TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 195, 1974

HOMOGENEITY AND EXTENSION PROPERTIES OF EMBEDDINGS OF S^1 IN E^3

BY

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ABSTRACT. Two properties of embeddings of simple closed curves in E^3 are explored in this paper. Let S^1 be a simple closed curve and $f(S^1) = S$ an embedding of S^1 in E^3. The simple closed curve S is *homogeneously embedded* or alternatively f is *homogeneous* if for any points p and q of S, there is an automorphism h of E^3 such that $h(S) = S$ and $h(p) = q$. The embedding f or the simple closed curve S is *extendible* if any automorphism of S extends to an automorphism of E^3. Two classes of wild simple closed curves are constructed and are shown to be homogeneously embedded. A new example of an extendible simple closed curve is constructed. A theorem of H. G. Bothe about extending orientation-preserving automorphisms of a simple closed curve is generalized.

1. Introduction. A topological space X is homogeneous if for any points p and q of X, there is a homeomorphism h of X onto itself such that $h(p) = q$. This property is a property of X and does not depend on how X is embedded in another space. In this paper we will discuss two properties, one very similar in description to homogeneity, but which do depend on an embedding.

Suppose S^1 is a simple closed curve and $f(S^1) = S$ is an embedding of S^1 in E^3. Since S^1 is homogeneous as a space, the following two definitions are nontrivial. An automorphism is an onto homeomorphism.

Definition 1.1. The simple closed curve S is *homogeneously embedded*, or alternatively f is homogeneous, if for any points p and q of S, there is an automorphism h of E^3 such that $h(S) = S$ and $h(p) = q$.

Definition 1.2. The embedding f or the simple closed curve S is *extendible* if any automorphism of S extends to an automorphism of E^3.

An extendible simple closed curve is clearly homogeneously embedded. It follows that tame simple closed curves in E^3 are homogeneously embedded. Invertible knots [6] are extendible, and orientation-preserving homeomorphisms of all tame simple closed curves in E^3 extend to E^3. To have either property,

Received by the editors September 1, 1971.

AMS (MOS) subject classifications (1970). Primary 55A30; Secondary 57A35.

Key words and phrases. Wild knot, wild simple closed curve, homogeneous embedding, homogeneity.

(1) This work is part of the author's doctoral dissertation written under Professor R. H. Bing at the University of Wisconsin.

a wild simple closed curve has to be wild at each of its points. Thus the Fox-Artin simple closed curve, for example, cannot have either property.

In this paper we will describe examples of simple closed curves and show that they have one or both of the properties. More specifically, in §2 we will describe two types of examples. In §3 we show that all of the examples are homogeneously embedded and prove that homogeneously embedded simple closed curves satisfy a stronger condition. Bothe has made a study of extendible simple closed curves in [3]. In §4 we describe Bothe's examples and then construct a new extendible simple closed curve. We also extend one of Bothe's major theorems.

Notation. If A is a set in E^3 then $\overline{A}$ is the closure of A. The usual metric on E^3 is ρ. If ϵ is positive, then $N(A, \epsilon)$ is the set of points in E^3 whose distance from A is less than ϵ.

Let $\mathbf{Z}$ be the integers and $\mathbf{Z}^+$ the positive integers. Suppose g_i are automorphisms of E^3 for $i \in \mathbf{Z}^+$. Then $\Pi_{i=1}^n g_i = g_n \circ g_{n-1} \circ \cdots \circ g_1$. If the limit as $j \to \infty$ of $f_j = \Pi_{i=1}^j g_i$ exists, then we call the limit the infinite left product and denote it $\Pi_{i=1}^\infty g_i$.

Suppose K_1 and K_2 are oriented knots. Then $K_1 + K_2$ is the oriented knot such that there is a 2-sphere R and an arc α in R such that:

(1) $R \cap K = \{x, y\}$ $(x \neq y)$.

(2) α is an arc from x to y.

(3) $(\operatorname{Int} R \cap K) \cup \alpha$ is the knot K_1.

(4) $(\operatorname{Ext} R \cap K) \cup \alpha$ is the knot K_2.

A simple closed curve S in E^3 is *tame* if there is an automorphism of E^3 taking S onto a polyhedral simple closed curve. A simple closed curve S is *wild* if it is not tame.

A *monotone map* is a map whose point inverses are compact and connected.

2. The examples. The examples we will give will all be toroidal, that is, the intersection of a decreasing sequence of solid tori. We describe two types of simple closed curves. Simple closed curves of Type 1 are actually a subset of those of Type 2 but are sufficiently interesting to be described separately. We show that Bing's example is a simple closed curve, and an analogous method works for the other examples.

Examples of Type 1. Let K_i be a knot and n_i an integer for i a nonnegative integer. Suppose $n_i \geq 1$ for i positive and $n_0 > 1$. We construct a simple closed curve (K_i, n_i) as follows.

Let T_0 be a tame solid torus tied in a K_0 knot. We divide T_0 up into n_0 solid cylinders $\{C_j^0\}$. In each C_j^0 we put n_1 solid cylinders $\{C_j^1\}$ each with the knot K_1 tied in it. This is done so that the union of the $\{C_j^1\}$ is another solid torus T_1, each C_j^1 lies in some C_r^0, and $T_1 \cap C_j^0 \cap C_{j+1}^0$ is a disk for $j = 1, 2,$

$\cdots, n_1$. Here and throughout the paper we take our subscript mod the appropriate integer. The torus T_i is constructed in a similar fashion with the knot K_i and integer n_i. If the diameters of C_j^i go to zero as $i \to \infty$ then $\bigcap_{i=1}^{\infty} T_i$ is a simple closed curve and we call it (K_i, n_i).

Examples of Type 2. For these examples we remove the restriction in the previous examples that each $T_i \cap C_j^{i-1} \cap C_{j+1}^{i-1}$ is one disk. We assume for each i that $T_i \cap C_j^{i-1} \cap C_{j+1}^{i-1}$ is a finite number of disks. We still assume that $T_{i+1} \cap C_j^i$ are congruent for all j and, in fact, the congruences between all adjacent $T_{i+1} \cap C_j^i$ are realized by a simple slide of T_i. To insure that we get a simple closed curve we also assume the diameters of the $\{C_j^i\}$ go to zero as $i \to \infty$ and that $T_{i+1} \cap C_j^i$ has only one spanning cell. That is, only one of the cells in $T_{i+1} \cap C_j^i$ hits both the left and right bases of C_j^i.

To better understand examples of Type 2 we refer to a well-known example of Bing. This example, J, was constructed by Bing in [2] as an example of a simple closed curve that pierces no disk. In this example T_0 is unknotted, and each stage is constructed in the same manner. The construction is illustrated in Figure 1.

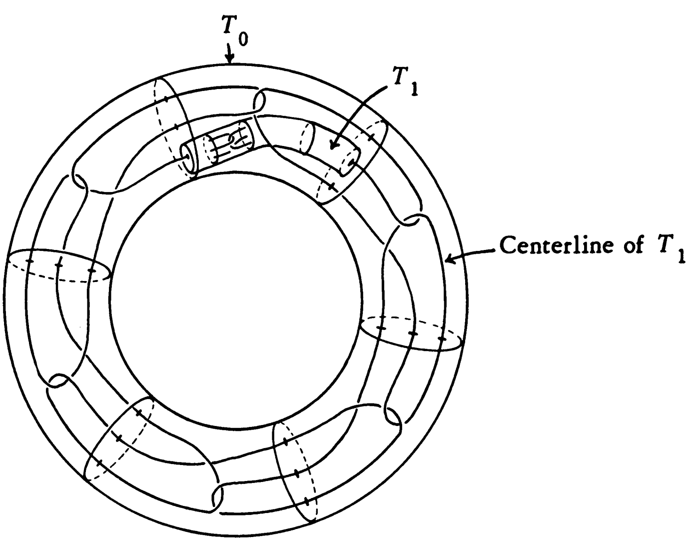

Figure 1(2)

To show J is a simple closed curve we will exhibit a specific homeomorphism f of S^1 onto J. This construction generalizes to simple closed curves of Type 2, and we use this fact in §3.

(2) This figure originally appeared in [2]. It is reprinted here with the permission of the author and the journal.

We construct our homeomorphism by defining it on a dense set of points. We think of S^1 as the unit circle in the plane, and we divide it into six equal parts with points $a_1^0, a_2^0, \cdots, a_6^0$. We put an orientation on each T_i so that as i increases the $\{C_j^i\}$ go around T_0 clockwise. For a given C_j^i we can define a point on $C_{j-1}^i \cap C_j^i$ as follows. In C_j^i there is a unique C_r^{i+1} which hits $C_{j-1}^i \cap C_j^i$ and lies on the spanning cell. Similarly, there is a unique such cylinder in C_r^{i+1}. The intersection of the sequence of such cylinders is a point of $C_{j-1}^i \cap C_j^i$. Thus, for each of the $\{C_j^0\}$ we get a point b_j^0 in this manner. We define $f(a_j^0) = b_j^0$. If there are m cylinders at the next stage we get points $b_1^1, b_2^1, \cdots, b_m^1$. Six of them will be the b_j^0's. We break S^1 into m pieces with points $a_1^1, a_2^1, \cdots, a_m^1$ with $a_0^1 = a_1^1$ and $a_0^r = a_1^{1+rm/6}$ for $r = 1, 2, \cdots, 5$. Equivalently, the a_j^0's are evenly spaced among the a_j^1's. The images under f of the a_j^1's will be the b_j^1's in such a way that going around S^1 clockwise corresponds to going around T_1 clockwise. Renumber the b_j^1's, if necessary, so that $f(a_j^1) = b_j^1$.

We repeat this process for all stages and we get a one-to-one function of a dense subset of S^1 onto a dense subset of J. This function is continuous on the union over all i and j of $\{a_j^i\}$ since, if two points are close enough, they will be forced into adjacent cylinders at a stage far out in the construction. Since the diameters of the cylinders go to zero, we get continuity. Thus we get a continuous function on all of S^1 by extending f to the closure of the union of the $\{a_j^i\}$. This extension is one-to-one, since distinct points of S^1 are separated by an a_r^i so they are mapped into different C_j^i's. Thus f is a one-to-one and continuous function of S^1 onto J which implies it is a homeomorphism.

For examples of Type 1 and Type 2 to be of interest in this paper we need to know they are wild. The fact that many examples of these types are wild follows from work done by Edwards in [5]. We state his Corollary 5.

Corollary 2.1 (Edwards). *Suppose that* $\{B_n\}_1^\infty$ *is a sequence of tame solid tori in* S^3 *with* $B_{n+1} \subset \operatorname{Int} B_n$ *for* $n \geq 1$, *such that no two of the tori* $\{B_n\}_1^\infty$ *are concentric. If* $\bigcap_{n=1}^\infty B_n$ *is a simple closed curve* K, *then* K *is wildly embedded in* S^3.

3. Homogeneously embedded simple closed curves. In this section we show that simple closed curves of Type 2 are homogeneously embedded. Following this we show that if a simple closed curve S is homogeneously embedded, then the automorphism h taking p to q can be assumed orientation preserving on S.

Theorem 3.1. *Simple closed curves of Type 2 are homogeneously embedded.*

Proof. Suppose S is a simple closed curve of Type 2. An immediate consequence of the construction of S is that for any integer m, positive integer i, and positive number ϵ, there is an automorphism h^i_m of E^3 such that:

(1) $h^i_m|E^3 - N(T_i, \epsilon) = \text{identity}$,

(2) $h^i_m(C^i_j) = C^i_{j+m}$ for all j,

(3) $h^i_m(T_r) = T_r$ for $r \geq i$.

This can be done with a slide along T_i fixed outside $N(T_i, \epsilon)$. From (3) we know $h^i_m(S) = S$.

As mentioned in §2 there is a homeomorphism f of S^1 onto S which takes the a^i_j's to the b^i_j's. We note that $f^{-1}h^i_m f$ takes a^i_j to a^i_{j+m} for all j. In fact $f^{-1}h^i_m f$ is just a rotation of S^1 taking a^i_j to a^i_{j+m}. This follows, since it takes all a^r_j's to other a^r_j's for $r \geq i$.

Suppose we have distinct points p and q of S. We must find an automorphism h of E^3 with $h(S) = S$ and $h(p) = q$. In S^1, $f^{-1}(p)$ and $f^{-1}(q)$ are limits of a^i_j's for increasing i, and we call them p_i's and q_i's for convenience. There is a rotation g_1 of S^1 which takes p_1 to q_1. There is a rotation g_2 of S^1 which takes $g_1(p_2)$ to q_2. Similarly there is a g_j taking $\Pi^{j-1}_{i=1} g_i(p_j)$ to q_j. The infinite left product $g = \Pi^{\infty}_{i=1} g_i$ exists and is an automorphism of S^1 taking $f^{-1}(p)$ to $f^{-1}(q)$.

Corresponding to each g_i there are h^i_m's so that $fg_i = h^i_m|S$. For each i we choose one such h^i_m, call it h_i, with the following properties. First by choosing ϵ small enough we insure that $h_i|E^3 - T_{i-1}$ is the identity. Secondly we choose the h_i's so that the distance that cylinders slide under h_i goes to zero as i increases. Then $h = \Pi^{\infty}_{i=1} h_i$ is an automorphism of E^3 such that $h(S) = S$ and $h(p) = q$. Clearly h is an automorphism on $E^3 - S$, since each point of $E^3 - S$ is moved finitely many times. We get that h is an automorphism of S, since $h|S = fgf^{-1}$. If h is continuous, we are done. This follows easily from the construction noting that points in the same cylinder at the ith stage stay in the same cylinder at the ith stage after applying $h_1, h_2, \cdots, h_i$. Thus S is homogeneously embedded.

We now prove a theorem that implies that homogeneously embedded simple closed curves satisfy a stronger condition.

Theorem 3.2. *Suppose S is a homogeneously embedded simple closed curve in E^3. Then for any points p and q of S there is an automorphism h on E^3 such that $h(S) = S$, $h(p) = q$, and h is orientation preserving on S.*

Proof. We let A_p be the set of all points x of S such that there is an automorphism h of E^3 with $h(p) = x$, $h(S) = S$, and h orientation preserving on S. Similarly, let B_p be the set such that h is orientation reversing on S. Since S is homogeneous, $A_p \cup B_p = S$. If B_p is empty we are done. Suppose f is an automorphism of E^3 with $f(S) = S$ and such that f reverses the orientation on S. Then f/S has a fixed point x.

We show $x \in A_p \cap B_p$. Since $x \in S$, x is either in A_p or B_p. If $x \in A_p$ then there is an orientation-preserving homeomorphism h such that $h(p) = x$. Then fh is orientation reversing and $fh(p) = x$ so $x \in B_p$. If $x \in B_p$, then there is an orientation-reversing homeomorphism h such that $h(p) = x$. Then fh is orientation preserving and $fh(p) = x$ so $x \in A_p$. Thus $x \in A_p \cap B_p$. Therefore there is an orientation-preserving h_1 and an orientation-reversing h_2 such that $h_1(p) = h_2(p) = x$. So $h_2^{-1}h_1(p) = p$ and $h_2^{-1}h_1$ is orientation reversing.

We can now show $B_p \subseteq A_p$ and thus $A_p = S$. Suppose g is any automorphism which reverses the orientation on S. Then $g(p)$ is in A_p, since $g(p) = g(h_2^{-1}h_1(p))$ and $gh_2^{-1}h_1$ is orientation preserving.

Conjecture. Suppose S is a homogeneously embedded simple closed curve in E^3. Then for any points p and q of S there is an automorphism of E^3 such that $h(S) = S$, $h(p) = q$, and h is orientation preserving on E^3. One might also require h to be orientation preserving on S as in Theorem 3.2.

4. Extendible simple closed curves. In [3], Bothe constructs examples of extendible simple closed curves. However, he uses the term homogeneous for this property. To prove his examples are extendible, he needs the following theorem.

Theorem 4.1 (Bothe). *Suppose S is a simple closed curve in E^3 with the following property: For any two distinct points p and q of S and an arc B between them on S, and positive number ϵ, there is an automorphism h of E^3 such that:*

(1) $h(S) = S$,

(2) $\rho(h(p), h(q)) < \epsilon$,

(3) $h(x) = x$ *for all* $x \in E^3 - N(B, \epsilon)$.

Then all orientation-preserving automorphisms of S extend to automorphisms of E^3.

In this section we describe a class of examples which satisfy the hypothesis of Theorem 4.1. We then describe Bothe's examples and give a new example of an extendible simple closed curve. Finally, we prove a theorem analogous to Theorem 4.1 but for monotone maps of S.

Theorem 4.2. *Suppose K is a knot and $K_i = K$ for $i > m$. Suppose n_i are integers greater than one for $i \geq 0$. Then all orientation-preserving automorphisms of (K_i, n_i) extend to automorphisms of E^3.*

Proof. Let S be some (K_i, n_i) as above. We will show that S satisfies the hypothesis of Theorem 4.1. Let p, q, ϵ, B be as in Theorem 4.1. Let B' be an arc of S with endpoints p' and q' such that $B \subseteq B' \subseteq S \cap N(B, \epsilon)$. We begin by building a chain of solid cylinders from p' to q' along B'. Let $r > m$ be so large that the diameter of each C_j^r is less than $\epsilon/2$. We begin our chain with all cylinders at the rth stage which contain no points of $S-B$. To these we add cylinders of $\{C_j^{r+1}\}$ which contain no points of $S-B$ and are not contained in cylinders already in the chain. If we continue this process for later stages, we get the required chain.

Naming one of the largest cylinders C_0, we number the cylinders going to p' $C_{-1}, C_{-2}, C_{-3}, \cdots$ and those going to q' $C_1, C_2, C_3, \cdots$. If $F = \bigcup_{i \epsilon Z} C_i \cup \{p', q'\}$ then F is a three cell in $N(B', \epsilon/2)$. There is an automorphism g of E^3 with the following properties:

(1) $g(S) = S$,

(2) $g(C_i) = C_{i+1}$,

(3) $g =$ identity on $(E^3 - N(F, \epsilon/2)) \cup \overline{S-B'}$.

The automorphism g slides the boundary of F along leaving p' and q' fixed. The interior of each C_i is moved to that of C_{i+1} with the possible necessity of shrinking or expanding. Note this process does rely on the fact that we have the same knot from some stage on. By repeated use of g we can take the C_i containing p to the one containing q and thus bring the image of p within ϵ of q. This composition is the automorphism necessary to show S satisfies the hypothesis of Theorem 4.1.

Bothe's examples. Let N be a knot and N' the reflection of N through a plane. Let $R_i = N + N'$ for all $i > 0$. That is, R_i is the knot N followed by its reflection. Suppose R_0 is the trivial knot. Bothe shows that $S = (R_i, n_i)$ is extendible.

By Theorem 4.2 any orientation-preserving automorphism of S extends to an automorphism of E^3. The simple closed curve S is constructed in such a way that there is a canonical automorphism R which reflects E^3 about a plane and takes S to S. The automorphism R also reverses the orientation on S. Suppose we have any orientation-reversing homeomorphism of S, g. Then Rg is orientation preserving on S so it extends to an automorphism G of E^3. Thus $R^{-1}G$ extends g, since $R^{-1}G|S = R^{-1}Rg = g$. This shows all automorphisms of S extend and S is extendible.

We now describe a new example of an extendible simple closed curve. The first two stages are illustrated in Figure 2.

Theorem 4.3. *Suppose L_0 is the trivial knot and L_i is the trefoil knot for all $i > 0$ and n_0 is even. Then if S is of the form (L_i, n_i) then S is extendible.*

Proof. From Theorem 4.2 we know all orientation-preserving automorphisms of S extend to E^3. As in Bothe's examples we must find one automorphism of E^3 which takes S onto itself and reverses the orientation on S. To find such an automorphism we note a particular property of the trefoil knot. If one takes a trefoil

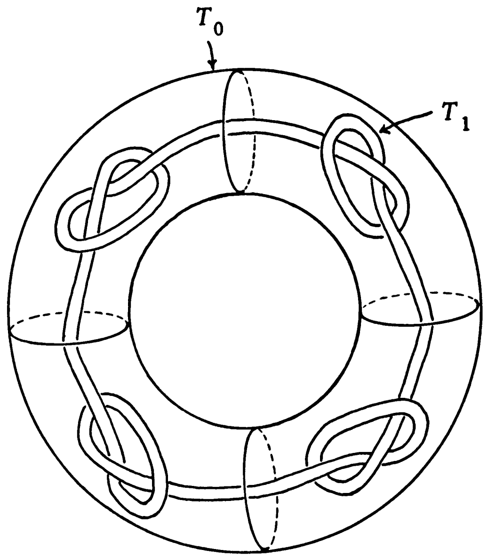

Figure 2

knot plus a trefoil knot and an axis through two points that separate the knots, then an appropriate rotation about the axis takes the simple closed curve to itself and reverses its orientation. Since n_0 is even, we can do the same thing for S using a simple rotation. This is the necessary automorphism of E^3 which reverses the orientation on S, so S is extendible.

The discussions in §§3 and 4 lead naturally to the question of which homogeneously embedded simple closed curves are not extendible.

We mention the following conjectures.

Conjecture. Let K_i be the square knot or another appropriate knot. Then no orientation-reversing automorphism of (K_i, n_i) extends to an automorphism of E^3.

Conjecture. Bing's simple closed curve is not extendible.

Conjecture. Let f be an orientation-preserving automorphism of Bing's example which extends to an automorphism of E^3. Then f is determined by where any one point goes under f. That is to say, only "rigid" automorphisms as described in §3 extend.

We now generalize Theorem 4.1.

Theorem 4.4. *Suppose S is a simple closed curve in E^3 with the following property: For any two distinct points p and q of S and any arc B between them on S, and any positive number ϵ, there is an automorphism h of E^3 such that:*

(1) $h(S) = S$,

(2) $\rho(h(p), q) < \epsilon$,

(3) $h(x) = x$ *for all* $x \in E^3 - N(B, \epsilon)$.

Then any orientation-preserving monotone map of S *onto* S *extends to* F *taking* E^3 *to* E^3 *and such that* $F/E^3 - S$ *is a homeomorphism.*

Before proving Theorem 4.4 we prove two necessary lemmas. We use p, q, ϵ, B as stated in Theorem 4.2. If a and b are points of S then $\overline{ab}$ is the appropriate arc between a and b on S as determined by the context.

Lemma 4.5. *Suppose* S *is a simple closed curve in* E^3 *satisfying the hypothesis of Theorem 4.4. Then there is an automorphism* h *of* E^3 *such that:*

(1) $h(S) = S$,

(2) $h(p) = q$,

(3) $h(x) = x$ *for* $x \in E^3 - N(B, \epsilon)$.

Proof. We construct h satisfying (1) and (2). This is done so that it is clear that a slight modification insures h satisfies (3) also.

We put an orientation on S so that, with respect to B, p is to the "left" of q. Let $M_i = N(p, 1/i)$ and $N_i = N(q, 1/i)$. We pick sequences $\{a_i\}$ and $\{b_i\}$ so that the following hold:

(1) $\{a_i\}$ are to the left and $\{b_i\}$ to the right of p,

(2) $\overline{a_i p}$ and $\overline{pb_i}$ are in M_i,

(3) a_i separates $\overline{a_{i-1}p}$ and b_i separates $\overline{pb_{i-1}}$.

Since S is a simple closed curve satisfying the hypothesis of Theorem 4.4, there is an automorphism g_1 of E^3 with $h(S) = S$ which fixes q but takes a_1 so close to q that $g_1(\overline{a_1 b_1}) \subseteq N_1$. We use the hypothesis again leaving $g_1(a_1)$ fixed, we drag $g_1(b_1)$ to the opposite side of q keeping the condition that $g_1(\overline{a_1 b_1}) \subseteq N_1$. We call the composition of the two automorphisms h_1.

We repeat the process to get h_2 which takes $\overline{h_1(a_2)h_1(b_2)}$ into N_2 with $h_1(a_2)$ and $h_1(b_2)$ on opposite sides of q. Since $\overline{h_1(a_2)h_1(b_2)}$ is in $N_1 \cap h_1(M_1)$ we can require that h_2 be fixed off of $N_1 \cap h_1(M_1)$. Similarly at the ith stage we get an automorphism h_i such that if $f_i = \Pi_{j=1}^{i} h_j$ we have

(1) $f_i(\overline{a_i b_i}) \subseteq N_i$,

(2) $f_i(a_i)$ is to the left of q,

(3) $f_i(b_i)$ is to the right of q,

(4) $h_i =$ identity on $E^3 - (f_{i-1}(M_{i-1}) \cap N_{i-1})$.

We must take the images of a_i and b_i to opposite sides of q to insure that at the following stage $q \in \overline{h_i(a_i)h_i(b_i)}$ and thus $q \in h_i(M_i) \cap N_i$.

We now show that $h = \Pi_{i=1}^{\infty} h_i$ is an automorphism of E^3 satisfying (1) and (2) of Lemma 4.5. Since each point of $E^3 - p$ is not in some M_i, it can only be moved by finitely many h_i by condition (4) above. Thus h is a homeomorphism of $E^3 - p$ with image in $E^3 - q$. Clearly S goes to S and p goes to q. We must show that h is continuous at p. This follows, since $hf_i^{-1}(N_r) = N_r$ for all $i > r$ and the diameters of the N_r go to zero. Thus (1) and (2) are satisfied. We note that by starting our construction with a sufficiently high i we could get condition (3).

Lemma 4.6. *Suppose S is a simple closed curve in E^3 satisfying the hypothesis of Theorem 4.4. Suppose $\overline{ab}$ is an arc of S and $\{x_i\}, \{y_i\}$, $i = 1, 2, \cdots, n$, are points of $\overline{ab}$ with $x_i \neq y_j$ for $i \neq j$, and so that the orientations of the points are $a, x_1, x_2, x_3, \cdots, x_n, b$ and $a, y_1, y_2, \cdots, y_n, b$. If A is an arc in the interior of $\overline{ab}$ containing $\{x_i\}$ and $\{y_i\}$ and ϵ is a positive number, then there is an automorphism h of E^3 such that:*

(1) $h(S) = S$,

(2) $h(x_i) = y_i$,

(3) $h = \textit{identity on } E^3 - N(A, \epsilon)$.

Proof. This lemma follows with repeated application of Lemma 4.5, never moving anything in $E^3 - N(A, \epsilon)$. We first take x_1 to y_1 on a small neighborhood of $\overline{x_1 y_1}$. By our orientation condition, y_1 cannot separate the image x_2 from y_2 in $\overline{ab}$, so we can take the image of x_2 to y_2 keeping y_1 fixed. We repeat n times to prove the lemma.

Corollary 4.7. *Suppose S is as in Lemma 4.6 and S is broken up into n disjoint arcs by n points $c_1, c_2, \cdots, c_n$. Suppose there are $\{x_i\}$ and $\{y_i\}$ as in Lemma 4.6 on each arc. Then there is an automorphism h such that:*

(1) $h(S) = S$,

(2) $h(x_i) = y_i$ *for all arcs*,

(3) $h = \textit{identity on } (E^3 - N(S, \epsilon)) \cup \{c_n\}$.

This follows by using Lemma 4.6 on the n arcs taking care not to let moves on one interfere with those on another.

Proof of Theorem 4.4. Suppose f is a monotone map of S onto S. We must extend f to a map of E^3 onto E^3 such that $F/E^3 - S$ is a homeomorphism. Since f is monotone, point inverses of f are compact and connected and thus must be points or closed arcs. Since S is separable, at most a countable number are arcs. We call them $\{\alpha_i\}$ and their images under f, $\{a_i\}$. Since $\{a_i\}$ is countable, $\bigcup_{i=1}^{\infty} a_i$ contains no open subset of S. Thus for each integer j we can pick a finite number of points $\{b_n^j\}$, $n = 1, 2, \cdots, n(j)$, such that:

(1) $b_n^j \in S - \{a_i\}\ \forall\ j$ and $n = 1, 2, \cdots, n(j)$,

(2) $\{b_n^j\}$ is oriented $b_1^j, b_2^j, \cdots, b_{n(j)}^j$,

(3) $b_r^j \in \{b_n^{j+1}\}$ $\forall\, j$ and $n = 1, 2, \cdots, n(j)$,

(4) The diameters of the components of $K - \{b_n^j\}$ are less than $1/2^j$.

To simplify notation we let $x_n^j = f^{-1}(b_n^j)$ and $N_i = N(S, 1/i)$. For each j we define an automorphism f_j of E^3 such that if $F_j = \Pi_{r=1}^j f_r$ and $F_0 =$ identity, we have

(1) $f_j(S) = S$.

(2) $f_j(F_{j-1}(x_n^j)) = b_n^j$ for $n = 1, 2, \cdots, n(j)$.

(3) f_j moves no point more than $1/2^{j-1}$.

(4) f_j is the identity on $E^3 - F_{j-1}(N_j)$.

The existence of the f_j's follows from Corollary 4.7. The arcs we use at the jth stage are

$$\overline{b_1^{j-1} b_2^{j-1}}, \quad \overline{b_2^{j-1} b_3^{j-1}}, \quad \cdots, \quad \overline{b_{n(j)}^{j-1} b_1^{j-1}}.$$

We take $F_{j-1}(x_n^j)$ to b_n^j in these arcs noting that the endpoints must be and are kept fixed. To get (3) we use the fact that the $\{b_i^j\}$ break S into components of diameter less than $1/2^j$.

We now show $F = \Pi_{i=1}^\infty f_i$ is a monotone map extending f and is a homeomorphism on $E^3 - S$. Condition (3) insures that F is well defined and continuous. Condition (4) insures that points of $E^3 - S$ are moved by finitely many of the f_i so F is a homeomorphism on $E^3 - S$. In fact, F takes $E^3 - S$ to itself since

$$F(E^3 - S) = F\left(\bigcup_{j=1}^{\infty} (E^3 - N_j)\right) = E^3 - \bigcup_{j=1}^{\infty} F(N_j) = E^3 - S.$$

We need only show that F extends f. By the construction we have $F(x_n^j) = F_j(x_n^j) = b_n^j$. The x_n^j are dense in $S - \bigcup_{i=1}^\infty \alpha_i$ so $F/S - \bigcup_{i=1}^\infty \alpha_i = f/S - \bigcup_{i=1}^\infty \alpha_i$. Furthermore, the α_i's are mapped to points, since the diameters of their images go to zero. By the continuity of F and f, $F/\alpha_i = f/\alpha_i$. Thus $F/S = f$.

Remark. By using a sufficiently small ϵ when applying Corollary 4.7 we can add the condition that F may be chosen so that it does not move points more than a given distance away from S.

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