LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH LAPLACE-STIELTJES TRANSFORMS AS COEFFICIENTS

BY

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ABSTRACT. The n-dimensional differential system $z' = (R + A(t))z$ is considered, where R is a constant $n \times n$ complex matrix and $A(t)$ is an $n \times n$ matrix whose entries $a(t)$ are complex valued functions which are representable as absolutely convergent Laplace-Stieltjes transforms, $\int_0^\infty e^{-st} d\alpha(s)$, for $t > 0$. The determining functions, $\alpha(s)$, are C valued, locally of bounded variation on $[0, \infty)$, continuous from the right, and $\alpha(+0) = \alpha(0) = 0$. Sufficient conditions on the determining functions are found which assure the existence of solutions of certain specified forms involving absolutely convergent Laplace-Stieltjes transforms for $t > 0$ and which behave asymptotically like certain solutions of the nonperturbed equation $z' = Rz$ as $t \to \infty$. Analogous results are proved for the nth order equation $\Pi_{i=1}^m (D - r_i)^{e(i)} z + \Sigma_{j=0}^{n-1} a_j(t) D^j z = 0$, where $r_i \in C$ and the $a_j(t)$ are like $a(t)$ above for $t > 0$.

0. **Introduction.** Consider the n-dimensional, first order, linear, ordinary differential system

(0.1)
$$y' = (R + A(t))y,$$

where R is an $n \times n$ constant complex matrix and $A(t)$ is an $n \times n$ matrix whose entries are complex valued functions representable as absolutely convergent Laplace-Stieltjes (or Laplace) transforms,

$$A(t) = \int_0^\infty e^{-st} d\alpha(s) \quad \text{for } t > 0.$$

The determining entries in the matrix $a(s)$ are complex valued functions, continuous from the right, locally of bounded variation, and $a(+0) = a(0) = 0$. The object of this paper is to determine sufficient conditions on the determining functions of the coefficients to assure the existence of solutions of a certain specified form involving absolutely convergent Laplace-Stieltjes or Laplace transforms and which behave asymptotically like certain solutions of $y' = Ry$ as $t \rightarrow \infty$.

The n-dimensional, first order case (0.1) has only been considered for $R = 0$ by Wintner in [4], and his results are contained in Theorem 1.1, cf. Remark 3.2. The nth order, one-dimensional equation

$$(0.2) \qquad \sum_{i=1}^{m} (D - r_i)^{e(i)} y + \sum_{i=0}^{n-1} a_i(t) D^i y = 0,$$

where $e(1) + \cdots + e(m) = n$, and the $a_i(t)$ are representable as Laplace-Stieltjes transforms as above, has been considered in several papers, in particular, Hartman [2]. Conditions have been determined which assure solutions corresponding to the root $r = 0$ if it appears and is in favorable geometric position with respect to the roots with smaller real part.

The case of (0.1) in which $A(t)$ is a power series in e^{-t} will correspond to a regular singular point at $z = 0$, with $z = e^{-t}$. Therefore the nature of our problem is not only one of asymptotic integration, but also of the "continuous" analogue of regular singular point theory. As in regular singular point theory, in order to specify the form of a solution corresponding to a certain root, assumptions will have to be made concerning its relative position with respect to the other roots and the nature of the coefficient functions. For example, we shall generally assume that $a(t) = 0$ on $[0, p]$, where p depends on the relative position of the roots. Under such conditions, we shall obtain certain specified forms for solutions corresponding to all the roots for the n-dimensional and nth order cases. It is clear from the theory of regular singular points that, without such assumptions, the solutions in the cases $p > 0$ of Theorems 1.1 and 6.1 could be expected to involve powers of $t = - \log z$.

In a subsequent paper we shall investigate the nature of solutions without such conditions in the case $p > 0$.

This paper is divided into two parts. In Part I, we consider the n-dimensional system (0.1). In Part II, the analogous results are given for the nth order case. The methods of this paper depend on extensions of the arguments of Dunkel [1] and of Hartman [2]. The latter, in turn, are suggested by Wintner [4]–[6]. They involve successive approximations whose convergence for large $t > 0$ is comparatively easy to prove. Convergence for $t > 0$ depends on considering t to be a complex variable and on the use of suitable "majorant" equations having completely monotone solutions. In the simpler cases, the choice of a majorant

equation is rather obvious. Wintner's proof for part of his main theorem on second order equations in [5] is incomplete as he failed to observe that the obvious majorant second order equation could not be used in all cases. In dealing with nth order equations, Hartman in [2] constructs certain majorant equations which are of high order. We adapt his procedure in our treatment of both (0.1) and (0.2).

I should like to thank Professor Hartman for suggesting the investigation of some of the questions treated here.

PART I. THE n-DIMENSIONAL FIRST ORDER CASE

1. The "smallest" solution corresponding to each Jordan block. Consider the first order differential system

$$(1.1) \qquad z'(t) = Rz(t) + A(t)z(t),$$

where R is an $n \times n$ constant complex matrix in Jordan canonical form

$$(1.2) \qquad R = \operatorname{diag}(J(1), \cdots, J(m)),$$

where $J(k)$ is the Jordan $e(k) \times e(k)$ block with r_k on the diagonal, 1 on the subdiagonal, and other entries 0,

$$(1.2') \qquad J(k) = \begin{bmatrix} r_k & & & & 0 \\ 1 & r_k & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & 1 & r_k \end{bmatrix},$$

$e(1) + \cdots + e(m) = n$.

Let $A(t)$ be an $n \times n$ matrix whose entries are representable as absolutely convergent Laplace-Stieltjes transforms of the form $\int_0^\infty e^{-st} d\alpha(s)$. In dealing with such functions, we shall always assume (but shall not always state) that the determining matrix of functions

$$(1.3) \qquad \begin{array}{l} \alpha(s) \text{ is complex valued on } [0, \infty), \text{ locally of bounded variation,} \\ \text{continuous from the right, and } \alpha(+0) = \alpha(0) = 0. \end{array}$$

We shall use the following block notation for $A(t)$:

$$(1.4) \qquad A(t) = (A_i^j(t)), \qquad 1 \le i, \ j \le m,$$

where $A_i^j(t)$ is a rectangular matrix with $e(j)$ columns and $e(i)$ rows, $A_i^j(t) = (a_{i\beta}^{j\delta}(t))$, where $1 \le \delta \le e(j)$, $1 \le \beta \le e(i)$, and $a_{i\beta}^{j\delta}(t)$ is complex valued. Similarly, for an n-dimensional vector or $n \times 1$ matrix z, we write

(1.4') $z = (z_1, \cdots, z_m)$,   where $z_k = (z_{k1}, \cdots, z_{ke(k)})$

is an $e(k)$-dimensional vector with complex components. We shall refer to $z_{k\delta}$ as the $(k\delta)$th component of z, where $1 \le k \le m$ and $1 \le \delta \le e(k)$. We shall let $1_{k\delta}$ be the n-dimensional vector with a $(k\delta)$th component 1 and other components 0. Correspondingly, $A^{j\beta}(t) = A(t)1_{j\beta}$ is the $(e(1) + \cdots + e(j) + \beta)$th column of $A(t)$ considered as an n-dimensional vector, $1 \le j \le m$ and $1 \le \beta \le e(j)$, so that the $(i\delta)$th component of $A^{j\beta}(t)$ is $a_{i\delta}^{j\beta}(t)$. In this notation, (1.1) can be written as

$$z_i' = J(i)z_i + \sum_{j=1}^{m} A_i^j(t)z_j \quad \text{for } 1 \le i \le m,$$

or in the form

$$z_{i\lambda}' = z_{i,\lambda-1} + r_i z_{i\lambda} + \sum_{j=1}^{m} \sum_{\delta=1}^{e(j)} a_{i\lambda}^{j\delta}(t)z_{j\delta} \quad \text{for } 1 \le i \le m, \ 1 \le \lambda \le e(i),$$

where $z_{i0} = 0$.

It is clear that if R in (1.1) is not in the form specified, a linear change of variables with constant coefficients will produce an equation of the above type, so that there is no loss of generality in our assumptions concerning the form of (1.1).

Corresponding to any of the blocks in R, we wish to find solutions of (1.1) which behave asymptotically like the solutions of $z'(t) = Rz(t)$ corresponding to the same block. The following theorem gives sufficient conditions to assure the existence of such solutions.

Theorem 1.1. *In the differential equation* (1.1), *let* R *be an* $n \times n$ *complex matrix in Jordan canonical form* (1.2). *For a fixed* κ, $1 \le \kappa \le m$, *let*

(1.5) $0 \le p = p_\kappa = \max \{r_\kappa - r_i \colon \operatorname{Im}(r_i) = \operatorname{Im}(r_\kappa), \ 1 \le i \le m\}$,

and let there be $\bar{v}$ *blocks whose diagonal entries are such that*

(1.6) $r_\kappa - r_{b(v)} = p \quad for \ 1 \le v \le \bar{v} \le m \le n.$

Assume that the $n \times n$ *matrix* $A(t)$ *is representable in the form*

(1.7) $A(t) = \int_0^\infty e^{-st} d\alpha(s)$,   *i.e.*, $a_{i\beta}^{j\delta}(t) = \int_0^\infty e^{-st} d\alpha_{i\beta}^{j\delta}(s)$,

where the integrals are absolutely convergent Laplace-Stieltjes transforms for $t > 0$,

(1.7') $\int_0^\infty e^{-st} |d\alpha_{i\beta}^{j\delta}(s)| < \infty$,

and the determining functions $\alpha_{i\beta}^{j\delta}(s)$ *satisfy* (1.3).

Case 1 $(p = 0)$. *Let* $p = 0$, *and let*

(1.8) $\gamma(k) \geq 0$, $\gamma(k, j) = e(k) - j + \gamma(k)$ [or $\gamma(k)$] *if* $r_k = r_\kappa$ [otherwise]

for $1 \leq k \leq m$, $1 \leq j \leq e(k)$. *Let*

(1.9) $$\int_0 s^{-\max[0,\gamma(k,j)-\gamma(i,b)]}|d\alpha_{k,j+1}^{i\beta}(s)| < \infty$$

for $1 \leq i$, $k \leq m$, $1 \leq \beta \leq e(i)$, $0 \leq j \leq e(k) - 1$, *and*

(1.9$_1$) $$\int_0 s^{-\gamma(k,j)}|d\alpha_{k,j+1}^{\kappa,e(\kappa)}(s)| < \infty \quad \text{for } 1 \leq k \leq m, \ 0 \leq j \leq e(k) - 1.$$

Then there exists a unique solution of (1.1) *of the form*

(1.10) $$z(t) = e^{r_\kappa t}(1_{\kappa e(\kappa)} + w(t)),$$

where

(1.11) $w(t) = (w_1(t), \cdots, w_m(t))$ *is an n-dimensional vector,*
          $w_i(t) = (w_{i1}(t), \cdots, w_{ie(i)}(t))$,

and

(1.11$_1$) $$w_{i\beta}(t) = \int_0^\infty e^{-st}d\omega_{i\beta}(s), \qquad \omega_{i\beta}(0) = \omega_{i\beta}(+0) = 0$$

is a Laplace-Stieltjes transform, which is absolutely convergent for $t > 0$,

(1.11$_2$) $$\int_0^\infty e^{-st}|d\omega_{i\beta}(s)| < \infty \quad \text{for } t > 0, \text{ and}$$

(1.11$_3$) $$\int_0 s^{-\gamma(i,\beta)}|d\omega_{i\beta}(s)| < \infty.$$

Case 2 $(p > 0)$. *Let* $p > 0$ *and let* $\gamma(k) \geq 0$;

(1.8$_1$) $\tilde{\gamma}(k, j) = e(k) - j + \gamma(k)$ [or $\gamma(k)$] *if* $k = b(v)$ *for some* v [otherwise];

(1.9$_2$) $$\int_0 s^{-\max[0,\tilde{\gamma}(k,j)-\tilde{\gamma}(i,\beta)]}|d\alpha_{k,j+1}^{i\beta}(s)| < \infty,$$

(1.9$_3$) $$\alpha^{\kappa,e(\kappa)}(s) = 0 \quad \text{on } [0, p],$$

and

(1.9$_4$) $$\int_p (s - p)^{-\tilde{\gamma}(k,j)}|d\alpha_{k,j+1}^{\kappa,e(\kappa)}(s)| < \infty,$$

where the indices take on the same values as in Case 1. Then there exists a unique solution of (1.1) of the form (1.10)–(1.11$_2$), and

$$(1.11_4) \qquad\qquad \omega_{i\beta}(s) = 0 \quad on \ [0, p].$$

If we substitute (1.10) into equation (1.1), we get

$$(1.12) \qquad\qquad w'(t) = (R - r_\kappa)w(t) + A(t)w(t) + A^{\kappa e(\kappa)}(t).$$

Thus Theorem 1.1 follows from the following result concerning a nonhomogeneous system.

Theorem 1.2. *In the linear differential system*

$$(1.13) \qquad\qquad w'(t) = Rw(t) + A(t)w(t) + f(t),$$

let R be an $n \times n$ complex matrix in Jordan canonical form (1.2), $r_\kappa = 0$ for some fixed κ, $1 \le \kappa \le m$. Let

$$(1.14) \qquad\qquad 0 \le p = p_\kappa = \max\{-r_i \colon \mathrm{Im}(r_i) = 0, \ 1 \le i \le m\},$$

and let there be $\bar{v}$ blocks whose diagonal entries are such that

$$(1.15) \qquad\qquad r_{b(v)} = -p, \qquad 1 \le v \le \bar{v} \le m \le n.$$

Let $A(t)$ be as in (1.4) and $f(t) = (f_1(t), \cdots, f_m(t))$, where $f_i(t) = (f_{i1}(t), \cdots, f_{ie(i)}(t))$,

$$(1.16) \qquad A(t) = \int_0^\infty e^{-st}\, d\alpha(s), \quad i.e., \quad a_{i\beta}^{j\delta}(t) = \int_0^\infty e^{-st}\, d\alpha_{i\beta}^{j\delta}(s),$$

and

$$(1.17) \qquad f(t) = \int_p^\infty e^{-st}\, d\phi(s), \quad i.e., \quad f_{i\beta}(t) = \int_p^\infty e^{-st}\, d\phi_{i\beta}(s)$$

are absolutely convergent Laplace-Stieltjes transforms for $t > 0$, such that their determining functions satisfy (1.3).

Case 1 $(p = 0)$. Assume that (1.9) is satisfied, and

$$(1.18) \qquad \int_0 s^{-\gamma(k,i)} |d(\phi_{k,j+1}(s) + c\alpha_{k,j+1}^{\kappa, e(\kappa)}(s))| < \infty$$

for $1 \le k \le m$, $0 \le j \le e(k) - 1$, where c is a constant. Then there exists a unique solution of (1.13) of the form

$$(1.19) \qquad\qquad w(t) = c1_{\kappa e(\kappa)} + (w_1(t), \cdots, w_m(t)),$$

$w_i(t) = (w_{i1}(t), \cdots, w_{ie(i)}(t))$, *and $w_{i\beta}(t)$ is representable as (1.11$_1$–1.11$_3$).*

Case 2 ($p > 0$). Assume that (1.9_2) is satisfied,

(1.20) $$c\alpha^{K,e(K)}(s) = 0 \quad on \ [0, p],$$

and

(1.21) $$\int_p (s - p)^{-\gamma(k,i)}|d(\phi_{k,j+1}(s) + c\alpha^{K,e(K)}_{k,j+1}(s))| < \infty.$$

Then there exists a unique solution of (1.13) of the form (1.19), where $w_{i\beta}(t)$ is representable as (1.11)–(1.11_2) and (1.11_4).

2. Proof of Theorem 1.2. The case where $c \neq 0$ is contained in the case where $c = 0$ if we replace $f(t)$ by $cA^{Kb(K)}(t) + f(t)$ in (1.13). We therefore assume that $c = 0$. If we substitute $w(t)$ into (1.13), we arrive at an integral equation for the entry $w_{k\lambda}(t)$ in the same manner as that used by Dunkel in [1]. We get

(2.1) $$\frac{dw_{k\lambda}}{dt} = r_k w_{k\lambda} + w_{k,\lambda-1} + \sum_{i=1}^m \sum_{\beta=1}^{e(i)} a_{k\lambda}^{i\beta} w_{i\beta} + f_{k\lambda}, \quad \text{where } w_{k0} = 0.$$

Therefore,

$$e^{r_k t}\left(\frac{d}{dt}\right)(e^{-r_k t}w_{k\lambda}) = w_{k,\lambda-1} + \sum_{i=1}^m \sum_{\beta=1}^{e(i)} a_{k\lambda}^{i\beta} w_{i\beta} + f_{k\lambda}$$

or

$$\int_t^\infty \left(\frac{d}{dt_1}\right)e^{-r_k t_1}w_{k\lambda}\,dt_1 = \int_t^\infty e^{-r_k t_1}w_{k,\lambda-1}\,dt_1$$

$$+ \int_t^\infty \sum_{i=1}^m \sum_{\beta=1}^{e(i)} e^{-r_k t_1}a_{k\lambda}^{i\beta}w_{i\beta}\,dt_1 + \int_t^\infty e^{-r_k t_1}f_{k\lambda}\,dt_1.$$

So we get

$$w_{k\lambda}(t) = e^{r_k t}\int_t^\infty e^{-r_k t_1}w_{k,\lambda-1}\,dt_1$$

$$+ e^{r_k t}\int_t^\infty \sum_{i=1}^m \sum_{\beta=1}^{e(i)} e^{-r_k t_1}a_{k\lambda}^{i\beta}w_{i\beta}\,dt_1 + e^{r_k t}\int_t^\infty e^{-r_k t_1}f_{k\lambda}\,dt_1.$$

Substituting the corresponding identity for $w_{k,\lambda-1}$ into this relation gives

$$w_{k\lambda}(t) = e^{r_k t}\int_t^\infty e^{-r_k t_1}\left[e^{r_k t_1}\int_{t_1}^\infty e^{-r_k t_2}w_{k,\lambda-2}\,dt_2\right.$$

$$+ e^{r_k t_1}\int_{t_1}^\infty \sum_{i=1}^m \sum_{\beta=1}^{e(i)} e^{-r_k t_2}a_{k,\lambda-1}^{i\beta}w_{i\beta}\,dt_2$$

$$+ \left. e^{r_k t_1}\int_{t_1}^\infty e^{-r_k t_2}f_{k,\lambda-1}\,dt_2\right]dt_1 + \cdots.$$

After λ such substitutions, keeping in mind that $w_{k0} = 0$, we get

$$
w_{k\lambda}(t) = e^{r_k t} \sum_{q=1}^{\lambda} \int_t^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{q-1}}^{\infty} e^{-r_k t_q} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} a_{k,\lambda+1-q}^{i\beta} w_{i\beta} dt_q
$$

(2.2)

$$
+ e^{r_k t} \sum_{q=1}^{\lambda} \int_t^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \cdots \int_{t_{q-1}}^{\infty} e^{-r_k t_q} f_{k,\lambda+1-q} dt_q,
$$

where the argument of $a_{k\delta}^{i\beta}$, $w_{i\beta}$, $f_{k\delta}$ in the integrands is t_q.

Case 1 ($p = 0$). Making use of the fact that $w_{i\beta}(t)$ is representable as (1.11_1) with $p = 0$, (2.2) becomes

$$
\int_0^{\infty} e^{-st} d\omega_{k\lambda}(s)
$$

$$
= e^{r_k t} \sum_{q=1}^{\lambda} \int_t^{\infty} dt_1 \cdots \int_{t_{q-1}}^{\infty} e^{-r_k t_q} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} \left(\int_0^{\infty} e^{-st} d\alpha_{k,\lambda+1-q}^{i\beta}(s) \right) \left(\int_0^{\infty} e^{-st} d\omega_{i\beta}(s) \right) dt_q
$$

$$
+ e^{r_k t} \sum_{q=1}^{\lambda} \int_t^{\infty} dt_1 \cdots \int_{t_{q-1}}^{\infty} e^{-r_k t_q} \left(\int_0^{\infty} e^{-st} d\phi_{k,\lambda+1-q}(s) \right) dt_q
$$

$$
= \sum_{q=1}^{\lambda} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} \int_0^{\infty} e^{-st}(s+r_k)^{-q} d \left\{ \int_0^s \alpha_{k,\lambda+1-q}^{i\beta}(s-u) d\omega_{i\beta}(u) \right\}
$$

$$
+ \sum_{q=1}^{\lambda} \int_0^{\infty} e^{-st}(s+r_k)^{-q} d\phi_{k,\lambda+1-q}(s).
$$

So formally we arrive at our main functional equation for $\omega_{k\lambda}$,

$$
d\omega_{k\lambda}(s) = \sum_{q=1}^{\lambda} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} (s+r_k)^{-q} d \left\{ \int_0^s \alpha_{k,\lambda+1-q}^{i\beta}(s-u) d\omega_{i\beta}(u) \right\}
$$

(2.3)

$$
+ \sum_{q=1}^{\lambda} (s+r_k)^{-q} d\phi_{k,\lambda+1-q}(s).
$$

Let V be the Banach space of $\mathbb{C}^n$ valued functions $\omega(t)$ of bounded variation on $[0, \infty)$, continuous from the right, such that $\omega(+0) = \omega(0) = 0$, with norm

$$
\|\omega\| = \sum_{k=1}^{m} \sum_{\lambda=1}^{e(k)} \left(\int_1^{\infty} + \int_0^1 s^{-\gamma(k,\lambda)} \right) |d\omega_{k\lambda}(s)|.
$$

Define a linear map $J: V \to V$ by

$$
(2.4) \quad (J\omega)_{k\lambda}(t) = \sum_{q=1}^{\lambda} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} \int_0^t (s+r_k)^{-q} d \left\{ \int_0^s \alpha_{k,\lambda+1-q}^{i\beta}(s-u) d\omega_{i\beta}(u) \right\}.
$$

Let

$$(2.5) \qquad H^{i\beta}_{k,\lambda+1-q}(s) = \int_0^s \alpha^{i\beta}_{k,\lambda+1-q}(s-u)\, d\omega_{i\beta}(u).$$

An integration by parts and a change of the integration variable $u \to s - u$ give

$$(2.6) \qquad H^{i\beta}_{k,\lambda+1-q}(s) = \int_0^s \left(\int_0^{s-u} d\omega_{i\beta}(g) \right) d\alpha^{i\beta}_{k,\lambda+1-q}(u);$$

in other words, (2.6) is the convolution $\omega_{i\beta} * \alpha^{i\beta}_{k,\lambda+1-q}$, where $\omega_{i\beta}(t) = \int_0^t d\omega_{i\beta}(g)$. Hence (2.6) and therefore $(J\omega)(t)$ are locally of bounded variation on $t \geq 0$.

Assume for the moment that ω is absolutely continuous. Then (2.6) is also, and $\omega(+0) = \omega(0) = 0 = \alpha(+0) = \alpha(0)$ imply that

$$(2.7) \qquad H^{i\beta'}_{k,\lambda+1-q}(s) = \int_0^s \omega'_{i\beta}(s-u)\, d\alpha^{i\beta}_{k,\lambda+1-q}(u),$$

where $H^{i\beta'}_{k,\lambda+1-q}(s)$ and $\omega'_{i\beta}(s)$ are Borel measurable. Therefore, by (2.4),

$$(2.8) \quad \|J\omega\| \leq \sum_{k\lambda qi\beta} \left(\int_0^1 s^{-\gamma(k,\lambda)} + \int_1^\infty \right) |s + r_k|^{-q} \left(\int_0^s |\omega'_{i\beta}(s-u)| |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \right) ds,$$

where $\sum_{k\lambda qi\beta} = \sum_{k=1}^m \sum_{\lambda=1}^{e(k)} \sum_{q=1}^{\lambda} \sum_{i=1}^m \sum_{\beta=1}^{e(i)}$. Let C be a constant such that $0 < C < 1$ and

$$(2.9) \qquad |s + r_k| > C > 0 \quad \text{for } s \geq 0 \text{ when } r_k \neq 0 = r_K.$$

Now for those k such that $r_k \neq r_K$, we see that

$$(2.10) \quad I = I_{k\lambda qi\beta} = \int_0^1 s^{-\gamma(k,\lambda)} |s + r_k|^{-q} \left(\int_0^s |\omega'_{i\beta}(s-u)| |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \right) ds$$

satisfies

$$(2.11) \quad \begin{aligned} I &\leq \int_0^1 s^{-\gamma(k)+\gamma(i,\beta)} C^{-q} \left(\int_0^s (s-u)^{-\gamma(i,\beta)} |\omega'_{i\beta}(s-u)| |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \right) ds \\ &\leq \int_0^1 u^{-\max[0,\gamma(k,\lambda-q)-\gamma(i,\beta)]} C^{-q} |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \left(\int_u^1 (s-u)^{-\gamma(i,\beta)} |d_s\omega_{i\beta}(s-u)| \right). \end{aligned}$$

For those k such that $r_k = r_K$, we get that

$$(2.12) \quad \begin{aligned} I &\leq \int_0^1 s^{-e(k)+\lambda-\gamma(k)-q} \left(\int_0^s |\omega'_{i\beta}(s-u)| |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \right) ds \\ &\leq \int_0^1 s^{-e(k)+\lambda-q-\gamma(k)+\gamma(i,\beta)} \left(\int_0^s (s-u)^{-\gamma(i,\beta)} |\omega'_{i\beta}(s-u)| |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \right) ds \\ &\leq \int_0^1 u^{-\max[0,\gamma(k,\lambda-q)-\gamma(i,\beta)]} |d\alpha^{i\beta}_{k,\lambda+1-q}(u)| \left(\int_0^1 s^{-\gamma(i,\beta)} |d\omega_{i\beta}(s)| \right). \end{aligned}$$

Also we get that

$$(2.13) \quad \sum_{\kappa\lambda} \int_1^\infty |dJ\omega_{k\lambda}(s)| \leq \sum_{k\lambda q i\beta} \int_1^\infty C^{-q} \int_0^s |\omega'_{i\beta}(s-u)||da^{i\beta}_{k,\lambda+1-q}(u)| \, ds$$

$$\leq \sum_{k\lambda q i\beta} C^{-q} \int_0^\infty |da^{i\beta}_{k,\lambda+1-q}(u)| \int_0^\infty |d\omega_{i\beta}(s)|.$$

Let θ be the number

$$\theta = B \sum_{k\lambda q i\beta} \left(\int_0^1 u^{-\max[0,\gamma(k,\lambda-q)-\gamma(i,\beta)]} + \int_1^\infty \right) |da^{i\beta}_{k,\lambda+1-q}(u)|,$$

where $B = 4(1 + C^{-n}) \sum_{k\lambda q i\beta} 1$. Since the set of absolutely continuous functions in V is dense in V, (2.11)–(2.13) imply that $J: V \to V$ is a bounded linear operator with norm $\|J\| \leq \theta$.

If we define g in V by

$$(2.14) \qquad g_{k\lambda}(t) = \sum_{q=1}^\lambda \int_0^t (s + r_k)^{-q} d\phi_{k,\lambda+1-q}(s),$$

then we can write (2.3) as the functional equation $\omega = g + J\omega$. Therefore if we assume that $\theta < 1$, the contraction principle implies that (2.3) has a unique solution ω in V given by

$$(2.15) \qquad\qquad \omega = (I - J)^{-1} g = \sum_{n=0}^\infty J^n g.$$

In the case where $p = 0$ and $\theta < 1$, it follows that the formal arguments leading to (2.3) can be reversed to give a solution of equation (1.13) of the form (1.19) satisfying (1.11_1).

If θ is not less than 1, fix U to be so large that

$$(2.16) \quad B \sum_{k\lambda q i\beta} \left(\int_0^1 u^{-\max[0,\gamma(k,\lambda-q)-\gamma(i,\beta)]} + \int_1^\infty \right) e^{-uU} |da^{i\beta}_{k,\lambda+1-q}(u)| < 1.$$

Replace t in (1.13) by $t + U$ where $t \geq 0$. The coefficients $a^{i\beta}_{k\delta}(t)$, $f_{k\delta}(t)$ are replaced by $a^{i\beta}_{k\delta}(t + U)$, $f_{k\delta}(t + U)$, and $da^{i\beta}_{k\delta}(s)$, $d\phi_{k\delta}(s)$ are replaced by $e^{-sU} da^{i\beta}_{k\delta}(s)$, $e^{-sU} d\phi_{k\delta}(s)$. By what we have already done, (1.13) has a solution $w(t)$ representable as (1.11) and (1.11_1) for $p = 0$, where the integral in (1.11_1) is absolutely convergent for $t \geq U$ and t is the original variable.

Suppose now that

$$(2.17) \qquad\qquad r_k \text{ is real}, \quad da^{j\delta}_{i\beta} \geq 0, \quad d\phi_{i\beta} \geq 0.$$

It is clear, from (2.15) and the definitions of J and g, that $d\omega \geq 0$. In this case it follows that the integral in (1.11_1) for $p = 0$ is absolutely convergent on the

half-line $0 \le a < t < \infty$ if the function w is regular analytic in the complex half-plane $\text{Re}(t) > a$; cf. [3, Theorem 5b, p. 58]. However, since the entries of $A(t)$ and $f(t)$ in (1.13) are regular analytic for $\text{Re}(t) > 0$, solutions of (1.13) are regular analytic for $\text{Re}(t) > 0$. Hence the integrals of (1.11_1) would be absolutely convergent for $t > 0$ in the case $d\omega$ is monotone.

If $d\omega$ is not monotone, consider the first order differential system

$$(2.18) \qquad w' = R^0 w + C^0 A^0(t)w + C^0 f^0(t).$$

Let the entries of $A^0(t)$ and $f^0(t)$ be of the type (1.16) and (1.17) for $p = 0$, where the integrals are absolutely convergent for $t > 0$, the determining functions satisfy (1.3), and

$$|d\alpha_{k\delta}^{i\beta}(s)| \le d\alpha_{k\delta}^{i\beta 0}(s), \qquad |d\phi_{k\delta}(s)| \le d\phi_{k\delta}^0(s).$$

For example, let $d\alpha_{k\delta}^{i\beta 0}(s) = |d\alpha_{k\delta}^{i\beta}(s)|$, $d\phi_{k\delta}^0(s) = |d\phi_{k\delta}(s)|$. Let $\alpha_{k\delta}^{i\beta 0}(s)$ and $\phi_{k\delta}^0(s)$ satisfy conditions (1.9) and (1.18) respectively. Let R^0 be the $n \times n$ constant matrix in the same Jordan canonical form as R, where $r_k^0 = |r_k|$. Let

$$(2.19) \qquad C^0 = \sup\{(s + r_k^0/|s + r_k|)^q : s \ge 0, \ r_k \ne 0, \ 1 \le q \le e(k)\}.$$

Let $w^0(t)$ be a solution of (2.18) of the form (1.11), (1.11_1) for $p = 0$ with $d\omega^0 \ge 0$ (since the conditions in (2.17) are satisfied) and the integrals absolutely convergent for $t > 0$. Then we have the equation

$$(2.20) \qquad d\omega_{k\lambda}^0(s) = C^0 \sum_{q=1}^{\lambda} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} (s + r_k^0)^{-q} d\left\{ \int_0^s \alpha_{k,\lambda+1-q}^{i\beta 0}(s - u) \, d\omega_{i\beta}^0 \right\}$$
$$+ \sum_{q=1}^{\lambda} (s + r_k^0)^{-q} d\phi_{k,\lambda+1-q}^0(s) \quad \text{for } s \ge 0.$$

We can now prove the existence of a unique solution ω for (2.3) by successive approximations $\omega_{(0)}, \omega_{(1)}, \cdots$ defined by

$$d\omega_{k\lambda(0)}(s) = \sum_{q=1}^{\lambda} (s + r_k)^{-q} d\phi_{k,\lambda+1-q}(s),$$

$$d\omega_{k\lambda(n)}(s) = \sum_{q=1}^{\lambda} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} (s + r_k)^{-q} d\left\{ \int_0^s \alpha_{k,\lambda+1-q}^{i\beta}(s - u) \, d\omega_{i\beta(n-1)}(u) \right\}$$

$$+ \sum_{q=1}^{\lambda} (s + r_k)^{-q} d\phi_{k,\lambda+1-q}(s).$$

If we define $\omega = \omega_{(0)} + \sum_{i=1}^{\infty} (\omega_{(i)} - \omega_{(i-1)})$, then we wish to show that

(2.21) $$|d\omega| \leq |d\omega_{(0)}| + \sum_{i=1}^{\infty} |d(\omega_{(i)} - \omega_{(i-1)})| \leq d\omega^0.$$

If we put $d\omega^{(0)} = |d\omega_{(0)}|$ and $d\omega^{(n)} = |d\omega_{(0)}| + \Sigma_{i=1}^{n} |d(\omega_{(i)} - \omega_{(i-1)})|$, we get that

$$d\omega_{k\lambda}^{(n)}(s) \leq \sum_{q=1}^{\lambda} \sum_{i=1}^{m} \sum_{\beta=1}^{e(i)} |s + r_k|^{-q} d\left\{ \int_0^s \alpha_{k,\lambda+1-q}^{i\beta 0} (s-u) d\omega_{i\beta}^{(n-1)}(s) \right\}$$

$$+ \sum_{q=1}^{\lambda} |s + r_k|^{-q} d\phi_{k,\lambda+1-q}^0(s).$$

Equation (2.20) and an induction on n show that $d\omega^{(n)} \leq d\omega^0$ for $t \geq 0$ and $n = 0, 1, \cdots$. Therefore the series $\omega_{(0)} + \Sigma_1^{\infty} (\omega_{(i)} - \omega_{(i-1)})$ converges absolutely for $t \geq 0$, and (2.21) holds. Also ω clearly satisfies (2.3). Therefore we get that the integral $\int_0^{\infty} e^{-st} d\omega(s)$ is absolutely convergent for $t > 0$, and the proof of existence in Theorem 1.2 is complete in the case $p = 0$. Uniqueness follows in the standard way.

    Case 2 $(p > 0)$. In this case, $w_{k\delta}(t)$ is of the form

(2.22) $$w_{k\delta}(t) = \int_p^{\infty} e^{-st} d\omega_{k\delta}(s) = e^{-pt} \int_0^{\infty} e^{-st} d\omega_{k\delta}(s+p) = e^{-pt} \widetilde{w}_{k\delta}(t),$$

(2.22') $$\widetilde{w}_{k\delta}(t) = \int_0^{\infty} e^{-st} d\widetilde{\omega}_{k\delta}(s), \qquad \widetilde{\omega}_{k\delta}(s) = \omega_{k\delta}(s+p).$$

Similarly let

(2.23) $$f_{k\delta}(t) = e^{-pt} \widetilde{f}_{k\delta}(t), \qquad \widetilde{f}_{k\delta}(t) = \int_0^{\infty} e^{-st} d\widetilde{\phi}_{k\delta}(s), \qquad \widetilde{\phi}_{k\delta}(s) = \phi_{k\delta}(s+p).$$

Substituting these expressions into (1.13), we get

(2.24) $$\widetilde{w}'(t) = (R+p)\widetilde{w}(t) + A(t)\widetilde{w}(t) + \widetilde{f}(t).$$

By the choice of p in (1.14), we have that $0 = \max\{-(r_i + p): \operatorname{Im}(r_i) = 0, 1 \leq i \leq m\}$ and we can apply Theorem 1.2, Case 1, where $p = 0$. We have that both

$$\int_0 s^{-\widetilde{\gamma}(k,j)} |d\widetilde{\phi}_{k,j+1}(s)| = \int_p (s-p)^{-\widetilde{\gamma}(k,j)} |d\phi_{k,j+1}(s)| < \infty,$$

and

$$\int_0 s^{-\max[0, \widetilde{\gamma}(k,j) - \widetilde{\gamma}(i,\beta)]} |d\alpha_{k,j+1}^{i\beta}(s)| < \infty \quad \text{for } 1 \leq k \leq m, \ 0 \leq j \leq e(k) - 1,$$

by assumption. Hence, we get a unique solution $\widetilde{w}(t)$ of (2.24) satisfying the analogue of (1.11) and (1.11$_1$) for $p = 0$. By (2.22), we obtain a unique solution

$w(t) = e^{-pt}\mathcal{W}(t)$ of (1.13) satisfying (1.11)–(1.11$_2$). This completes the proof of Theorem 1.2. Notice also that

$$\int_p (s - p)^{-\gamma(i,\beta)} |d\omega_{i\beta}(s)| < \infty \quad \text{for } 0 \leq i \leq m,\ 1 \leq \beta \leq e(i).$$

3. Remarks and corollaries. Remark 3.1. Notice that since the proofs of Theorem 1.2 and therefore of Theorem 1.1 (and in fact of all the results to be obtained in this paper) are based upon majorizations, everything remains true if the real half-line $t > 0$ is replaced by the complex half-plane $\text{Re}(t) > 0$. Also if we assume that the determining functions $\alpha(s)$ and $\phi(s)$ of Theorems 1.1 and 1.2 are of finite total variation on $t \geq 0$, then the resulting Laplace-Stieltjes integral solutions are absolutely convergent for $t \geq 0$. In this case the real half-line can be replaced by the imaginary axis $\text{Re}(t) = 0$. Accordingly, we arrive at the following Fourier analogue of Theorem 1.1, which we shall state only for the case $p = 0$. Similar corollaries can be formulated for all the results to follow in Parts I and II but will be left unstated.

Corollary 3.1. *In equation (1.1) of Theorem 1.1, let the entries of the coefficient matrix $A(t)$ satisfy*

$$a^{j\beta}_{k\delta}(t) = \int_0^\infty e^{its}\, d\alpha^{j\beta}_{k\delta}(s) \quad \text{for } -\infty < t < \infty,\ \int_0^\infty |d\alpha^{j\beta}_{k\delta}(s)| < \infty,$$

for any j, k, β, δ. Assume that conditions (1.8), (1.9) and (1.9$_1$) are satisfied. Then there exists a unique solution of (1.1) of the form

$$z(t) = e^{ir_\kappa t}(1_{\kappa e(\kappa)} + w(t)),$$

$w(t)$ of the form (1.11), where

$$w_{k\beta}(t) = \int_0^\infty e^{its}\, d\omega_{k\beta}(s) \quad \text{for } -\infty < t < \infty,\ \omega_{k\beta}(0) = \omega_{k\beta}(+0) = 0,$$

the integral is absolutely convergent for $t \geq 0$, and $\omega_{k\beta}(s)$ satisfies (1.11$_3$).

Remark 3.2. Theorem 1.1 contains the theorem of Wintner [4, p. 163] concerning the existence of a fundamental system of solutions of the differential equation $z' = A(t)z$, corresponding to $R = 0$. He assumes that the determining functions of the entries of $A(t)$ are of finite total variation on $[0, \infty)$ and, as above, obtains solutions whose determining functions are also of finite total variation on $[0, \infty)$.

Remark 3.3. Notice that p_κ in (1.5) is the distance between r_κ and the farthest diagonal entry with the same imaginary part lying to the left of r_κ (i.e., having a smaller real part). Thus if $\text{Im}(r_i) \neq \text{Im}(r_j)$ for $i \neq j$, and the entries of

$A(t)$ are of the form (1.7) and satisfy the conditions of Theorem 1.1 with $p = 0$, then Theorem 1.1 states that, for any $1 \leq \kappa \leq m$, we can find a solution of (1.1) of the form (1.10) with $p = 0$. Also, in any case, we can always find a solution (or solutions) of the form (1.10) with $p = 0$ for the block (or blocks) whose diagonal entries have the smallest real part without assuming that the determining functions in the entries of $A(t)$ are zero on some neighborhood $[0, p]$ of the origin.

If the determining functions for the entries of $A(t)$ are stepfunctions, Theorem 1.1 implies the following result for almost periodic functions. Similar corollaries can be formulated for all the results to follow in Parts I and II but will be left unstated.

Corollary 3.2. *In equation (1.1), let the entries of $A(t)$ be uniformly almost periodic functions with Fourier expansions*

$$(3.01) \qquad A(t) \sim \sum_{1}^{\infty} A_n e^{-\gamma_n t}$$

absolutely convergent for $\mathrm{Re}\,(t) > 0$, *where* A_n *is a constant complex matrix. Assume that the set of exponents* $\{\gamma_n\}$ *is an additive semigroup and* $p < \gamma_n \to \infty$ *as* $n \to \infty$. *Then there exists a solution of (1.1) of the form (1.10) and (1.11), such that the entries* $w_{i\beta}(t)$ *are uniformly almost periodic functions with Fourier expansions*

$$(3.02) \qquad w_{i\beta}(t) \sim \sum_{1}^{\infty} w_{i\beta}^{(n)} e^{-\gamma_n t}$$

absolutely convergent for $\mathrm{Re}\,(t) > 0$.

Notice that we lose no generality in assuming that the set of γ_n's is an additive semigroup because we can always insert amplitude matrices A_n into expression (3.01), all of which are the zero matrix, in order that the $\{\gamma_n\}$ becomes an additive semigroup.

The following statement is a corollary of Theorem 1.2. There is an analogous corollary for Theorem 1.1 which will not be stated.

Corollary 3.3. (i) *In Theorem 1.2, if*

$$(3.03) \qquad R \text{ is a real } n \times n \text{ matrix, } d\alpha_{k\delta}^{i\beta} \geq 0 \text{ and } d(c\alpha^{\kappa e(\kappa)} + \phi) \geq 0,$$

then $d\omega \geq 0$.

(ii) *If* $d(c\alpha^{\kappa e(\kappa)} + \phi) = 0$ *on* $[p, S]$, *then* $d\omega = 0$ *on* $[p, S]$.

(iii) *The discontinuities of* $\omega(t)$ *are contained in the additive semigroup generated by the discontinuities of* α *and* ϕ.

(iv) *If* $ca^{Ke(\kappa)} + \phi$ *is continuous* [*or absolutely continuous*] *for* $s \geq p$, *then* $\omega(s)$ *is continuous* [*or absolutely continuous*] *for* $s \geq p$.

(v) *If* $ca^{Ke(\kappa)} + \phi$ *is* $C^1([p, \infty))$ *and* α *is* $C^0([p, \infty))$, *then* ω *is* $C^1((p, \infty))$.

(vi) *Let* R^0 *be an* $n \times n$ *constant matrix in the same Jordan canonical form as* R *in Theorem 1.2, and let* $r_k^0 = |r_k + p| - p$. *Let*

$$C^0 = \sup\{(s + |r_k + p|/|s + r_k + p|)^q : s \geq 0, \; r_k \neq -p, \; 1 \leq q \leq e(k)\}.$$

In equation (2.18), let the entries of $A^0(t)$ *and* $f^0(t)$ *satisfy the conditions of Theorem 1.2, and be completely monotone (i.e.,* $da^0 \geq 0$, $d\phi^0 \geq 0$*). Let* $z^0(t)$ *be a solution of (2.18) with the representation*

$$z^0(t) = |c| 1_{Ke(\kappa)} + \int_p^\infty e^{-st} d\omega^0(s).$$

Then, for $s \geq p$, $|da_{k\delta}^{i\beta}| \leq da_{k\delta}^{i\beta 0}$, *and* $|d\phi_{k\delta}| \leq d\phi_{k\delta}^0$ *implies that* $|d\omega| \leq d\omega^0$.

Proof of Corollary 3.3. (i), (ii), (iii), and (vi) follow directly from the proof of Theorem 1.2. For (iv), suppose $\phi + ca^{Ke(\kappa)}$ is continuous and assume that $p = 0$. Then for $J: V \to V$ in the proof of Theorem 1.2, the space V can be replaced by $V \cap C^0([0, \infty))$ without changing the proof. Also if $\phi + ca^{Ke(\kappa)}$ is absolutely continuous, we can replace V by its subset of absolutely continuous functions; cf. (2.7) which shows that $(J\omega)(t)$ in (2.4) is absolutely continuous. The case $p > 0$ can be transformed to the case $p = 0$ just as in the proof of Theorem 1.2, Case 2.

(v) follows from (2.3)–(2.8), (2.14), and the smoothing properties of convolutions.

4. A system of solutions corresponding to a block. If a block in the complex matrix R in (1.1) is $e(\kappa) \times e(\kappa)$ where $e(\kappa) > 1$, then in this section and the next, we determine conditions under which solutions involving powers of t can be found corresponding to the block in question.

Theorem 4.1. *In the homogeneous equation (1.1), let* $R, \kappa, p, r_{b(v)}, \bar{v}, \gamma(k),$ $\gamma(k, j),$ *and* $\tilde{\gamma}(k, j)$ *be as in Theorem 1.1. Let* Q *be an integer such that* $0 \leq Q < e(\kappa)$. *Let* ζ *be a number such that* $\zeta \geq Q$.

Case 1 $(p = 0)$. *Assume that* $A(t)$ *satisfies (1.9), and*

$$(4.1) \quad \int_0 s^{-\gamma(i,b) - \zeta + j} |da_{i,b+1}^{\kappa, e(\kappa) - j}(s)| < \infty \quad for \; 1 \leq i \leq m, \; 0 \leq b \leq e(i) - 1.$$

Then there exist $Q + 1$ n-*dimensional functions* $w(t) = w^j(t)$ *representable as in (1.11)–(1.11$_2$), and*

(4.2) $$\int_0 s^{-\gamma(i,\beta)+j-\zeta} |d\omega^j_{i\beta}(s)| < \infty \quad \text{for } j = 0, \cdots, Q$$

such that

(4.3) $$z(t) = e^{r_\kappa t}\left[(1_{\kappa e(\kappa)} + w^0(t))t^i/i! + \sum_{j=1}^{i} (1_{\kappa,e(\kappa)-j} + w^j(t))t^{i-j}/(i-j)!\right] \cdot$$

is a solution of the homogeneous equation (1.1) for $i = 0, \cdots, Q$.

Case 2 ($p > 0$). *Assume that* $A(t)$ *satisfies* (1.9_2),

(4.1_1) $$\alpha^{\kappa,e(\kappa)-j}(s) = 0 \quad \text{on } [0, p] \text{ for } 0 \le j \le Q,$$

and

(4.1_2) $$\int_p (s-p)^{-\tilde{\gamma}(i,b)-\zeta+j} |d\alpha^{\kappa,e(\kappa)-j}_{i,b+1}(s)| < \infty \quad \text{for } 1 \le i \le m, \ 0 \le b \le e(i) - 1.$$

Then there exist $Q + 1$ *functions* $w(t) = w^j(t)$ *as in* $(1.11)-(1.11_2)$ *and* (1.11_4), *and* (4.3) *is a solution of* (1.1) *for* $i = 0, \cdots, Q$.

Proof of Theorem 4.1. *Case 1* ($p = 0$). The result for $Q = 0$ is contained in Theorems 1.1 and 1.2 if we replace $\gamma(k)$ by $\gamma(k) + \zeta$. Suppose that $0 < N \le Q$ ($< e(\kappa)$), that $w^0, \cdots, w^{N-1}$ exist as specified, and that $z(t)$ in (4.3) is a solution of (1.1) for $i = 0, \cdots, N - 1$. In order that $z(t)$ in (4.3) be a solution of (1.1) for $i = N$, it is necessary and sufficient that w^N satisfy the equation

(4.4) $$w^{N\prime} = ((R - r_\kappa) + A(t))w^N + A^{\kappa,e(\kappa)-N}(t) - w^{N-1}.$$

Consequently, the existence of w^N follows from Theorem 2.1 if it is verified that

(4.5) $$\int_0 s^{-\gamma(i,b)-\zeta+N} |d\alpha^{\kappa,e(\kappa)-N}_{i,b+1}(s)| < \infty,$$

(4.6) $$\int_0 s^{-\gamma(i,b)-\zeta+N} |d\omega^{N-1}_{i,b+1}(s)| < \infty.$$

(4.5) follows by assumption, and (4.6) follows because the left side of the inequality is less than or equal to $\int_0 s^{-\gamma(i,b)-\zeta+(N-1)} |d\omega^{N-1}_{ib}(s)|$ since $0 \ge -\gamma(i) - \zeta + N > -\gamma(i) - \zeta + N - 1$ and $-e(i) + h - \gamma(i) - \zeta + N = -\gamma(i, b+1) - \zeta + (N-1)$. The proof for Case 2 ($p > 0$) follows similarly.

5. Another form for a system of solutions corresponding to a block. As mentioned in Hartman [2] for the nth order case, the theory of asymptotic integration of linear differential equations suggests that an analogous result (with a weaker conclusion than that of Theorem 4.1) should hold if the extra conditions (4.1) thru (4.1_2) are reduced somewhat.

Theorem 5.1. *In the homogeneous equation* (1.1), *let* R, κ, p, $r_{b(v)}$, $\bar{v}$, $\gamma(k)$, $\gamma(k, j)$ *and* $\tilde{\gamma}(k, j)$ *be as in Theorem 1.1.*

Case 1 $(p = 0)$. *Let* $A(t)$ *be as in* (1.9) *and assume that*

$$(5.1) \qquad \int_0 s^{-\gamma(k,j)+\lambda} |d\alpha_{k,j+1}^{\kappa, e(\kappa)-\lambda}(s)| < \infty \quad \text{for } 1 \le k \le m, \ 0 \le j \le e(k) - 1.$$

Then for any i, $0 < i < e(\kappa)$, (1.1) *has a solution of the form*

$$(5.2) \qquad\qquad z(t) = e^{r_\kappa t}[t^i w(t) + x(t)],$$

where $w(t)$ *is representable as* (1.11)–(1.11$_3$), *and* $x(t)$ *is an n-dimensional vector*

$$(5.3) \qquad x(t) = (x_1(t), \cdots, x_m(t)), \qquad x_k(t) = (x_{k1}(t), \cdots, x_{ke(k)}(t)),$$

where $x_{k\lambda}(t) = 0$ *for* $k \ne \kappa$, $x_{\kappa \cdot \lambda}(t) = t^{i-e(\kappa)+\lambda}/(i - e(\kappa) + \lambda)!$ *for* $\lambda \ge e(\kappa) - i$, *and* $x_{\kappa\lambda}(t) = 0$ *for* $\lambda < e(\kappa) - i$, *i.e.,*

$$x(t) = (0, \cdots, 0, \underbrace{0, \cdots, 0, 1, t, \cdots, t^{i-1}/(i-1)!, t^i/i!,}_{\kappa th \ block} 0, \cdots, 0),$$

Case 2 $(p > 0)$. *Let* $A(t)$ *be as in* (1.9$_2$). *Assume that*

$$(5.1_1) \qquad\qquad \alpha^{\kappa, e(\kappa)-\lambda}(s) = 0 \quad \text{on } [0, p],$$

and

$$(5.1_2) \qquad\qquad \int_p (s - p)^{-\gamma(k,j)+\lambda} |d\alpha_{k,j+1}^{\kappa, e(\kappa)-\lambda}(s)| < \infty$$

for $0 \le \lambda \le e(\kappa) - 1$, $1 \le k \le m$, $0 \le j \le e(k) - 1$. *Then for any* i, $0 < i < e(\kappa)$, (1.1) *has a solution of the form* (5.2) *where* $w(t)$ *is representable as* (1.11)–(1.11$_2$) *and* (1.11$_4$).

Proof of Theorem 5.1. Substituting (5.2) into (1.1) gives

$$(5.4) \qquad (t^i w)' = (R - r_\kappa)t^i w + t^i \left[A(t)w + \sum_{j=0}^{i} A^{\kappa, e(\kappa)+j-i}(t) t^{j-i}/j! \right].$$

We can write (5.4) as

$$(5.5) \qquad\qquad t^{-i}(t^i w)' = Rw + A(t)w + f(t),$$

where

$$f_{k,b+1}(t) = \sum_{j=0}^{i} A_{k,b+1}^{\kappa, e(\kappa)+j-i}(t) t^{j-i}/j! = \int_p^\infty e^{-st} d\phi_{k,b+1}(s).$$

That is

$$d\phi_{k,b+1}(s) = \sum_{j=0}^{i} d \int_{p}^{s} (s-u)^{-j+i} d\alpha_{k,b+1}^{\kappa,e(\kappa)+j-i}(u)/j!(-j+i)!.$$

Therefore by (5.1)–(5.1$_2$),

$$\int_{p}^{} (s-p)^{-\widetilde{\gamma}(k,i)} |d\phi_{k,j+1}(s)| < \infty, \qquad 1 \le k \le m,\ 0 \le j \le e(k)-1.$$

Consequently, Theorem 5.1 is contained in the following

Theorem 5.2. *In (5.5), let R, A(t) and f(t) satisfy the conditions of Theorem 1.2, Case 1 [or Case 2]. Then (5.5) has a solution w(t) of the form (1.11)–(1.11$_3$) [or (1.11)–(1.11$_2$) and (1.11$_4$)].*

Proof of Theorem 5.2. It is enough to prove the theorem for $p = 0$. For if $p > 0$, and we let $w = e^{-pt}\widetilde{w}$ and $f = e^{-pt}\widetilde{f}$ as in (2.22) and (2.23), (5.5) becomes

$$t^{-i}(t^{i}\widetilde{w})' = (R+p)\widetilde{w} + A(t)\widetilde{w} + \widetilde{f}(t),$$

which clearly satisfies the conditions of Theorem 5.2 for $p = 0$.

Let Λ be the Banach space of $\mathbf{C}^n$ valued functions ϕ, locally of bounded variation on $t \ge 0$, continuous from the right, $\phi(0) = \phi(+0) = 0$, with norm

$$(5.6)\quad \|\phi\|_{\Lambda} = \sum_{k=1}^{m} \sum_{\lambda=1}^{e(k)} \sum_{q=1}^{\lambda} \left(\int_{0}^{1} s^{-\gamma(k,\lambda)} + \int_{1}^{\infty} \right) |s + r_k|^{-q} |d\phi_{k,\lambda+1-q}(s)| < \infty.$$

Define the linear operator T_i on Λ by

$$(5.7)\quad \omega_{k\lambda}(s) = (T_i\phi)_{k\lambda}(s) = \sum_{q=1}^{\lambda} \left\{ \int_{0}^{\infty} \left(\frac{d}{du}\right)^{i} \left[\frac{u^{i}}{(u+r_k)^{q}i!}\right] d\phi_{k,\lambda+1-q}(u) \right.$$
$$\left. - \int_{s}^{\infty} \left(\frac{d}{du}\right)^{i} \left[\frac{(u-s)^{i}}{(u+r_k)^{q}i!}\right] d\phi_{k,\lambda+1-q}(u) \right\};$$

or equivalently, $\omega_{k\lambda}(0) = 0$, and

$$(5.8)\quad d\omega_{k\lambda}(s) = \sum_{q=1}^{\lambda} \left[\frac{d\phi_{k,\lambda+1-q}(s)}{(s+r_k)^{q}} + \left\{ \int_{s}^{\infty} \left(\frac{d}{du}\right)^{i} \left[\frac{(u-s)^{i-1}}{(u+r_k)^{q}(i-1)!}\right] d\phi_{k,\lambda+1-q}(u) \right\} ds \right].$$

We shall verify the following

Lemma 5.1. *For $0 < i < e(\kappa)$, $T_i : \Lambda \to V$ is a bounded linear operator, where V is defined after (2.3). Furthermore, the function (1.11), (1.11$_1$) with $p = 0$ and $\omega = T_i\phi$, satisfies the differential equation*

(5.9)
$$t^{-i}(t^i w)' = Rw + f(t).$$

Proof of Lemma 5.1. In the last integral of (5.7), write $[(u-s)^i - u^i] + u^i$ for $(u-s)^i$ to obtain $\omega_{k\lambda}(s)$ as the sum of integrals

$$\omega_{k,\lambda,-1}(s) = \sum_{q=1}^{\lambda} \int_0^s \left(\frac{d}{du}\right)^i \left[\frac{u^i}{(u+r_k)^q i!}\right] d\phi_{k,\lambda+1-q}(u),$$

$$\omega_{k,\lambda,n}(s) = \sum_{q=1}^{\lambda} (-1)^n C_{in} s^{i-n} \int_s^\infty \left(\frac{d}{du}\right)^i \left[\frac{u^n}{(u+r_k)^q i!}\right] d\phi_{k,\lambda+1-q}(u),$$

$n = 0, \cdots, i-1$, where $C_{in} = i!/n!(i-n)!$. If C_1 is a sufficiently large constant, then

$$|d\omega_{k,\lambda,-1}(s)| \le C_1 \sum_{q=1}^{\lambda} |s+r_k|^{-q} |d\phi_{k,\lambda+1-q}(s)|,$$

since $|(d/du)^i u^i/(u+r_k)^q| \le C_1 |1/(u+r_k)^q|$ for large u. Also, for $n = 0, \cdots, i-1$,

$$|d\omega_{k,\lambda,n}(s)| \le C_1 \sum_{q=1}^{\lambda} \left\{ s^{i-n-1} \left(\int_s^\infty |u^{i-n}(u+r_k)^q|^{-1} |d\phi_{k,\lambda+1-q}(u)|\right) ds \right.$$
$$\left. + |s+r_k|^{-q} |d\phi_{k,\lambda+1-q}(s)|\right\}$$
$$\le C_1 \sum_{q=1}^{\lambda} \left\{\int_s^\infty |u(u+r_k)^q|^{-1} |d\phi_{k,\lambda+1-q}(u)| ds + |s+r_k|^{-q} |d\phi_{k,\lambda+1-q}(s)|\right\}.$$

Hence the boundedness of the operator $T_i: \Lambda \to V$ follows from

$$\sum_{k\lambda q} \left(\int_0^1 s^{-\gamma(k,\lambda)} + \int_1^\infty\right) |s+r_k|^{-q} |d\phi_{k,\lambda+1-q}(s)| < \infty.$$

In order to verify (5.9), suppose first that $\phi \in C^\infty$ has compact support in $t > 0$. Differentiate (5.7) to obtain

$$\omega'_{k\lambda}(s) = \sum_{q=1}^{\lambda} \phi'_{k,\lambda+1-q}(s)/(s+r_k)^q$$

(5.10)
$$+ \sum_{q=1}^{\lambda} \int_s^\infty (d/du)^i [(u-s)^{i-1}/(u+r_k)^q (i-1)!] \phi'_{k,\lambda+1-q}(u)\, du.$$

An integration by parts gives

$$\omega'_{k\lambda}(s) = - \sum_{q=1}^{\lambda} \int_s^\infty (d/du)^i [(u-s)^{i-1}/(u+r_k)^q (i-1)!] \phi''_{k,\lambda+1-q}(u)\, du.$$

Successive differentiations and integrations by parts give

$$\omega_{k\lambda}^{(i+1)}(s) = \sum_{q=1}^{\lambda} (s + r_k)^{-q} \phi_{k,\lambda+1-q}^{(i+1)}(s).$$

So, if we define

$$w_i(t) = \int_0^\infty e^{-st} \omega^{(i+1)}(s)\, ds \quad \text{and} \quad f_i(t) = \int_0^\infty e^{-st} \phi^{(i+1)}(s)\, ds,$$

we get $w_i' = R w_i + f_i$. Therefore, by (1.11) and (1.11_1) with $p = 0$, the fact that ϕ has compact support in $t > 0$, and the standard rules for Laplace-Stieltjes transforms,

$$f_i(t) = t^i f(t) = t^i \int_0^\infty e^{-st}\, d\phi(s), \qquad w_i(t) = t^i w(t) = t^i \int_0^\infty e^{-st}\, d\omega(s).$$

Equation (5.9) follows from the last two displays.

In order to obtain (5.9) for arbitrary $\phi \in \Lambda$, let $\phi_n \in C^\infty$ have compact support in $t > 0$ and $\phi_n \to \phi$ in Λ as $n \to \infty$. Then $\omega_n = T_i \phi_n \to \omega$ in Λ as $n \to \infty$, and so the functions

$$w_n(t) = \int_0^\infty e^{-st}\, d\omega_n(s), \qquad f_n(t) = \int_0^\infty e^{-st}\, d\phi_n(s)$$

tend to $w(t)$, $f(t)$ in C^j on compact subsets of $t > 0$ for every j. Since $t^{-i}(t^i w_n)' = R w_n + f_n(t)$, (5.9) follows, and the proof of Lemma 5.1 is complete.

Lemma 5.2. *The map* $\Psi: V \to \Lambda$ *given by*

(5.11) $$(\Psi\omega)_{k\lambda}(t) = \sum_{i\beta} \int_0^t \alpha_{k\lambda}^{i\beta}(t - s)\, d\omega_{i\beta}(s),$$

is bounded, and its norm satisfies

(5.12) $\|\Psi\| \leq B \displaystyle\sum_{k\lambda q i\beta} \left(\int_0^1 u^{-\max[0,\gamma(k,\lambda-q)-\gamma(i,\beta)]} + \int_1^\infty \right) |d\alpha_{k,\lambda+1-q}^{i\beta}(u)| \equiv \theta.$

See the proof of Theorem 1.2 and note that $\|\Psi\omega\|_\Lambda = \|J\omega\|_V$.

Completion of the proof of Theorem 5.2. *Step* (A). If (1.11), (1.11_1) with $p = 0$ is substituted into (5.5), we formally obtain the functional equation $\omega = T_i\phi + T_i\Psi\omega$. If the right side of inequality (5.12) is sufficiently small, $T_i\Psi$: $V \to V$ is bounded with a norm less than 1. Hence, arguing as in the proof of Theorem 1.2, there exists a constant $U \geq 0$ and a function $\omega(t)$ such that (1.11) is absolutely convergent and a solution of (5.5) for $t \geq U$.

Step (B). If the operator T_i in (5.7) were monotone in the sense that $d\phi \geq 0$

implies that $d\omega \geq 0$, then by an argument similar to that of Theorem 1.2, we could complete the proof. We now define an operator $\tilde{T}_i$ which is monotone, majorizes T_i, and can be used to complete the proof. To this end, write (5.8) as

$$d\omega_{k\lambda}(s) = \sum_{q=1}^{\lambda} \left[d\phi_{k,\lambda+1-q}(s)/(s+r_k)^q \right.$$
(5.13)
$$\left. + \left\{ \sum_{\rho\sigma\tau} a_{\rho\sigma\tau} s^{\rho} \int_s^{\infty} [u^{-1-\rho}(u+r_k)^q]^{-1}[u^{\tau}/(u+r_k)^{q\sigma}]d\phi_{k,\lambda+1-q}(u) \right\} ds \right]$$

where $a_{\rho\sigma\tau}$ is a constant, $\rho = 0, \cdots, i-1$; $\sigma = 0, \cdots, i$; $\tau = 0, \cdots, \sigma q$. Define $\tilde{T}_i$ by replacing $a_{\rho\sigma\tau}$ by $|a_{\rho\sigma\tau}|$ and $u+r_k$ by $u+r_k^0$, where $r_k^0 = |r_k|$. Therefore, we have

$$d(\tilde{T}_i\phi)_{k\lambda}(s) = d\omega_{k\lambda}(s)$$

(5.14)
$$= \sum_{q=1}^{\lambda} \left[d\phi_{k,\lambda+1-q}(s)/(s+r_k^0)^q \right.$$
$$\left. + \left\{ \sum_{\rho\sigma\tau} |a_{\rho\sigma\tau}| s^{\rho} \int_s^{\infty} [u^{-1-\rho}(u+r_k^0)^q]^{-1}[u^{\tau}/(u+r_k^0)^{q\sigma}]d\phi_{k,\lambda+1-q}(u) \right\} ds \right].$$

It is clear that $\tilde{T}_i : \Lambda \rightarrow V$ is a bounded operator. Let

$$w_{k\lambda}(t) = \int_0^{\infty} e^{-st} d\omega_{k\lambda}(s) = \sum_{q=1}^{\lambda} w_{k\lambda,q}(t), \quad \text{where}$$

(5.15)

$$w_{k\lambda,q}(t) = \int_0^{\infty} e^{-st} \left(d\phi_{k,\lambda+1-q}(s)/(s+r_k^0)^q \right.$$
$$\left. + \left\{ \sum_{\rho\sigma\tau} |a_{\rho\sigma\tau}| s^{\rho} \int_s^{\infty} [u^{-1-\rho}(u+r_k^0)^q]^{-1}[u^{\tau}/(u+r_k^0)^{q\sigma}]d\phi_{k,\lambda+1-q}(u) \right\} ds \right).$$

Step (C). We now verify that $w_{k\lambda,q}(t)$ satisfies a differential equation of the form

(5.16)
$$D^N[P(D)]^M[t^B P(D)]w_{k\lambda,q} = Q_q(t,D)f_{k,\lambda+1-q},$$

where $\mathrm{Re}\,(t) > 0$, $1 \leq k \leq m$, $1 \leq \lambda \leq e(k)$, $1 \leq q \leq \lambda$, $Q_q(t,y)$ is a polynomial in t and y, $P(t) = \Pi_{k=1}^m (t+r_k^0)^{e(k)}$, B, M, N are sufficiently large integers,

(5.17)
$$Q_q(t,D)f_{k,\lambda+1-q} = ct^B f_{k,\lambda+1-q}^{(n-q)+N+Mn} + \{\cdots\},$$

where $\{\cdots\}$ indicates lower order derivatives of $f_{k,\lambda+1-q}$, and c is a constant. (Throughout this section, c will denote a constant, however, not necessarily the same constant in each case.) In the verification of (5.16) we can assume that $f_{k,\lambda+1-q}$ has compact support in $t > 0$. Note that

(5.18) $(D + r_k^0)^q w_{k\lambda,q} = cf_{k,\lambda+1-q} + \sum_{\rho\sigma\tau} |a_{\rho\sigma\tau}| D^\rho (D + r_k^0)^q b_{\rho\sigma\tau},$

where $b = b_{\rho\sigma\tau}$ is

$$b(t) = \int_0^\infty e^{-st}\left\{\int_s^\infty u^{-1}[u^\rho(u + r_k^0)^q]^{-1}[u^\tau/(u + r_k^0)^{q\sigma}]\,d\phi_{k,\lambda+1-q}\right\}ds$$

$$= t^{-1}\int_0^t z(s)\,ds$$

and $z = z_{\rho\sigma\tau}$ is

(5.19) $z(t) = \int_0^\infty e^{-st}[s^\rho(s + r_k^0)^q]^{-1}[s^\tau/(s + r_k^0)^{q\sigma}]d\phi_{k,\lambda+1-q}(s).$

Therefore we get that

(5.20) $D^\rho(D + r_k^0)^{q(\sigma+1)}z = cf_{k,\lambda+1-q}^{(\tau)}(s),$ where $\tau \leq \sigma q.$

If $B \geq e(\kappa) + i + 1 \geq e(\kappa) + \sigma + 1$, then (5.18) and the display following it show that $t^B P(D)w_{k,\lambda,q} = ct^B f_{k,\lambda+1-q}^{(n-q)} + t^B$ (lower derivatives of $f_{k,\lambda+1-q}$) plus terms of the type $P_{-1}(t)\int_0^t z(s)\,ds$, $P_0(t)z(t), \cdots, P_{\rho+n-1}(t)z^{(\rho+n-1)}(t)$, where $P_j(t)$ is a polynomial of degree $\leq B - 1$. Thus

$$D^B t^B P(D)w_{k\lambda,q} = cD^B t^B f_{k,\lambda+1-q}^{(n-q)} + D^B(t^B \text{ (lower order derivatives of } f_{k,\lambda+1-q}))$$
$$+ \text{ terms of the form } q_j(t)z^{(j)},$$

where $j = 0, \cdots, B + \rho + n - 1$ and $q_j(t)$ is a polynomial. Note that, by (5.20),

$$D^\rho(s + r_k^0)^{q(\sigma+1)}z^{(j)} = cf_{k,\lambda+1-q}^{(\tau+j)}, \quad \text{where } \tau + j \leq \sigma q + B + \rho + q - 1.$$

Therefore

$$D^N[P(D)]^M[t^B P(D)]w_{k\lambda,q} = D^{N-B}[P(D)]^M D^B t^B P(D)w_{k\lambda,q}$$

(5.21) $= D^{N-B}[P(D)]^M(cD^B t^B f_{k,\lambda+1-q}^{(n-q)} + c_1 D^B(t^B \cdot \text{ lower derivatives of } f_{k,\lambda+1-q}))$

$$+ D^{N-B}[P(D)]^M\left(\sum_j q_j(t)z^{(j)}\right).$$

The contribution of the last term of (5.21) to the right side of (5.16) is a differential operator on $f_{k,\lambda+1-q}$ of order $N - B - \rho + Mn - q(\sigma + 1) + [\sigma q + B + \rho + q - 1] = N + Nm - 1$, while the main term of (5.16) comes from the first term of (5.21) and is of the form $ct^B f_{k,\lambda+1-q}^{((n-q)+N+Mn)}.$ This verifies the statement concerning (5.16).

Step (D). If we let $d\tilde{\phi}_{k\lambda} = |d\phi_{k\lambda}|$ and $d\tilde{a}_{k\lambda}^{i\beta} = |da_{k\lambda}^{i\beta}|$ in (5.5) and

$$(\tilde{\Psi}\omega)_{k\lambda}(t) = \sum_{qi\beta}\int_0^t \tilde{\alpha}_{k,\lambda+1-q}^{i\beta}(t - s)\,d\omega_{i\beta}(s),$$

then the functional equation $\omega^0 = C^{0(i+1)}(\tilde{T}_i \tilde{f} + \tilde{T}_i \tilde{\Psi}\omega)$ with C^0 as in (2.19), in which $(\tilde{df}, \tilde{d\alpha}, d\omega^0)$ are replaced by $e^{-Us}(\tilde{df}, \tilde{d\alpha}, d\omega^0)$, has a solution $\omega^0(s)$ such that $d\omega^0 \geq 0$ and

$$(5.22) \qquad w^0(t) = \int_0^\infty e^{-st} d\omega^0(s), \qquad \omega^0(+0) = \omega^0(0) = 0,$$

is absolutely convergent for $t \geq U$, for some $U \geq 0$. But w^0 is a solution of the differential system

$$D^N[P(D)]^M[t^B P(D)](w^0_{\kappa\lambda}) = C^{0(i+1)}\left(\sum_{q=1}^\lambda Q_q(t, D)\left\{ \tilde{f}_{k,\lambda+1-q} + \sum_{i,\beta} \tilde{a}^{i\beta}_{k,\lambda+1-q} w^0_{i\beta} \right\} \right),$$

where the leading terms on the left are of the form $ct^B w^{0(N+Mn+n)}_{i\beta}$, and the highest order terms on the right are of the form $ct^B w^{0(n+N+Mn-q)}_{i\beta}$, where $q \geq 1$. Therefore the equation is nonsingular, and $w^0(t)$ is absolutely convergent for $t > 0$. The proof of Theorem 5.2 can now be completed by the same arguments used at the end of Theorem 1.2, and the fact that the $\omega(t)$ constructed in Step (A) satisfies $|d\omega_{i\beta}| < d\omega^0_{i\beta}$.

PART II. THE nTH ORDER, ONE-DIMENSIONAL CASE

In Part II of this paper, we determine sufficient conditions for the existence of solutions for the nth order, one-dimensional, linear, ordinary differential equations with Laplace-Stieltjes coefficients, which correspond to the solutions determined for the n-dimensional, first order system in Part I. Instead of deriving these results from those of Part I, we shall obtain them directly from the results and techniques of Hartman [2].

6. One solution corresponding to each root. Consider the differential equation

$$(6.1) \qquad [(D - r_1)^{e(1)}(D - r_2)^{e(2)} \cdots (D - r_m)^{e(m)}]y + \sum_{i=0}^{n-1} a_i(t)y^{(i)} = 0,$$

where $D = d/dt$, $e(1) + \cdots + e(m) = n$, and the r_i are complex constants, $r_i \neq r_j$ for $i \neq j$. Let the coefficients $a_i(t)$ be representable as absolutely convergent Laplace-Stieltjes transforms for $t > 0$. Given any of the roots r_κ, we are interested in finding solutions of (6.1) which behave asymptotically like certain solutions to the unperturbed part of (6.1), namely, $\Pi_{i=1}^m (D - r_i)^{e(i)}y = 0$ as $t \to \infty$. With this in mind we state the following

Theorem 6.1. *In equation (6.1), for a fixed κ, $1 \leq \kappa \leq m$, let*

$$(6.2) \qquad 0 \leq p = p_\kappa = \max\{r_\kappa - r_i \colon \operatorname{Im}(r_i) = \operatorname{Im}(r_\kappa), \ 1 \leq i \leq m\},$$

and denote by r_b *the root such that*

(6.3) $r_K - r_b = p, \quad and \ e(b) \ its \ multiplicity.$

Assume that the $a_i(t)$ *are representable in the form*

(6.4) $a_i(t) = \int_p^\infty e^{-st} \, d\alpha_i(s),$

where the integrals are absolutely convergent Laplace-Stieltjes transforms for $t > 0$,

(6.4') $\int_p^\infty e^{-st} |d\alpha_i(s)| < \infty$

and the determining functions $\alpha_i(s)$ *satisfy* (1.3). *For* $p = 0$, *let*

(6.5) $\sum_{\lambda=j}^{n-1} \int_0 s^{j-e(K)} |r_K^{\lambda-j}| |d\alpha_\lambda(s)| < \infty \quad for \ 0 \le j \le e(K);$

for $p > 0$, *let*

(6.5') $\sum_{\lambda=0}^{n-1} \int_p (s-p)^{-e(b)} |r_K^\lambda| |d\alpha_\lambda(s)| < \infty.$

Then there exists a unique solution of (6.1) *of the form*

(6.6) $y(t) \doteq e^{r_K t} \left(c - \int_p^\infty e^{-st} \, d\omega(s) \right), \quad \omega(p) = \omega(+p) = 0,$

where c *is a complex constant, and the integral is an absolutely convergent Laplace-Stieltjes transform,*

(6.6') $\int_p^\infty e^{-st} |d\omega(s)| < \infty \quad for \ t > 0.$

Proof of Theorem 6.1. Substituting (6.6) into equation (6.1) gives

(6.7) $\prod_{i=1}^m [D - (r_i - r_K)]^{e(i)} w + \sum_{j=0}^{n-1} (-1)^{e(b)+j} g_j(t) D^j w = (-1)^{e(b)+1} c g_0(t),$

where

(6.8) $w = w(t) = -\int_p^\infty e^{-st} \, d\omega(s), \quad \omega(p) = \omega(+p) = 0,$

and

(6.9) $g_j(t) = (-1)^{e(b)+j} \sum_{\lambda=j}^{n-1} \binom{\lambda}{j} r_K^{\lambda-j} a_\lambda(t) \equiv (-1)^{e(b)+j} \int_p^\infty e^{-st} \, d\gamma_j(s).$

Theorem 6.1 will follow from (6.7)–(6.9) and the following

Theorem 6.2. *Consider the differential equation*

$$(6.10) \qquad \prod_{i=1}^{m} [(D - r_i)^{e(i)}]w + \sum_{j=0}^{n-1} (-1)^{e(b)+j} g_j(t) D^j w = (-1)^{e(b)+1} f(t),$$

where the r_i are complex constants, $r_i \neq r_j$ for $i \neq j$, $e(1) + \cdots + e(m) = n$. For a fixed κ, $1 \leq \kappa \leq m$, let $r_\kappa = 0$ and let

$$(6.11) \qquad\qquad 0 \leq p = \max_i \{-r_i : r_i \text{ is real}\}.$$

Let r_b be the root such that

$$(6.12) \qquad\qquad -r_b = p, \quad \text{and } e(b) \text{ its multiplicity.}$$

Let g_j and f be representable as

$$(6.13) \qquad g_j(t) = \int_p^\infty e^{-st} d\gamma_j(s) \quad \text{for } 0 \leq j \leq n - 1,$$

$$(6.14) \qquad f(t) = \int_p^\infty e^{-st} d\phi(s),$$

absolutely convergent Laplace-Stieltjes transforms for $t > 0$, such that their determining functions satisfy (1.3). Assume that, for $p = 0$,

$$(6.15) \qquad \int_0 s^{-e(\kappa)+j} |d\gamma_j(s)| < \infty \quad \text{for } 0 \leq j \leq e(\kappa),$$

and, for $p > 0$,

$$(6.16) \qquad \int_p (s - p)^{-e(b)} |cd\gamma_0(s) + d\phi(s)| < \infty.$$

Then there exists a unique solution of (6.10) for $t > 0$ of the form

$$(6.17) \qquad w(t) = c - \int_p^\infty e^{-st} d\omega(s), \quad \omega(p) = \omega(+p) = 0,$$

so that $w(\infty) = c$, where c is a complex constant and the integral is an absolutely convergent Laplace-Stieltjes transform for $t > 0$,

$$(6.18) \qquad \int_p^\infty e^{-st} |d\omega(s)| < \infty.$$

There is a corollary of Theorem 6.2 analogous to that of Corollary 3.3 which we shall not state. However, we shall need the following:

Corollary 6.1. *In Theorem 6.2, if $\phi + c\gamma_0$ satisfies*

$$\int_p (s - p)^{-e(b)-\zeta} |d(\phi + c\gamma_0)| < \infty \quad \text{for some constant } \zeta \geq 0,$$

then

$$\int_p (s - p)^{-\zeta} |d\omega(s)| < \infty.$$

7. **Proof of Theorem 6.2.** The case where $c \neq 0$ is contained in the case where $c = 0$ if we replace f in (6.10) by $cg_0 + f$. So we seek a solution w of the form (6.8). If $p = 0$, then (6.11) implies that $r_b = r_\kappa = 0$, $e(b) = e(\kappa)$, and $\Pi_{i=1}^m (D - r_i)^{e(i)} = D^{e(\kappa)} P(-D)$, where $P(t)$ is a polynomial of degree $n - e(\kappa)$ and $P(t) \neq 0$ for $t \geq 0$. Our desired result follows directly in this case from Hartman [2, Theorem 2.1].

If $p > 0$, then

(7.1)
$$w(t) = -\int_p^\infty e^{-st} d\omega(s) = -e^{-pt} \int_0^\infty e^{-st} d\omega(s + p) = e^{-pt}\widetilde{w}(t),$$

$$\widetilde{w}(t) = -\int_0^\infty e^{-st} d\widetilde{\omega}(s), \quad \text{where } \widetilde{\omega}(s) = \omega(s + p).$$

Similarly, we put

(7.2) $$f(t) = e^{-pt}\widetilde{f}(t), \quad \widetilde{f}(t) = \int_0^\infty e^{-st} d\widetilde{\phi}(s), \quad \text{where } \widetilde{\phi}(s) = \phi(s + p).$$

Substituting these expressions into (6.10), we get

(7.3)
$$D^{e(b)}\widetilde{P}(-D)\widetilde{w} + \sum_{j=0}^{n-1} (-1)^{e(b)+j}\widetilde{g}_j(t) D^j \widetilde{w} = (-1)^{e(b)+1}\widetilde{f},$$

where $\widetilde{P}(t)$ is a polynomial of degree $n - e(b)$, $\widetilde{P}(t) = \Pi_{i=1; i \neq b}^m (t + r_i + p)^{e(i)}$, $\widetilde{P}(t) \neq 0$ for $t \geq 0$, and

$$\widetilde{g}_j(t) = (-1)^{e(b)+j} \sum_{\lambda=j}^{n-1} \binom{\lambda}{j} p^{\lambda - j} g_\lambda(t) \equiv \int_0^\infty e^{-st} d\widetilde{\gamma}_j(s)$$

where $\widetilde{\gamma}_j(s) = 0$ on $[0, p]$. Again we can apply Hartman [2, Theorem 2.1] to equation (7.3) because $\widetilde{\gamma}_j = 0$ on $[0, p]$ implies $\int_0 s^{-e(b)+j} |d\widetilde{\gamma}_j(s)| < \infty$ for $0 \leq j \leq e(b)$, and

$$\int_0 s^{-e(b)} |d\widetilde{\phi}(s)| = \int_p (s - p)^{-e(b)} |d\phi(s)| < \infty,$$

by assumption. Hence, we get a unique solution $\widetilde{w}$ for (6.21) of the form $\widetilde{w}(t) = \int_0^\infty e^{-st} d\widetilde{\omega}(s)$, where $\widetilde{\omega}(+0) = \widetilde{\omega}(0) = 0$, or a unique solution w for (6.10) of the form (6.8), where the integral is absolutely convergent for $t > 0$.

Proof of Corollary 6.1. If $p = 0$, this is contained in Corollary 2.1(vi) of [2]. If $p > 0$, then a transformation as in the proof of Theorem 6.2 will reduce this case to $p = 0$.

8. A system of solutions corresponding to each root. Our next result is the following

Theorem 8.1. *Let* κ, p, r_b *and* $e(b)$ *be as in Theorem 6.1. Let* Q *be an integer such that* $0 \leq Q < e(\kappa)$, *and* ζ *a constant. If* $p = 0$, *assume that* $\zeta \geq Q$ *and that the coefficients of (6.1) satisfy the conditions of Theorem 6.1, and in addition.*

$$(8.1) \qquad \sum_{\lambda=j}^{n-1} \int_0 s^{j-e(\kappa)-\zeta} |r_\kappa^{\lambda-j}| |da_\lambda(s)| < \infty \quad \text{for } 0 \leq j \leq Q,$$

where $e(\kappa)$ *is the multiplicity of* r_κ. *Then there exist* $Q + 1$ *functions* $w_0, \cdots,$ w_Q *representable as absolutely convergent Laplace-Stieltjes transforms*

$$(8.2)\ w_j(t) = \int_0^\infty e^{-st} d\omega_j(s) \quad \text{for } t > 0, \quad \omega_j(+0) = \omega_j(0) = 0, \quad \int_0 s^{j-\zeta} |d\omega_j(s)| < \infty,$$

such that

$$(8.3) \qquad y(t) = e^{r_\kappa t} w(t) = e^{r_\kappa t} \left[(1 + w_0(t)) t^i / i! + \sum_{j=1}^{i} w_j(t) t^{i-j} / (i-j)! \right]$$

is a solution of the homogeneous equation (6.1) for $i = 0, \cdots, Q$.

If $p > 0$, *assume that* $\zeta \geq Q e(b)$ *and that the coefficients of (6.1) satisfy the conditions of Theorem 6.1, and in addition*

$$(8.1') \qquad \sum_{\lambda=j}^{n-1} \int_p (s-p)^{-e(b)+je(b)-\zeta} |r_\kappa^{\lambda-j}| |da_\lambda(s)| < \infty \quad \text{for } 0 \leq j \leq Q,$$

where $e(b)$ *is as in (6.3). Then there exist* $Q + 1$ *functions* $w_0, \cdots, w_Q$ *representable as Laplace-Stieltjes transforms*

$$(8.2') \qquad \begin{aligned} w_j(t) &= \int_p^\infty e^{-st} d\omega_j(s) \quad \text{for } t > 0, \quad \omega(+p) = \omega(p) = 0, \\ &\int_p (s-p)^{je(b)-\zeta} |d\omega_j(s)| < \infty, \end{aligned}$$

such that (8.3) is a solution of (6.1) for $i = 0, \cdots, Q$.

Proof of Theorem 8.1. If we substitute $y = e^{r_\kappa t} w$ into (6.1), we obtain the equation (6.7) where $c = 0$, and the $g_j(t)$ are given by (6.9). Therefore the conclusions of Theorem 8.1 will follow directly from the case $r_\kappa = 0$ of Theorem 8.1. Therefore write (6.1) as

$$(8.4) \qquad D^{e(\kappa)} P(-D) w + \sum_{j=0}^{n-1} (-1)^{e(\kappa)+j} g_j(t) D^j w(t) = 0,$$

where the g_j satisfy the conditions of Theorem 6.2 or equivalently those of Theorem 6.1 for $r_K = 0$, and condition (8.1) or (8.1') above. Let $P(t) = \sum_{i=0}^{n-e(\kappa)} a_i t^i$. If $p = 0$, then $r_b = r_K = 0$, $e(b) = e(\kappa)$ and our result is contained in [2, Corollary 2.4]. If $p > 0$, we wish to prove the existence of $Q + 1$ functions $w_0, \cdots, w_Q$ satisfying (8.2') such that

$$(8.5) \qquad w(t) = \cdot \left[(1 + w_0(t)) t^i / i! + \sum_{j=1}^{i} w_j(t) t^{i-j} / (i-j)! \right]$$

is a solution of (8.4) for $0 \le i \le Q$. The case where $Q = 0$ is contained in Theorem 6.2 and Corollary 6.1. For if $c = 1$, the solution (6.17) can be written in the form

$$w = 1 + w_0, \qquad w_0(t) = \int_p^\infty e^{-st} d\omega_0(s), \qquad \int_p (s-p)^{-\zeta} |d\omega_0(s)| < \infty.$$

Suppose that $0 < N \le Q \ (< e(\kappa))$, that $w_0, \cdots, w_{N-1}$ exist as specified, and that (8.5) is a solution of (8.4) for $i = 0, \cdots, N-1$. In order to simplify notation, write (8.4) as

$$(8.6) \qquad L[w] = \sum_{\sigma=0}^{n} b_\sigma(t) w^\sigma = 0$$

where $b_\sigma(t) = (-1)^{\sigma+e(\kappa)} g_\sigma(t)$ for $0 \le \kappa \le e(\kappa) - 1$ and $b_{\sigma+e(\kappa)}(t) = (-1)^\sigma [a_\sigma + g_{\sigma+e(\kappa)}(t)]$ for $0 \le \sigma \le n - e(\kappa)$, and $g_n(t) = 0$. Also let $v_0 = 1 + w_0$, $v_j = w_j$ for $j > 0$. In [2], Hartman showed that $w = w_N$ (together with w_i, $i = 0, \cdots, N-1$) in (8.5) satisfies (8.4) if and only if

$$L[w] = (-1)^{e(\kappa)+1} f,$$

where $(-1)^{e(\kappa)} f = \sum_{\sigma=1}^{n} b_\sigma \sum_{\tau=1}^{\min(N,\sigma)} C_{\sigma\tau} v_{N-\tau}^{(\sigma-\tau)}$, $C_{\sigma\tau} = \sigma! / \tau! (\sigma-\tau)!$. In other words,

$$(8.7) \qquad f = \sum_{\sigma=e(\kappa)}^{n} \sum_{\tau=1}^{N} c_{\sigma\tau}^1 w_{N-\tau}^{(\sigma-\tau)} + \sum_{\sigma=1}^{n-1} \sum_{\tau=1}^{\min(N,\sigma)} c_{\sigma\tau}^2 g_\sigma w_{N-\tau}^{(\sigma-\tau)} + c^3 g_N,$$

where the c_{jk}^i are constants. The function f is representable as an absolutely convergent Laplace-Stieltjes transform with determining function ϕ given by

$$(8.7') \qquad d\phi(t) = \sum_{\sigma=e(\kappa)}^{n} \sum_{\tau=1}^{N} c_{\sigma\tau}^1 t^{\sigma-\tau} d\omega_{N-\tau}(t)$$

$$+ \sum_{\sigma=1}^{n-1} \sum_{\tau=1}^{\min(N,\sigma)} c_{\sigma\tau}^2 d \left\{ \int_p^{t-p} \gamma_\sigma(t-s) s^{\sigma-\tau} d\omega_{N-\tau}(s) \right\} + c^3 \gamma_N(t).$$

Our desired result follows from Theorem 6.2 and Corollary 6.1 if

$$(8.8) \qquad \int_p (s-p)^{-e(b)+Ne(b)-\zeta} |d\phi(s)| < \infty.$$

But this follows because

$$\int_p (s-p)^{-e(b)+Ne(b)-\zeta} \left| \sum_{\sigma=e(\kappa)}^{n} \sum_{\tau=1}^{N} s^{\sigma-\tau} d\omega_{N-\tau}(s) \right|$$

$$\leq \sum_{\sigma} \sum_{\tau} c_{\sigma\tau} \int_p (s-p)^{(N-\tau)e(b)-\zeta} |d\omega_{N-\tau}(s)| < \infty$$

where $c_{\sigma\tau}$ is a constant, since $0 \geq (N-1)e(b) - \zeta \geq (N-\tau)e(b) - \zeta$. The second term on the right-hand side of (8.7′) satisfies (8.8) because the determining functions are convolutions of two functions which are 0 on $[0, p]$, and are therefore 0 on $[0, 2p]$. Finally,

$$\int_p (s-p)^{-e(b)+Ne(b)-\zeta} |d\gamma_N(s)| < \infty \quad \text{by assumption.}$$

9. Another form for a system of solutions corresponding to each root.

In this section, we prove

Theorem 9.1. *In the equation (6.1), let the coefficients $a_i(t)$ satisfy the conditions of Theorem 6.1. Then for $0 \leq i < e(\kappa)$, (6.1) has a solution of the form*

$$(9.1). \quad y(t) = e^{\tau_\kappa t} t^i w(t) = e^{\tau_\kappa t} t^i \left(1 - \int_p^\infty e^{-st} d\omega(s) \right), \qquad \omega(+p) = \omega(p) = 0,$$

where the integral is absolutely convergent for $t > 0$.

Proof of Theorem 9.1. Arguing as in the proof of Theorem 8.1, it is clearly sufficient to prove the theorem for the case where $r_\kappa = 0$. Write (6.1) as the homogeneous equation (8.4), where the g_j satisfy the conditions of Theorem 6.2.

The case where $i = 0$ is just Theorem 6.2. Let $0 < i < e(\kappa)$ and make the change of variables $w \rightarrow t^i w$ in (8.4). The resulting differential equation is of the form

$$(9.2) \quad t^{-i}(D-p)^{e(b)} P(-D)(t^i w) + \sum_{j=0}^{n-1} (-1)^{e(b)+j} g_{0j}(t) D^j w = (-1)^{e(b)+1} f_0(t),$$

where the coefficient functions $g_{0j}(t)$ are of the form

(9.3) $$g_{0j}(t) = \sum_m c_{jm} g_m(t) t^{-(n-j)}, \qquad j \le m \le \min(i+j, n-1),$$

the c_{jm} are constants, and $f_0(t) = 0$. Notice that

(9.4) $$g_{0j}(t) = \int_p^\infty e^{-st} d\gamma_{0j}(s), \qquad f_0(t) = \int_p^\infty e^{-st} d\phi_0(s),$$

where $\phi_0(s) = 0$ and

(9.3') $$\gamma_{0j}(s) = \sum_m c_{jm} \int_p^s (s-u)^{n-j} d\gamma_m(u)/(n-j)!.$$

Consequently, Theorem 9.1 for the case where $r_\kappa = 0$ is contained in the following result:

Theorem 9.2. *Let $e(b)$, $e(\kappa)$, $0 < i < e(\kappa)$, and $P(t)$ be as in Theorem 6.2. In (9.2), let $g_{0j}(t)$, $f_0(t)$ be representable as absolutely convergent Laplace-Stieltjes transforms for $t > 0$, and let $g_{0j}(t)$, $f_0(t)$ satisfy the conditions of Theorem 6.2. Then, for every constant c, (9.2) has a solution representable as an absolutely convergent Laplace-Stieltjes transform*

(9.5) $$w(t) = c - \int_p^\infty e^{-st} d\omega(s) \quad \text{for } t > 0, \quad \omega(+p) = \omega(p) = 0.$$

Proof of Theorem 9.2. For $p = 0$, $e(b) = e(\kappa)$ and the theorem is contained in Theorem 4.1 of [2], and the result follows from the fact that from (9.3'),

$$\int_0 s^{-e(\kappa)+j} |d\gamma_{0j}(s)| < \infty, \qquad \int_0 s^{-e(\kappa)} |d\phi_0(s)| < \infty,$$

and $\gamma_{0j}(+0) = \gamma_{0j}(0) = 0 = \phi_0(+0) = \phi_0(0) = 0$.

If $p > 0$ and if we write $w(t)$ as $e^{-pt}\tilde{w}(t)$ and $f_0(t)$ as $e^{-pt}\tilde{f}_0(t)$ as in (6.19) and (6.20), then (9.2) becomes

$$t^{-i}(D)^{e(b)}\tilde{P}(-D)(t^i\tilde{w}) + \sum_{j=0}^{n-1} (-1)^{e(b)+j}\tilde{g}_{0j}(t)D^j\tilde{w} = (-1)^{e(b)+1}\tilde{f}_0(t)$$

where, as in the display after (6.21),

$$\tilde{g}_{0j}(t) = \int_0^\infty e^{-st} d\tilde{\gamma}_{0j}(s), \qquad \tilde{\gamma}_{0j}(s) = 0 \text{ on } [0, p].$$

But now we are reduced to the case where $p = 0$, because $\tilde{P}(t) \ne 0$ for $t \ge 0$,

$$\int_0 s^{-e(b)} |d\tilde{\phi}_0(s)| < \infty \quad \text{and} \quad \int_0 s^{-e(b)+j} |d\tilde{\gamma}_{0j}(s)| < \infty$$

because $\tilde{\gamma}_{0j}(s) = 0$ on $[0, p]$.

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