2(X)$  be the projection  $\alpha(1 + a_1 + a_2) = a_1$  and let  $\beta: H^4(X) \rightarrow \tilde{G}(X)$  be the inclusion  $\beta(a_2) = 1 + (-a_2)$ . The sequence  $0 \rightarrow H^4(X) \xrightarrow{\beta} \tilde{G}(X) \xrightarrow{\alpha} H^2(X) \rightarrow 0$  is clearly exact.

Lemma 3. (i)  $\text{Im } \beta = G(X)_2^\gamma$ . (ii)  $G(X)_n^\gamma = 0$  for  $n > 2$ . (iii) The product in  $\Gamma G(X)$  from  $(G(X)_1^\gamma / G(X)_2^\gamma) \times G(X)_1^\gamma / G(X)_2^\gamma \rightarrow G(X)_2$  is (isomorphic to) the cup product  $H^2(X) \times H^2(X) \rightarrow H^4(X)$ .

Proof. By formal algebra it follows that

$$(1) \quad \gamma^n[0, a] = [0, 1 + (-1)^{n-1}(n-1)!a_n + \dots],$$

$$(2) \quad [0, 1 + a_n + \text{higher terms}] \otimes [0, 1 + b_n + \text{higher terms}] = [0, 1 + c_{m+n} + \text{higher terms}].$$

From (1), (2), it follows that  $G(X)_n^\gamma = 0$  for  $n > 2$  and since  $\gamma^2[0, 1 + a_2] = [0, 1 - a_2]$ , we see that  $\text{Im } \beta \subset G(X)_2^\gamma$ . But  $G(X)_2^\gamma$  in this case is generated by elements of the form  $\gamma^2[0, 1 + a_1 + a_2]$  or  $[0, 1 + a_1][0, 1 + b_1]$ , i.e. by elements of the form  $[0, 1 + a_2]$ , whence  $\text{Im } \beta = G(X)_2^\gamma$ . Finally a simple calculation shows

$$[0, 1 + a_1 + a_2] \otimes [0, 1 + b_1 + b_2] = [0, 1 + (-a_1 b_1)]$$

which completes the proof of the lemma.

4. A real version. We would like to prove a theorem analogous to Theorem 1 for  $KO$ -theory using the corresponding characteristic classes. Since Stiefel-Whitney classes do not carry enough information even on spheres, we try Pontryagin classes. However, there is a technical difficulty to overcome, namely that Pontryagin classes do not obey a Whitney-sum formula. However, if  $E, F$  are real vector bundles over  $X$  and if  $p_i(E) \in H^{4i}(X, \mathbb{Z})$  denotes the  $i$ th Pontryagin class then  $p_n(E \oplus F) - \sum_{i+j=n} p_i(E)p_j(F)$  is an element of order 2 in  $H^{4n}(X, \mathbb{Z})$ . Let  $H^{4*}(X) = \bigoplus_{n \geq 0} H^{4n}(X, \mathbb{Z})$ .  $H^{4*}: \mathcal{U}_* \rightarrow \text{graded rings}$ , so we can define

$$(GO)^\sim(X) = \tilde{\Lambda} H^{4*}(X), \quad GO(X) = \Lambda H^{4*}(X), \quad H^{4*}(X) = DH^{4*}(X).$$

(Alternatively, we can define  $(GO)^\sim(X)$  as the subring of  $\tilde{G}(X)$  consisting of elements with zero components in odd degrees:

$$(GO)^\sim(X) = \{a \in \tilde{G}(X) : a_n = 0 \text{ if } n \text{ is odd}\}.$$

Let  $q: (KO)^\sim(X) \rightarrow \tilde{G}(X)$  be the composite

$$q: KO(X) \xrightarrow{\text{complexification}} K(X) \xrightarrow{c} G(X) \xrightarrow{\psi^2} G(X)$$

where  $\psi^2$  is the Adams operation.

Clearly  $q$  is a ring homomorphism.

**Lemma 4.** *If  $F \rightarrow X$  is a real vector bundle, then*

$$(q(E))_{2n+1} = 0, \quad (q(E))_{2n} = (-4)^n p_n(E).$$

**Proof.** If  $F$  is the complexification of  $E$ , with Chern classes  $c_1, \dots, c_n$ , then  $p_i(E) = (-1)^i c_{2i}$ . If  $F$  were a sum of line bundles  $F = L_1 \oplus \dots \oplus L_n$  with  $c_1(L_j) = \alpha_j$  then

$$\begin{aligned} \psi^2 c(F) &= c\psi^2(F) = c(L_1^2 \oplus \dots \oplus L_n^2) \\ &= [1, 1 + 2\alpha] \oplus [1, 1 + 2\alpha_2] \oplus \dots \oplus [1, 1 + 2\alpha_n] \\ &= [n, 1 + 2c_1 + 2^2 c_2 + 2^3 c_3 + \dots] \end{aligned}$$

so by the splitting principle for complex vector bundles, we see that  $(q(E))_i = (\psi^2 c(F))_n = 2^n c_n(F)$ .

If  $i$  is odd, then  $2c_i(F) = 0$ , so  $(q(E))_i = 0$ . If  $i = 2n$ , then  $(q(E))_{2n} = 2^{2n} c_{2n}(E) = 2^{2n} (-1)^n p_n(E)$ .

Thus the image of  $q$  is contained in  $GO(X)$ . By naturality it induces a map  $\tilde{q}: (KO)^\sim(X) \rightarrow (GO)^\sim(X)$ .

**Theorem 2.** *If  $X$  is a finite CW-complex of dimension  $\leq 4n + 3$ , then*

- (i)  $\tilde{q}: (KO)^\sim(X) \rightarrow (GO)^\sim(X)$  is an isomorphism modulo  $(l_{2n} \cup \{2\})$ -torsion.
- (ii)  $q: KO(X) \rightarrow GO(X)$  is an isomorphism modulo  $(l_{2n+1} \cup \{2\})$ -torsion.
- (iii)  $\sigma q: KO(X) \rightarrow DH^{4*}(X)$  is an isomorphism modulo  $(l_{2n+1} \cup \{2\})$ -torsion.

**Proof.** As before it suffices to examine the maps on spheres. On  $S^t$ , where  $t \not\equiv 0 \pmod{4}$ ,  $(KO)^\sim(S^t)$  is either  $\mathbb{Z}_2$  or 0, but  $(GO)^\sim(S^t)$  and  $H^{4*}(S^t)$  are zero. The map from  $(KO)^\sim(S^{4n}) \rightarrow (GO)^\sim(S^{4n}) \cong H^{4n}(S^{4n}, \mathbb{Z})$  is  $4^n$  times the  $n$ th Pontryagin class and again by theorems of Borel-Hirzebruch the  $n$ th Pontryagin class is a multiple of  $(2n - 1)! \text{GCD}(n + 1, 2)$ , and moreover every such multiple arises. Thus this map from  $\mathbb{Z}$  to  $\mathbb{Z}$  is multiplication by some power of 2 times  $(2n - 1)!$

whose cokernel is thus a  $(l_{2n} \cup \{2\})$ -torsion group. The result now follows.

We briefly state two corollaries of Theorems 1 and 2.

**Corollary 1.** (1) *If  $X$  is a finite CW-complex of dimension  $\leq 2n+1$  and  $H^{ev}(X)$  has no  $l_{n+1}$ -torsion then*

$$K(X) \cong H^{ev}(X) \text{ as abelian groups.}$$

(2) *If  $X$  is a finite CW-complex of dimension  $\leq 4n+3$  and  $H^{4*}(X)$  has no  $l_{2n+1}$  torsion then*

$$KO(X) \cong H^{4*}(X) \text{ modulo 2 torsion, as abelian groups.}$$

**Proof.** (1)  $K(X)$  and  $H^{ev}(X)$  have the same ranks and the same  $p$ -torsion for  $p \neq l_n$ . From [6] by a simple spectral sequence argument we see that if  $H^{ev}(X)$  has no  $p$ -torsion then  $K(X)$  has no  $p$ -torsion.

(2) Proof similar.

**Corollary 2.** *If  $X$  is a finite CW-complex of  $\dim \leq 2n$  and  $H^{2k}(X)$  has no  $l_k$ -torsion then  $c: K(X) \rightarrow G(X)$  is an isomorphism on non- $l_n$ -torsion and a monomorphism on  $l_n$ -torsion.*

**Proof.** By Corollary 1, it suffices to show that  $\tilde{c}$  is a monomorphism. Suppose  $\tilde{c}(x) = 0$ . Then since  $\dim X \leq 2n$ , we can represent  $x$  by  $E - n$  for some  $n$ -dimensional complex vector bundle  $E$ . Then  $\tilde{c}(E) = 0$ , so by a theorem of Peterson [7]  $E$  is trivial whence  $\tilde{c}$  is a monomorphism.

**5. Bott periodicity and an exact sequence.** By Brown's theorem,  $\tilde{K}$  and  $\tilde{G}$  are representable functors. Let  $\tilde{K}(\ ) = [ \ , E ]$  and  $\tilde{G} = [ \ , B ]$  where  $E, B$  are  $H$ -spaces with multiplication  $m$ . Then  $\tilde{c}: \tilde{K} \rightarrow \tilde{G}$  defines a map  $p: E \rightarrow B$  which we can assume without loss of generality to be a fibration, and which is an  $H$ -map, since  $\tilde{c}$  is a homomorphism.

Let  $i: F \rightarrow E$  be the fibre of  $p: E \rightarrow B$ . The composite  $F \times F \rightarrow_{i \times i} E \times E \xrightarrow{m} E \xrightarrow{p} B$  is homotopic to the composite  $F \times F \rightarrow_{i \times i} E \times E \xrightarrow{p \times p} B \times B \xrightarrow{m} B$  which is the constant map. Let  $H: * \simeq pm(i \times i)$ .

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ \uparrow m(i \times i) & \searrow G & \uparrow H \\ F \times F & \xrightarrow{\quad} & F \times F \times I \end{array}$$

There exists a map  $G: F \times F \times I \rightarrow E$  making the diagram commute. Let  $m = G_1: F \times F \rightarrow F$ . Thus  $m$  defines an  $H$ -space structure on  $F$  such that  $i: F \rightarrow E$  is an  $H$ -map.

Define  $\tilde{U} = [ \ , F ]: \mathcal{U}_* \rightarrow \text{Abelian groups}$ . The Puppe sequence

$$\dots \rightarrow \Omega^n F \xrightarrow{\Omega^n i} \Omega^n E \xrightarrow{\Omega^n p} \Omega^n B \rightarrow \dots \rightarrow \Omega B \rightarrow F \xrightarrow{i} E \xrightarrow{p} B$$

induces a long exact sequence

$$\begin{aligned} \dots \rightarrow [X, \Omega^n F] &\rightarrow [X, \Omega^n E] \rightarrow [X, \Omega^n B] \rightarrow \dots \\ &\rightarrow [X, \Omega B] \rightarrow [X, F] \rightarrow [X, E] \rightarrow [X, B] \end{aligned}$$

which can be rewritten as

$$(S) \quad \begin{aligned} \dots \rightarrow \tilde{U}(\Sigma^n X) &\rightarrow \tilde{K}(\Sigma^n X) \xrightarrow{\tilde{c}(\Sigma^n X)} \tilde{G}(\Sigma^n X) \rightarrow \dots \\ &\rightarrow \tilde{U}(X) \rightarrow \tilde{K}(X) \xrightarrow{\tilde{c}(X)} \tilde{G}(X). \end{aligned}$$

A simple calculation when  $X = S^k$  gives the following lemma.

- Lemma 5.** (i)  $\tilde{U}(S^{2n}) = 0$ .  
(ii)  $\tilde{U}(S^{2n+1}) = \mathbb{Z}_n$ .  
(iii) If  $X$  is a finite CW-complex of dimension  $\leq 2n+1$  with  $\nu_r$   $r$ -cells then  $|\tilde{U}(X)|$  divides  $\prod_{r \leq n} (r!)^{\nu_{2r+1}}$ .

The purpose of this section is to obtain an exact sequence

$$(E) \quad \begin{aligned} \tilde{U}(\Sigma^2 X) &\rightarrow K(X) \xrightarrow{s} H(X) \rightarrow \tilde{U}(\Sigma X) \rightarrow K^1(X) \\ &\xrightarrow{t} H^{\text{odd}}(X) \rightarrow \tilde{U}(X) \rightarrow K(X) \xrightarrow{c} G(X) \end{aligned}$$

where  $t$  is defined as follows: If  $E \rightarrow SX$  is an  $n$ -dimensional vector bundle, then it is determined by a map  $X \xrightarrow{f} U(n)$ . The cohomology of  $U(n)$  is an exterior algebra on generators  $x_i \in H^{2i-1}(U(n))$  where  $1 \leq i \leq n$ . Define  $t(E) = (f^*x_1, f^*x_2, \dots, f^*x_n)$ .

First we observe that since cup products vanish on suspensions the natural bijection  $\tilde{H}^{\text{ev}}(\Sigma X) \rightarrow \tilde{G}(\Sigma X)$  is an isomorphism of abelian groups, so that  $\tilde{G}(\Sigma X) \rightarrow H^{\text{odd}}(X)$ , and  $\tilde{G}(\Sigma^2 X) \cong \tilde{H}^{\text{ev}}(X)$ , so we can insert these groups into the sequence (S). That the map

$$t: K^1(X) = \tilde{K}(\Sigma X) \xrightarrow{\tilde{c}} \tilde{G}(\Sigma X) \cong H^{\text{odd}}(X)$$

is given by the above construction we leave to the reader. Essentially it remains to prove the following lemma.

**Lemma 6.** The following diagram commutes:

$$\begin{array}{ccc}
 & \tilde{K}(X) & \\
 \beta \swarrow & & \searrow * \\
 \tilde{K}(\Sigma^2 X) & & \tilde{K}(X) \\
 \downarrow \tilde{c} & & \uparrow s \\
 \tilde{G}(\Sigma^2 X) & \xrightarrow{\cong} & \tilde{H}(X)
 \end{array}$$

where  $\beta$  is Bott periodicity and  $*$ :  $\tilde{K}(X) \rightarrow \tilde{K}(X)$  is conjugation.

**Proof.** Let  $L \rightarrow X$  be a line bundle with  $c_1(L) = l \in H^2(X)$  and let  $H \rightarrow S^2$  be the Hopf bundle with  $c_1(H) = b$ . From the commutative diagram

$$\begin{array}{ccc}
 \tilde{K}(X) \otimes \tilde{K}(S^2) & \xrightarrow{\tilde{c} \otimes \tilde{c}} & \tilde{G}(X) \otimes \tilde{G}(S^2) \\
 \downarrow & & \downarrow \\
 \tilde{K}(\Sigma^2 X) & \xrightarrow{\tilde{c}} & \tilde{G}(\Sigma^2 X)
 \end{array}$$

we see that

$$\begin{aligned}
 \tilde{c}\beta(L-1) &= \tilde{c}(L-1) \otimes \tilde{c}(H-1) \\
 &= (1+l) \otimes (1+b) \\
 &= (1+\hat{b}+\hat{l}) \ominus (1+\hat{b}) \ominus (1+\hat{l})
 \end{aligned}$$

where  $\hat{b}, \hat{l}$  are the images of  $b, l$  in  $H^2(S^2 \times X)$ . That is,  $\tilde{c}\beta(L-1) = (1+\hat{b}+\hat{l}) \oplus (1-\hat{b}) \oplus (1-\hat{l}+\hat{l}^2-\hat{l}^3+\dots)$  since  $\hat{b}^n = 0$  for  $n > 1$ , i.e.  $\tilde{c}\beta(L-1) = (1+\hat{l}-\hat{b}\hat{l}) \oplus (1-\hat{l}+\hat{l}^2-\dots)$ . The component in dimension  $n$ , that is in  $H^{2n}(S^2 \times X)$ , is

$$(-1)^{n-1}\hat{l}^{n-1}\hat{l} - (\hat{b}\hat{l})(-1)^{n-2}\hat{l}^{n-2}.$$

The term  $(-1)^{n-1}\hat{l}^n$  lies in  $H^{2n}(S^2 \vee X)$  and so contributes nothing in  $H^{2n}(\Sigma^2 X)$  and the term  $(-1)^{n-1}\hat{b}\hat{l}^{n-1}$  on desuspending maps to  $(-1)^{n-1}l^{n-1} \in H^{2n-2}(X)$ , so that

$$\tilde{c}\beta(L-1) = (-l, l^2, -l^3, \dots) = s(L-1)$$

and so by the splitting principle and the universal definition of  $s$ , we see that  $\tilde{c}\beta * = s$ .

The exact sequence (E) is now obtained from (S) by replacing  $\tilde{K}(\Sigma^2 X) \rightarrow \tilde{G}(\Sigma^2 X)$  by  $K(X) \xrightarrow{s} H^{ev}(X)$  which is isomorphic to  $\tilde{K}(\Sigma^2 X) \oplus Z \xrightarrow{\tilde{c} \oplus 1} \tilde{G}(\Sigma^2 X) \oplus Z$  and so preserves exactness.

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