TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 195, 1974

THE CLOSURE OF THE SPACE OF HOMEOMORPHISMS ON A MANIFOLD

BY

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ABSTRACT. The space, $\overline{H}(M)$, of all mappings of the compact manifold M onto itself which can be approximated arbitrarily closely by homeomorphisms is studied. It is shown that $\overline{H}(M)$ is homogeneous and weakly locally contractible. If M is a compact 2-manifold without boundary, then $\overline{H}(M)$ is shown to be locally contractible.

1. Let M be a compact manifold and $H(M)$ denote the space of all homeomorphisms of M onto itself. We shall study the space, $\overline{H}(M)$, of all continuous functions of M onto itself which can be approximated arbitrarily closely by elements of $H(M)$. All function spaces on compact spaces will be assumed to have the supremum metric, ρ; i.e., if X and Y are spaces with d the metric on Y and f and g are functions from X into Y, then $\rho(f, g) = \sup_{x \in X}\{d(f(x), g(x))\}$. Since M is compact, the topology thus generated agrees with the compact-open topology.

A mapping of an n-manifold, M^n, onto itself is said to be cellular if for each $y \in M^n$, $f^{-1}(y)$ can be expressed as the intersection of a nested sequence of n-cells. Armentrout ($n \leq 3$) [4] and Siebenmann ($n \geq 5$) [20] have recently shown that $\overline{H}(M^n)$, $n \neq 4$, is precisely the space of all cellular mappings of M^n onto itself. Hence most of the results of this paper could be stated in terms of spaces of cellular mappings. Cellular mappings have been studied extensively (cf., Lacher [15], [16]).

Let $H_\partial(M)$ denote the space of all homeomorphisms of M onto itself which equal the identity when restricted to the boundary of M and, following our previous notation, let $\overline{H}_\partial(M)$ denote the space of all continuous functions of M onto itself which can be approximated arbitrarily closely by elements of $H_\partial(M)$. We shall state some of the major results concerning $H(M)$ and $H_\partial(M)$ and then indicate which of the analogous theorems can be proven for $\overline{H}(M)$ and $\overline{H}_\partial(M)$:

Received by the editors December 20, 1972 and, in revised form, July 9, 1973.

AMS (MOS) subject classifications (1970). Primary 54H15, 57E05; Secondary 57A60, 57A20.

Key words and phrases. Spaces of homeomorphisms, cellular mappings, closure of the space of homeomorphisms, compact manifolds, infinite dimensional manifolds, homogeneous spaces.

(1) Research partially supported by NSF Grant GP 33872.

(i) It is well known that for any compact manifold M each of $H(M)$ and $H_\partial(M)$ is a separable metric space, a topological group under composition of functions and topologically complete.

(ii) Let B^n be the standard n-ball. The Alexander isotopy [1], first used in 1923, is very useful in dealing with $H_\partial(B^n)$ and combined with the fact that $H_\partial(B^n)$ is a topological group provides a trivial proof that $H_\partial(B^n)$ is locally contractible. Mason [19] showed that $H_\partial(B^2)$ is an absolute retract and Anderson [3] proved that $H_\partial(B^1)$ is homeomorphic to l_2 (separable Hilbert space).

(iii) Recently Černavskiĭ [6] and Edwards-Kirby [7] proved that for any compact manifold each of $H(M)$ and $H_\partial(M)$ is locally contractible.

(iv) Of current interest is the problem of whether $H(M)$ is an l_2-manifold (i.e., locally homeomorphic to l_2). Geoghegan [9] has shown that $H(M) \times l_2 \approx H(M)$ and $H_\partial(M) \times l_2 \approx H_\partial(M)$.

We shall discuss in this paper the state of the corresponding statements for $\overline{H}(M)$ and $\overline{H}_\partial(M)$:

(i) $\overline{H}(M)$ and $\overline{H}_\partial(M)$ are obviously separable metric spaces. In addition they are not merely topologically complete, but are complete under the supremum metric. Neither space is a topological group, under composition of functions, since the inverse of a cellular map need not be even well defined. However, we do prove (§2) that $\overline{H}(M)$ and $\overline{H}_\partial(M)$ are homogeneous.

(ii) Making use of an Alexander-type homotopy and the fact that $\overline{H}_\partial(B^n)$ is homogeneous, we give a simple proof (§3) that this space is locally contractible. Elsewhere the author [13] has shown that $\overline{H}_\partial(B^2)$ is an AR and Geoghegan [9] has proven that $\overline{H}_\partial(B^1)$ is homeomorphic to l_2.

(iii) It is unknown whether $\overline{H}(M)$ is locally contractible for an arbitrary compact manifold. In §4, it is shown that $\overline{H}(M)$ and $\overline{H}_\partial(M)$ are weakly locally contractible. Then by modifying slightly the techniques of Edwards-Kirby we show in §5 that if M^2 is a compact 2-manifold then $\overline{H}_\partial(M^2)$ is locally contractible.

(iv) Geoghegan and Henderson [10] proved that $\overline{H}(M) \times l_2 \approx \overline{H}(M)$. In §4, using a theorem of Anderson [2], we give an easy proof of the fact that if $\overline{H}(M)$ is an l_2-manifold for a particular compact manifold M, then $H(M)$ is an l_2-manifold.

If $f \in \overline{H}(M)$ and $\epsilon > 0$, let $N_\epsilon(f) = \{g \in \overline{H}(M) | \rho(g, f) < \epsilon\}$. When we wish to speak of a neighborhood in $H(M)$ we write $N_\epsilon(f) \cap H(M)$ to denote $\{g \in H(M) | \rho(g, f) < \epsilon\}$. If X and Y are spaces with $X \subset Y$, the complement of X in Y will be denoted by $\tilde{X}$ when there is no possibility of confusion. The boundary of X is written ∂X, the closure of X is written $\overline{X}$, and 1_X denotes the identity map on X. We use the symbol Int X to denote the interior of the set X and I to denote the closed unit interval.

The work of this paper is an extension of a portion of the author's doctoral dissertation written under the direction of Louis F. McAuley at State University of New York at Binghamton; the proof of Lemma 5.1 was contained in that dissertation.

2. In this section we shall use a result of Edwards-Kirby [7] and an isotopy control devise motivated by a technique of Mason [18] to prove that if M is a compact manifold, then $\overline{H}(M)$ and $\overline{H}_\partial(M)$ are homogeneous. (A space, X, is homogeneous if given $x, y \in X$, there exists a homeomorphism $\phi: X \to X$ such that $\phi(x) = y$.)

Lemma 2.1. *Suppose $g_0 \in \overline{H}(M)$, $\{\epsilon_i\}_{i=1}^\infty$ is a sequence of positive numbers such that $\epsilon_{i+1} < \epsilon_i/2$, $\epsilon_i < 1/2^i$ for each i, and $\{\phi_i: \overline{H}(M) \to \overline{H}(M)\}_{i=0}^\infty$ is a sequence of homeomorphisms satisfying*:

(a) $\rho(\phi_i \circ \cdots \circ \phi_0(g_0), 1_M) < \epsilon_{i+2}$, *for* $i \geq 0$.

(b) *If* $\rho(f, \phi_{i-1} \circ \cdots \circ \phi_0(g_0)) \geq \epsilon_i$, *then* $\phi_i(f) = f$, *for* $i \geq 1$.

(c) *If* $\rho(f, \phi_{i-1} \circ \cdots \circ \phi_0(g_0)) \leq \epsilon_{i+1}$, *then* $\rho(\phi_i(f), \phi_i \circ \cdots \circ \phi_0(g_0)) = \rho(f, \phi_{i-1} \circ \cdots \circ \phi_0(g_0))$, *for* $i \geq 1$.

(d) *If* $\rho(f, \phi_{i-1} \circ \cdots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$, *then* $\rho(\phi_i(f), \phi_i \circ \cdots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$, *for* $i \geq 0$.

(e) *If* $f \in \overline{H}(M)$, $\rho(f, g_0) = \rho(\phi_0(f), \phi_0(g_0))$.

Then $\phi = \lim_{i\to\infty} \phi_i \circ \cdots \circ \phi_0$ is a homeomorphism of $\overline{H}(M)$ onto itself taking g_0 to 1_M.

Proof. Property (a) implies that $\phi(g_0) = 1_M$. To see that ϕ is onto let $f \in \overline{H}(M)$ and suppose $f \neq 1_M$. Choose a large enough integer, i, so that $\rho(f, 1_M) \geq \epsilon_i$. Since $\phi_i \circ \cdots \circ \phi_0$ is a homeomorphism, there is an element $\tilde{f}$ of $\overline{H}(M)$ so that $\phi_i \circ \cdots \circ \phi_0(\tilde{f}) = f$.

However, $\rho(f, \phi_i \circ \cdots \circ \phi_0(g_0)) \geq \rho(f, 1_M) - \rho(\phi_i \circ \cdots \circ \phi_0(g_0), 1_M) \geq \epsilon_i - \epsilon_{i+2} > \epsilon_{i+1}$ and hence, by property (b), $\phi(\tilde{f}) = \phi_i \circ \cdots \circ \phi_0(\tilde{f}) = f$.

Similarly, since $\phi|(\phi_i \circ \cdots \circ \phi_0)^{-1}(\{f|\rho(f, 1_M) > \epsilon_i\}) = \phi_i \circ \cdots \circ \phi_0|(\phi_i \circ \cdots \circ \phi_0)^{-1}(\{f|\rho(f, 1_M) > \epsilon_i\})$, ϕ is 1-1 and continuous on $\phi^{-1}(\overline{H}(M) - \{1_M\})$. To show that ϕ is indeed 1-1, we need to show that if $f \neq g_0 \in \overline{H}(M)$, then $\phi(f) \neq 1_M$. Let i be the smallest integer so that $\epsilon_{i+1} \leq \rho(f, g_0)$. By properties (c) and (e), $\rho(\phi_i \circ \cdots \circ \phi_0(f), \phi_i \circ \cdots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$. And hence by property (d), $\rho(\phi_i \circ \cdots \circ \phi_0(f), \phi_i \circ \cdots \circ \phi_0(g_0)) \geq \epsilon_{i+1}$. But this implies that

$$\rho(\phi(f), 1_M) \geq \rho(\phi(f), \phi_i \circ \cdots \circ \phi_0(g_0)) - \rho(\phi_i \circ \cdots \circ \phi_0(g_0), 1_M)$$
$$\geq \epsilon_{i+1} - \epsilon_{i+2} > \epsilon_{i+1}/2,$$

which shows that $\phi(f) \neq 1_M$.

To show that ϕ is continuous at g_0, note that if $\rho(f, g_0) < \epsilon_i$, then

$$\rho(\phi(f), 1_M) \le \rho(\phi(f), \phi_{i-1} \circ \cdots \circ \phi_0(f)) + \rho(\phi_{i-1} \circ \cdots \circ \phi_0(f), \phi_{i-1} \circ \cdots \circ \phi_0(g_0)) + \rho(\phi_{i-1} \circ \cdots \circ \phi_0(g_0), \phi(g_0))$$

$$< \left(\sum_{j=i}^{\infty} 2\epsilon_j\right) + \epsilon_i + \epsilon_{i+1} < 2\left(\sum_{j=i}^{\infty} \frac{1}{2^j}\right) + \frac{1}{2^i} + \frac{1}{2^{i+1}} < \frac{1}{2^{i-3}}.$$

Finally, ϕ^{-1} is obviously continuous on $\overline{H}(M) - \{1_M\}$ and we have shown that if $\rho(f, g_0) \ge \epsilon_{i+1}$, then $\rho(\phi(f), 1_M) > \epsilon_{i+1}/2$. Hence ϕ^{-1} is continuous.

As a corollary to their main theorem Edwards-Kirby [7, p. 80] obtain the following result: Let $\{B_i | 1 \le i \le p\}$ be an open cover of M. Then there exists a neighborhood, Q, of 1_M in $H(M)$ and a map $\phi: Q \times [0, p] \to H(M)$ such that: For each $h \in Q$ and each j, $j = 1, \cdots, p$, if $j - 1 \le t \le j$, then $\phi(h, t)|\widetilde{B}_j = \phi(h, j)|\widetilde{B}_j$; $\phi(h, 0) = h$ for all $h \in H(M)$; $\phi(h, p) = 1_M$ for all $h \in H(M)$; and $\phi(1_M, t) = 1_M$ for each $t \in [0, p]$. We will make use of the following immediate corollary to the above statement.

Lemma 2.2 (Edwards-Kirby). *Given $\eta > 0$, there is a $\delta > 0$ such that if $h \in H(M)$ and $\rho(h, 1_M) < \delta$, then there is a map $H: [0, p] \to H(M)$ such that $h_0 = h$, $h_p = 1_M$, $\rho(h_t, 1_M) < \eta$ for all $t \in [0, p]$ and for each j, $j = 1, 2, \cdots, p$, if $j - 1 \le t \le j$, then $h_t|\widetilde{B}_j = h_j|\widetilde{B}_j$ (where $H(t)$ is denoted h_t).*

The map H can be defined so that in addition if $h|\partial M = 1_{\partial M}$, then $h_t|\partial M = 1_{\partial M}$ [7, p. 64].

Lemma 2.3. *Let M be a compact manifold. For every $\epsilon > 0$ there exists a cover $\{B_1, \cdots, B_p\}$ of M and an $\epsilon' > 0$ such that if $f \in \overline{H}(M)$ and $(f, 1_M) > \epsilon$, then for each j, $j = 1, 2, \cdots, p$, $\rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) \ge \epsilon'$.*

Proof. It suffices to prove the following statement: Let M be a compact manifold. Given $\epsilon > 0$ there is an $\epsilon' > 0$ such that if $h \in H(M)$ with $\rho(h, 1_M) > \epsilon$, then there exist $x, y \in M$ so that $d(x, y) \ge \epsilon'$, $d(h(x), x) \ge \epsilon'$ and $d(h(y), y) \ge \epsilon'$. (To complete the proof of the lemma, first notice that it suffices to deal only with elements of $H(M)$, and then choose an open cover $\{B_1, \cdots, B_p\}$ of M of mesh less than ϵ'.)

The above statement is obviously true if M is a compact 0-manifold. Assume inductively that it has been demonstrated for all compact manifolds of dimension $\le n - 1$. Let M be a compact n-manifold and suppose $\epsilon > 0$ is given. Pick $\eta > 0$ such that $\eta < \epsilon/4$ and such that if S is a subset of M of diameter $< 2\eta$, then S is contained in a ball of diameter less than $\epsilon/8$.

Using the inductive hypothesis and the fact that M can be covered by a finite number of coordinate patches we next choose $\epsilon' > 0$ such that

(a) $\epsilon' < \eta/4$;

(b) For each $x \in M$, there is a ball, B_x, containing x such that if $w \in \partial B_x \cap \text{Int } M$, then $\epsilon' < d(x, w)$ and if $w \in \partial B_x$, $d(x, w) < \eta/2$;

(c) If $g \in H(\partial M)$ with $\rho(g, 1_{\partial M}) > \eta/2$, then there exist elements $x, y \in \partial M$ so that $d(x, y) \geq \epsilon'$, $d(h(x), x) \geq \epsilon'$, and $d(h(y), y) \geq \epsilon'$.

Now, suppose $h \in H(M)$ with $\rho(h, 1_M) > \epsilon$. Pick $x \in M$ such that $d(h(x), x) > \epsilon$. Choose a ball, B_x, so that $x \in B_x$, if $w \in \partial B_x \cap \text{Int } M$ then $d(x, w) > \epsilon'$ and if $w \in \partial B_x$, then $d(x, w) < \eta/2$.

Now either (i) there is a $y \in \partial B_x \cap \partial M$ such that $d(h(y), x) \geq \eta$ or (ii) there is a $y \in \partial B_x \cap \text{Int } M$ such that $d(h(y), x) \geq \eta$ or (iii) $h(\partial B_x) \subset N_\eta(x)$.

In case (i), $d(h(y), y) \geq d(h(y), x) - d(x, y) \geq n - n/2 > n/2$. Hence by property (c) of the definition of ϵ', the conclusion of the inductive statement is satisfied.

In case (ii), $d(x, y) > \epsilon'$ and $d(h(y), y) > \eta/2 > \epsilon'$. Hence x and y are the desired points.

In case (iii), let B be a ball of diameter less than $\epsilon/8$ bounded by $h(\partial B_x)$. Now, $h(x) \notin B$, for otherwise $d(x, h(x)) < d(x, B) + \text{diam } B < \eta + \epsilon/8 < \epsilon/4 + \epsilon/8 < \epsilon$. Therefore, $h(\widetilde{B}_x) = B$ and we can choose $y \in \widetilde{B}_x$ such that $d(y, x) \geq \epsilon/2$ (this is possible since diam $M > \epsilon$). Then $d(y, h(y)) \geq d(y, x) - d(x, h(y)) \geq \epsilon/2 - \eta - \epsilon/8 > \epsilon/2 - \epsilon/4 - \epsilon/8 = \epsilon/8 > \epsilon'$.

Lemma 2.4. *Let M be a compact manifold and let $\epsilon > 0$ be given. Then there is a $\delta = \delta(\epsilon) > 0$ such that if $h \in H(M)$ and $\rho(h, 1_M) < \delta$, then there is a homeomorphism $\psi: \overline{H}(M) \to \overline{H}(M)$ such that*

(a) *if $\rho(f, 1_M) \geq \epsilon$, then $\psi(f) = f$;*

(b) *if $\rho(f, 1_M) < \delta$, then $\psi(f) = fh^{-1}$.*

Proof. By Lemma 2.3 we can choose a cover $\{B_1, \cdots, B_p\}$ and a number $\eta > 0$ such that if $\rho(f, 1_M) \geq \epsilon$, then $\rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) \geq 4\eta$, for $j = 1, \cdots, p$.

Then by Lemma 2.2, there is a δ, $0 < \delta \leq \eta$ such that if $\rho(h, 1_M) < \delta$ then there exists a map $H: [0, p] \to H(M)$ such that $h_0 = h$, $h_p = 1_M$, $\rho(h_t, 1_M) < \eta$ for every $t \in [0, p]$ and for each j, $j = 1, \cdots, p$, if $j - 1 \leq t \leq j$, then $h_t|\widetilde{B}_j = h_j|\widetilde{B}_j$. Next for each j, we define a map λ_j: $\overline{H}(M) \to [0, 1]$ by

$$\lambda_j(f) = \begin{cases} 0, & \text{if } \rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) \leq 3\eta, \\ \dfrac{\rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j})}{\eta} - 3, & \text{if } 3\eta \leq \rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) \leq 4\eta, \\ 1, & \text{if } \rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) \geq 4\eta. \end{cases}$$

For each j, $1 \leq j \leq p$, define ψ_j: $\overline{H}(M) \to \overline{H}(M)$ by

$$\psi_j(f) = fh_{j-1}^{-1}h_{j-\lambda_j(f)}.$$

Define $\psi: \overline{H}(M) \to \overline{H}(M)$ by $\psi = \psi_p \circ \cdots \circ \psi_1$.

To prove that ψ is a homeomorphism it suffices to show that for each j, ψ_j is a homeomorphism. The fact that if $j-1 \leq t \leq j$, then $h_t|\widetilde{B}_j = h_j|\widetilde{B}_j$ implies that for each positive number, s, $\psi_j|\{f \in \overline{H}(M) | \rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) = s\}$ is a homeomorphism of $\{f \in \overline{H}(M) | \rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) = s\}$ onto itself. Therefore ψ_j is 1-1 and onto. This fact and the continuity of λ_j proves that ψ_j and its inverse are continuous.

To see that condition (a) is met, note that if $\rho(f, 1_M) \geq \epsilon$, then for each j, $\rho(f|\widetilde{B}_j, 1_{\widetilde{B}_j}) \geq 4\eta$ and hence $\psi_j(f) = f$.

Now suppose $\rho(f, 1_M) < \delta$. Then $\rho(f, 1_M) < \eta$ and hence $\psi_1(f) = fh^{-1}h_1$. Assume inductively that $\psi_{j-1} \circ \cdots \circ \psi_1(f) = fh^{-1}h_{j-1}$; then since

$$\rho(fh^{-1}h_{j-1}|\widetilde{B}_j, 1_M|\widetilde{B}_j) \leq \rho(fh^{-1}h_{j-1}, 1_M) \leq \rho(f, 1_M) + \rho(h^{-1}, 1_M) + \rho(h_{j-1}, 1_M) \leq 3\eta,$$

$$\psi_j(fh^{-1}h_{j-1}) = fh^{-1}h_{j-1}h_{j-1}^{-1}h_j = fh^{-1}h_j.$$

Therefore

$$\psi(f) = \psi_p(\psi_{p-1}(\cdots(\psi_1(f))\cdots)) = fh^{-1}h_p = fh^{-1}.$$

It is interesting to note that the statement and proof of Lemma 2.4 remain valid if $\overline{H}(M)$ is replaced throughout by $H(M)$.

Lemma 2.5. *Given $a > 0$ there exists $b' = b'(a) > 0$ such that if $b < b'$, $g \in \overline{H}(M)$ such that $\rho(g, 1_M) < b$, and $c > 0$ are given then there exists a homeomorphism $\psi: \overline{H}(M) \to \overline{H}(M)$ such that*

(i) $\rho(\psi(g), 1_M) < c$;

(ii) *if* $\rho(f, g) \geq a$, *then* $\psi(f) = f$;

(iii) *if* $\rho(f, g) \leq b$, *then* $\rho(\psi(f), \psi(g)) = \rho(f, g)$;

(iv) *if* $\rho(f, g) \geq b$, *then* $\rho(\psi(f), \psi(g)) \geq b$.

Proof. In Lemma 2.4, let $\epsilon = a/2$. Then let $b' = \min(\delta(\epsilon)/2, a/2)$. Suppose $b < b'$ and $g \in \overline{H}(M)$ are given with $\rho(g, 1_M) < b$. Choose $h \in H(M)$ such that $\rho(h, g) < \min(b, c)$ and $\rho(h, 1_M) < b$. Then by Lemma 2.4, there exists a homeomorphism $\psi: \overline{H}(M) \to \overline{H}(M)$ such that if $\rho(f, 1_M) \geq a/2$, then $\psi(f) = f$ and if $\rho(f, 1_M) < 2b$, then $\psi(f) = fh^{-1}$. It is trivial to check that ψ satisfies conditions (i)–(iv).

Theorem 2.6. *If M is a compact manifold, $\overline{H}(M)$ is homogeneous.*

Proof. Let g_0 be an arbitrary element of $\overline{H}(M)$. It is sufficient to show that there exist sequences $\{\epsilon_i\}_{i=1}^{\infty}$ and $\{\phi_i: \overline{H}(M) \to \overline{H}(M)\}_{i=0}^{\infty}$ satisfying the hypothesis of Lemma 2.1 and hence that there exists a homeomorphism $\phi: \overline{H}(M) \to \overline{H}(M)$ taking g_0 to 1_M.

Choose a sequence of positive numbers $\{\epsilon_i\}_{i=1}^\infty$ such that for each i, $\epsilon_{i+1} < \epsilon_i/2$, $\epsilon_i < 1/2^i$ and $\epsilon_{i+1} \le b'(\epsilon_i)$, where $b'(\epsilon_i)$ is the number promised in Lemma 2.5 for $a = \epsilon_i$. Let $h \in H(M)$ be chosen so that $\rho(h, g_0) < \epsilon_2$. Define the homeomorphism $\phi_0: \overline{H}(M) \to \overline{H}(M)$ by $\phi_0(f) = fh^{-1}$. Note that $\rho(\phi_0(g_0), 1_M) = \rho(g_0 h^{-1}, 1_M) = \rho(g_0, h) < \epsilon_2$ and that if $f \in \overline{H}(M)$, $\rho(\phi_0(f), \phi_0(g_0)) = \rho(f, g_0)$.

Now assume inductively that homeomorphisms $\phi_0, \phi_1, \cdots, \phi_{j-1}$ have been defined satisfying conditions (a)–(d) of Lemma 2.1.

In Lemma 2.5, let $a = \epsilon_j$, $b = \epsilon_{j+1}$, $c = \epsilon_{j+2}$ and $g = \phi_{j-1} \circ \cdots \circ \phi_0(g_0)$. By the inductive hypothesis, $\rho(g, 1_M) = \rho(\phi_{j-1} \circ \cdots \circ \phi_0(g_0), 1_M) < \epsilon_{j+1}$. Therefore, since $\epsilon_{j+1} < b'(\epsilon_j)$, by Lemma 2.5, there exists a homeomorphism $\phi_j: \overline{H}(M) \to \overline{H}(M)$ such that:

(i) $\rho(\phi_j \circ \cdots \circ \phi_0(g_0), 1_M) < \epsilon_{j+2}$;

(ii) if $\rho(f, \phi_{j-1} \circ \cdots \circ \phi_0(g_0)) \ge \epsilon_j$, then $\phi_j(f) = f$;

(iii) if $\rho(f, \phi_{j-1} \circ \cdots \circ \phi_0(g_0)) \le \epsilon_{j+1}$, then;

$$\rho(\phi_j(f), \phi_j \circ \cdots \circ \phi_0(g_0)) = \rho(f, \phi_{j-1} \circ \cdots \circ \phi_0(g_0));$$

(iv) if $\rho(f, \phi_{j-1} \circ \cdots \circ \phi_0(g_0)) \ge \epsilon_{j+1}$, then $\rho(\phi_j(f), \phi_j \circ \cdots \circ \phi_0(g_0)) \ge \epsilon_{j+1}$.

But these are precisely conditions (a)–(d) of Lemma 2.1 that the homeomorphism ϕ_j was to satisfy (condition (e) refers only to ϕ_0). The proof of Theorem 2.6 is completed.

In order to simplify notation we used the symbol $\overline{H}(M)$ throughout this section. The identical proofs also show that $\overline{H}_\partial(M)$ is homogeneous (recall the comment following the statement of Lemma 2.2).

Theorem 2.7. *Let M be a compact manifold. Then $\overline{H}_\partial(M)$ is homogeneous.*

3. In this section we consider $\overline{H}_\partial(B^n)$, where B^n is the Euclidean n-ball.

Theorem 3.1. *$\overline{H}_\partial(B^n)$ is locally contractible.*

Proof. Since $\overline{H}_\partial(B^n)$ is homogeneous, it suffices to show that $\overline{H}_\partial(B^n)$ is locally contractible at 1_{B^n}. We show, using an Alexander-type homotopy, that $N_\epsilon(1_{B^n})$ is contractible within itself to 1_{B^n}.

For any $f \in \overline{H}_\partial(B^n)$, define $\hat{f}: R^n \to R^n$ by

$$\hat{f}(x) = f(x), \quad x \in B^n,$$
$$= x, \quad x \notin B^n.$$

Next define $A: \overline{H}_\partial(B^n) \times I \to \overline{H}_\partial(B^n)$ by

$$A(f, t)(x) = \frac{1-t}{1+t}\hat{f}\left(\frac{1+t}{1-t}x\right), \qquad 0 \le t < 1,$$
$$= x, \qquad\qquad t = 1.$$

We note that A is continuous, $A(f, 0) = f$, $A(f, 1) = 1_{B^n}$ and $A(f, t) \in \overline{H}_\partial(B^n)$ for all $f \in \overline{H}_\partial(B^n)$ and $t \in I$. Furthermore, since $\rho(f, 1_{B^n}) < \epsilon$ implies that $\rho(A(f, t), 1_{B^n}) < \epsilon$, A contracts $N_\epsilon(1_{B^n})$ within itself to 1_{B^n} for every $\epsilon > 0$.

Actually it is possible to prove that $\overline{H}_\partial(B^n)$ is locally contractible without knowing that $\overline{H}_\partial(B^n)$ is homogeneous. See [12].

For the special case, $n = 4$, it is not known whether $\overline{H}_\partial(B^4)$ is equal to $Ce_\partial(B^4) = \{f\colon B^4 \to B^4 | f|\partial B^4 = 1_{\partial B^4}$ and f is cellular$\}$. However, the map A does show that $Ce_\partial(B^4)$ is locally contractible at 1_{B^4}.

It was mentioned in the introduction that $\overline{H}_\partial(B^2)$ is an AR [13]. The following theorem is also contained in [13]:

Theorem 3.2. *Let α be an open cover of $\overline{H}_\partial(B^2)$. Then there exists a locally finite polyhedron, P, and maps $b\colon \overline{H}_\partial(B^2) \to P$, $g\colon P \to \overline{H}_\partial(B^2)$ and θ: $\overline{H}_\partial(B^2) \times I \to \overline{H}_\partial(B^2)$ such that*

(a) *for each $f \in H_\partial(B^2)$ there is an element, U_f, of α such that $\theta(f, t) \in U_f$, for each $t \in I$;*

(b) *$\theta(f, 1) = f$, for each $f \in \overline{H}_\partial(B^2)$;*

(c) *$\theta(f, 0) = gb(f)$, for each $f \in \overline{H}_\partial(B^2)$;*

(d) *$\theta(f, t) \in H_\partial(B^2)$ for each $f \in \overline{H}_\partial(B^2)$ and $t \in [0, 1)$.*

This theorem will be used in §5. Theorem 3.2 implies that the inclusion map $i\colon H_\partial(B^2) \to \overline{H}_\partial(B^2)$ is a homotopy equivalence. Siebermann [20] has asked whether $i\colon H(M) \to \overline{H}(M)$ is a homotopy equivalence, for an arbitrary compact manifold M.

4. In this section we obtain some general topological results concerning the closure of a uniformly locally contractible space (compare with [8]). These results are then used in order to give partial solutions to the following unsolved problems:

(i) Let M be a compact manifold. Given $\delta > 0$ does there exist a continuous function $\phi_\delta\colon \overline{H}(M) \to H(M)$ with the property that for each $g \in \overline{H}(M)$, $\rho(g, \phi_\delta(g)) < \delta$?

(ii) Let M be a compact manifold. Is $\overline{H}(M)$ locally contractible?

Proposition 4.1. *Let Y be a metric space and X be a uniformly locally contractible subset of Y. Let $\delta > 0$ be given and let $f\colon P \to \overline{X}$ be a map of an arbitrary locally finite polyhedron, P, into $\overline{X}$. Then there exists a map $\phi\colon P \times I \to \overline{X}$ so that for each $p \in P$:*

(a) $\phi(p, 0) = f(p)$;

(b) *if* $t \neq 0$, $\phi(p, t) \in X$;

(c) *if* $t \in I$, $d(f(p), \phi(p, t)) < \delta$.

Proof. Let $\delta_1, \delta_2, \cdots$ be a decreasing sequence of positive numbers such that $\delta_1 \leq \delta/3$, $\delta_n \leq 1/3n$, and if $A_n \subset X$ with $\operatorname{diam}(A_n) < 3\delta_{n+1}$, then $i: A_n \to X$ is null-homotopic in a subset of X of diameter less than δ_n.

Suppose $P \times (0, 1]$ has a fixed locally finite triangulation. If τ is a simplex of $P \times (0, 1]$, define the two positive integers, m_τ and n_τ as follows:

$$m_\tau = \max\{\dim \sigma | \tau < \sigma\},$$

$$n_\tau = \min\{n | n \text{ is an integer and if } \tau < \sigma, \text{ then } \sigma \subset P \times [1/n, 1]\}.$$

Note that if $\tau' < \tau$, $m_{\tau'} \geq m_\tau$ and $n_{\tau'} \geq n_\tau$.

Consider $P \times (0, 1]$ to have a locally finite triangulation such that if τ is a simplex of $P \times (0, 1]$, then $\operatorname{diam}(f(\pi_1(\tau))) < \delta_{m_\tau + n_\tau}$ (where τ_1 is projection on the first coordinate).

We will define a map $\psi: P \times (0, 1] \to X$ by induction on the skeleta of $P \times (0, 1]$. Define $\psi_0: (P \times (0, 1])^0 \to X$ as follows. If σ is a 0-simplex of $P \times (0, 1]$, let $\psi_0(\sigma)$ be an element of X such that $\rho(f(\pi_1(\sigma)), \psi_0(\sigma)) < \delta_{m_\sigma + n_\sigma}$.

Assume inductively that there exist maps $\psi_1, \cdots, \psi_{k-1}$ with the following properties for $j = 1, \cdots, k-1$:

(i) ψ_j maps $(P \times (0, 1])^j$ into X;

(ii) ψ_j extends ψ_{j-1};

(iii) if τ is a j-simplex of $P \times (0, 1]$, then $\operatorname{diam} \psi_j(\tau) < \delta_{m_\tau + n_\tau - j}$.

We define $\psi_k: (P \times (0, 1])^k \to X$ as follows: If τ is a j-simplex of $(P \times (0, 1])^k$, $j < k$, let $\psi_k|\tau = \psi_{k-1}|\tau$. If τ is a k-simplex of $(P \times (0, 1])^k$, we note that $\operatorname{diam}(\psi_{k-1}(\operatorname{bdry} \tau)) < 3\delta_{m_\tau + n_\tau - (k-1)}$. If $k \neq 1$, this is true by the inductive hypothesis since if $\tau' < \tau$, $\operatorname{diam}(\psi_{k-1}(\tau')) < \delta_{m_{\tau'} + n_{\tau'} - (k-1)} < \delta_{m_\tau + n_\tau - (k-1)}$. (Remember, if $\tau' < \tau$, then $m_{\tau'} \geq m_\tau$ and $n_{\tau'} \geq n_\tau$.) In the special case $k = 1$, suppose $\tau = \langle \tau', \tau'' \rangle$. Then

$$\begin{aligned}
\operatorname{diam}(\psi_{k-1}(\operatorname{bdry} \tau)) &= \rho(\psi_0(\tau'), \psi_0(\tau'')) \\
&\leq \rho(\psi_0(\tau'), f(\pi_1(\tau'))) + \rho(f(\pi_1(\tau')), f(\pi_1(\tau''))) + \rho(f(\pi_1(\tau'')), \psi_0(\tau'')) \\
&< \delta_{m_{\tau'} + n_{\tau'}} + \operatorname{diam}(f(\pi_1(\tau))) + \delta_{m_{\tau''} + n_{\tau''}} \\
&< \delta_{m_\tau + n_\tau} + \delta_{m_\tau + n_\tau} + \delta_{m_\tau + n_\tau} = 3\delta_{m_\tau + n_\tau}.
\end{aligned}$$

In either case, $\psi_k|\operatorname{bdry} \tau = \psi_{k-1}|\operatorname{bdry} \tau$ can be extended to a map, $\psi_k|\tau$, in such a way that $\psi_k(\tau)$ is a subset of X of diameter less than $\delta_{m_\tau + n_\tau - (k-1) - 1} = \delta_{m_\tau + n_\tau - k}$. We have shown that ψ_k satisfies the inductive hypothesis.

Then define $\psi: P \times (0, 1] \to X$ by $\psi(p, t) = \lim_{j\to\infty} \psi_j(p, t)$.

Finally define $\phi: P \times [0, 1] \to X$ by

$$\begin{aligned} \phi(p, t) &= \psi(p, t), \quad \text{if } t \neq 0, \\ &= f(p), \quad \text{if } t = 0. \end{aligned}$$

For $t \neq 0$, local continuity is assured since $P \times (0, 1]$ is locally finite. If $(p, t) \in P \times (0, 1]$ and $t < 1/n$, let σ be a simplex containing (p, t) and suppose $\dim \sigma = s$. The diameter of $\phi(\sigma)$ is less than $\delta_{m_\sigma + n_\sigma - s}$ by property (c) of the inductive statement. But, $s \leq m_\sigma$ and $n_\sigma \geq n$; hence $\operatorname{diam}(\phi(\sigma)) < \delta_{n_\sigma} < \delta_n$. Also, if σ' is a vertex of σ, $\rho(f(\pi_1(\sigma')), \phi(\sigma')) < \delta_{n_\sigma + m_\sigma} < \delta_n$. Therefore

$$\begin{aligned} \rho(\phi(p, t), \phi(p, 0)) &\leq \rho(\phi(p, t), \phi(\sigma')) + \rho(\phi(\sigma'), f(\pi_1(\sigma'))) + \rho(f(\pi_1(\sigma')), f(p)) \\ &< 3\delta_n < 3(1/3n) = 1/n. \end{aligned}$$

We have thereby shown that ϕ is continuous.

Finally, if (p, t) is any element of $P \times (0, 1]$, $\rho(\phi(p, t), f(p)) = \rho(\phi(p, t), \phi(p, 0)) < 3\delta_1 < \delta$.

Proposition 4.2. *Let γ be a metric space and X be a uniformly locally contractible subset of γ. Let $\delta > 0$ be given and let $f: A \to \overline{X}$ be a map of an arbitrary ANR, A, into $\overline{X}$. Then given $\delta > 0$, there exists a map $\psi: A \times I \to \overline{X}$ so that for each $a \in A$:*

(a) $\psi(a, 0) = f(a)$;

(b) $\psi(a, 1) \in X$;

(c) $d(f(a), \psi(a, t)) < \delta$ *for all* $t \in I$.

Proof. Choose a cover, $\mathfrak{B}$, of A with the property that if $B \in \mathfrak{B}$, then $\operatorname{diam} f(B)$ is less than $\delta/2$.

Since A is an ANR, by a theorem of Hanner [11], there are a locally finite polyhedron P, maps $g: A \to P$ and $w: P \to A$ and a homotopy $H: A \times I \to A$ such that $H(a, 0) = a$, $H(a, 1) = wg(a)$ and for each $a \in A$, $H(a, I) \subset B$, for some $B \in \mathfrak{B}$.

By Proposition 4.1, there exists a homotopy $\phi: P \times I \to \overline{X}$ such that $\phi(p, 0) = fw(p)$, $\phi(p, 1) \in X$ and $d(fw(p), \phi(p, t)) < \delta/2$. Define $\psi: A \times I \to X$ by

$$\begin{aligned} \psi(a, t) &= f(H(a, 2t)), \quad 0 \leq t \leq ½, \\ &= \phi(g(a), 2t - 1), \quad ½ \leq t \leq 1. \end{aligned}$$

Note that ψ is continuous, since $f(H(a, 1)) = fwg(a) = \phi(g(a), 0)$. Also, $\psi(a, 0) = f(H(a, 0)) = f(a)$ and $\psi(a, 1) = \phi(g(a), 1) \in X$. By the definition of H, if $0 \leq t \leq ½$, then $d(\psi(a, 0), \psi(a, t)) < \delta/2$ and by the definition of ϕ, if $½ \leq t \leq 1$, then $d(\psi(a, ½), \psi(a, t)) < \delta/2$.

Proposition 4.3. *Let Y be a metric space and X be a uniformly locally contractible subset of Y. Given $\epsilon > 0$, there is a $\delta > 0$ such that if $y \in \overline{X}$ and $f: A \to N_\delta(y) \cap \overline{X}$ is a map of an arbitrary ANR, A, into $N_\delta(y) \cap \overline{X}$, then there is a map $G: A \times I \to N_\epsilon(y) \cap \overline{X}$ such that for all $a \in A$, $G(a, 0) = f(a)$ and $G(a, 1) = y$.*

Proof. Let $\epsilon > 0$ be given and choose $\delta > 0$ small enough so that there exists a homotopy $H:(N_{2\delta}(y) \cap X) \times I \to N_\epsilon(y) \cap \overline{X}$ such that $H(x, 0) = x$ and $H(x, 1) = y$, for all $x \in N_{2\delta}(y) \cap X$. Now suppose A is an arbitrary ANR and $f: A \to N_\delta(y) \cap \overline{X}$ is given. We will make use of the map ψ defined in Proposition 4.2 to define $G: A \times I \to N_\epsilon(y)$.

Let

$$G(a, t) = \psi(a, 2t), \qquad 0 \le t \le \tfrac{1}{2},$$
$$= H(\psi(a, 1), 2t - 1), \qquad \tfrac{1}{2} \le t \le 1.$$

Then G is continuous, maps $A \times I$ into $N_\epsilon(y) \cap \overline{X}$, $G(a, 0) = \psi(a, 0) = f(a)$ for all $a \in A$ and $G(a, 1) = H(\phi(a, 1), 1) = y$ for all $a \in A$.

As mentioned in the introduction, if M is a compact manifold, Černavskiĭ [6] and Edwards-Kirby [7] have shown that $H(M)$ and $H_\partial(M)$ are locally contractible spaces. Using the fact that if f, g, h are arbitrary elements of $H(M)$ then $\rho(f, g) = \rho(fh^{-1}, gh^{-1})$, it is trivial to show that $H(M)$ and $H_\partial(M)$ are uniformly locally contractible. Therefore, Propositions 4.1, 4.2, and 4.3 hold where X is replaced by $H(M)$ (or $H_\partial(M)$) and $\overline{X}$ by $\overline{H}(M)$ (or $\overline{H}_\partial(M)$). A space Z is said to be weakly locally contractible at $z \in Z$ if given any open set U containing z, there exists an open set V with $z \in V \subset U$ such that if P is any locally finite polyhedron and $f: P \to V$ any mapping, then there is a map $G: P \times I \to U$ such that $G(p, 0) = f(p)$ and $G(p, 1) = z$ for all $p \in P$. To indicate our partial solutions to the problems discussed at the beginning of this section we will restate some of the results of the preceding propositions in the following theorem:

Theorem 4.4. *Let M be a compact manifold.*

(i) *Given $\delta > 0$ and a map $F: A \to \overline{H}(M)$ of an ANR, A, into $\overline{H}(M)$, then there exists a continuous function $\phi_\delta: A \to H(M)$ with the property that $\rho(F(a), \phi_\delta(a)) < \delta$, for all $a \in A$. (This statement also holds if $\overline{H}(M)$ is replaced by $\overline{H}_\partial(M)$ and $H(M)$ by $\overline{H}_\partial(M)$.)*

(ii) *$\overline{H}(M)$ and $\overline{H}_\partial(M)$ are weakly locally contractible.*

A closed set K of a space X is called Z-set if for any nonempty homotopically trivial open set U in X, $U - K$ is nonempty and homotopically trivial. R. D. Anderson [2] has shown that if $\{Z_i\}_{i>0}$ is a countable collection of Z-sets in l_2, then $l_2 - \bigcup_{i>0} N_i$ is homeomorphic to l_2.

Proposition 4.5. *If M is a compact manifold and $\overline{H}(M)$ is locally homeomorphic to l_2, then $H(M)$ is locally homeomorphic to l_2.*

Proof. Suppose N is a neighborhood of 1_M in $\overline{H}(M)$ that is homeomorphic to l_2. We will show that $N \cap H(M)$ is homeomorphic to l_2, thereby demonstrating the proposition.

For each positive integer i, let $Z_i = \{f \in N |$ there exists $x \in M$ with $\operatorname{diam} f^{-1}(x) \geq 1/i\}$. Now, $N \cap H(M) = N - \bigcup_{i>0} Z_i$. So to show that $N \cap H(M)$ is homeomorphic to l_2 it suffices, by Anderson's theorem, to prove that for each i, Z_i is a Z-set. By standard arguments (cf., [14, p. 57]), Z_i is a closed (rel N) subset of N. Suppose U is a nonempty homotopically trivial open subset of N and let $f: S^{n-1} \to U - Z_i$ be given. Then choose a map $g: B^n \to U$ so that $g|S^{n-1} = f$. By Proposition 4.1 (applied to the case where $X = H(M)$, $P = B^n$ and $\delta = \rho(g(B^n), \widetilde{U}))$ there exists a map $\phi: B^n \times I \to U$ such that, for each $w \in B^n$, $\phi(w, t) \in H(M)$ if $t \neq 1$ and $\phi(w, 1) = g(w)$. Now label the points of B^n radially so that $B^n = \{tx | x \in S^{n-1}, 0 \leq t \leq 1\}$. Define $F: B^n \to U - Z_i$ by $F(tx) = \phi(tx, t)$. Note that $F|S^{n-1} = f$ and for each $t < 1$, $F(tx) = \phi(tx, t) \in H(M) \cap U \subset U - Z_i$.

5. In this section we show that if M^2 is a compact 2-manifold, then $\overline{H}_\partial(M^2)$ is locally contractible. Lemma 5.1 will be proven using a slight modification of the lifing process of Edwards-Kirby (and is valid in all dimensions). We then make use of the canonical approximation result for $\overline{H}_\partial(B^2)$ (Theorem 3.2) to show that $\overline{H}_\partial(M^2)$ is locally contractible.

If U is a subset of a manifold M, a proper imbedding of U into M is an imbedding $h: U \to M$ such that $h^{-1}(\partial M) = U \cap \partial M$. If C and U are compact subsets of M with $C \subset U$, let $I(U, C; M)$ denote the set of proper imbeddings of U into M which are the identity when restricted to C. Let $\overline{I}(U, C; M)$ denote the set of all mappings of U into M which can be approximated arbitrarily closely by elements of $I(U, C; M)$.

The statement of Lemma 5.1 corresponds to that of Lemma 4.1 of [7] except that in the situation under consideration it is not possible to obtain a homotopy by using the Alexander isotopy as in the Edwards-Kirby paper. (The inversion devise of Siebenmann (see [20, Main Idea]) was developed to handle a similar situation and is valid in all dimensions. Unfortunately, it also does not lead to the desired homotopy.) We are, however, able to obtain a homotopy for the 2-manifold case (Proposition 5.3).

We have omitted the details of the proof of Lemma 5.1 in those places where the argument parallels that of [7].

Lemma 5.1. *Let positive numbers a, b, δ be given with $1 < a < b \leq 2$. Then there exists a positive number $\epsilon_{(a,b,\delta)} \leq \delta$ so that if*

$$\overline{I}_{(a,b,\delta)} = \{f \in \overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n) | \rho(f|B^k \times ((a+b)/2)B^n, 1_{B^k \times ((a+b)/2)B^n}) < \epsilon_{(a,b,\delta)}\}$$

then there exists a continuous function

$$\phi_{(a,b,\delta)}: \overline{I}_{(a,b,\delta)} \to \overline{I}(D^k - 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$$

such that for all $f \in \overline{I}_{(a,b,\delta)}$:

(1) $\phi_{(a,b,\delta)}(f)|B^k \times (4B^n - bB^n) = 1_{B^k \times (4B^n - bB^n)}$;

(2) $\phi_{(a,b,\delta)}(f)|B^k \times aB^n = f|B^k \times aB^n$;

(3) $\rho(\phi_{(a,b,\delta)}(f), 1_{B^k \times 4B^n}) < \delta/2$.

Proof. As in the proof of Lemma 4.1 of [7] it suffices to show that there exists an $\epsilon'_{(a,b,\delta)} > 0$ and a map $\phi: \overline{I}'_{(a,b,\delta)} \to \overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ satisfying conditions (1)–(3), where

$$\overline{I}'_{(a,b,\delta)} = \{f \in \overline{I}(B^k \times 4B^n, (\partial B^k \times 4B^n) \cup ([\delta/16, 1]B^k \times 3B^n); B^k \times R^n) | \rho(f|B^k \times ((a+b)/2)B^n, 1_{B^k \times ((a+b)/2)B^n}) < \epsilon'_{(a,b,\delta)}\}$$

plays the role of $\overline{I}_{(a,b,\delta)}$.

We shall produce such a map ϕ by assigning to each pair (h, i), where $h \in \overline{I}'_{(a,b,\delta)} \cap I(B^k \times 4B^n, (\partial B^k \times 4B^n) \cup ([\delta/16, 1]B^k \times 3B^n); B^k \times R^n) = \overline{I}'_{(a,b,\delta)} \cap I$ and i is a positive integer, an imbedding $h^i: B^k \times 4B^n \to B^k \times R^n$ in such a way that:

(i) $h^i|B^k \times aB^n = h|B^k \times aB^n$;

(ii) $h^i|(\partial B^k \times 4B^n) \cup B^k \times (4B^n - bB^n) = 1_{(\partial B^k \times 4B^n) \cup B^k \times (4B^n - bB^n)}$;

(iii) given $\eta > 0$ there exists $\delta > 0$ and an integer N such that if $i, j > N$ and $d(h(x), g(x)) < \delta$ for all $x \in B^k \times ((a+b)/2) B^n$, then $d(h^i(x), g^j(x)) < \eta$ for all $x \in B^k \times 4B^n$;

(iv) if $h|B^k \times ((a+b)/2)B^n = 1_{B^k \times ((a+b)/2)B^n}$, then $\rho(h^i, 1_{B^k \times 4B^n}) < \delta/4$.

The construction of a collection of such imbeddings would complete the proof of Lemma 5.1 as the following argument indicates: Let $f \in \overline{I}'_{(a,b,\delta)}$ and choose a sequence of elements of $\overline{I}'_{(a,b,\delta)} \cap I$, $\{h_i\}$, which converges to f. Then define $\phi(f)$ to be $\lim_{i\to\infty} h_i^i$. By property (i), $\phi(f)|B^k \times aB^n = f|B^k \times aB^n$ since for any $x \in B^k \times aB^n$, $h_i^i(x) = h_i(x)$ and $\{h_i(x)\}$ converges to $f(x)$. Similarly, property (ii) assures that $\phi(f)|(\partial B^k \times 4B^n) \cup B^k \times (4B^n - bB^n)$ is the identity map. Property (iii) guarantees that ϕ is well defined (independent of the choice of the sequence $\{h_i\}$), $\phi(f)$ is an element of $\overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$, and that ϕ is continuous. Property (iv) guarantees that if $f|B^k \times ((a+b)/2)B^n =$

$1_{B^k \times ((a+b)/2)B^n}$, then $\rho(\phi(f), 1_{B^k \times 4B^n}) \leq \delta/4$. Hence, we can choose $\epsilon'_{(a,b,\delta)}$ small enough (making use of the continuity of ϕ) and thereby redefine $\overline{I}'_{(a,b,\delta)}$ so that $\rho(\phi(f), 1_{B^k \times 4B^n}) < \delta/2$ for all $f \in \overline{I}'_{(a,b,\delta)}$.

Given a pair (h, i), we shall make use of a modification of the lifting diagram of [7] to obtain the map h^i.

$$\begin{array}{ccc}
B^k \times R^n & \xrightarrow{\quad h^i \quad} & B^k \times R^n \\
\uparrow \gamma^{-1} & & \uparrow \gamma^{-1} \\
B^k \times R^n & \xrightarrow{\quad \tilde{h}^i \quad} & B^k \times R^n \\
\downarrow e & & \downarrow e \\
B^k \times T^n & \xrightarrow{\quad \bar{h}^i \quad} & B^k \times T^n \\
\uparrow \mathrm{id} & & \uparrow w_i \\
B^k \times T^n & \xrightarrow{\quad \bar{h} \quad} & B^k \times T^n \\
\uparrow i' & & \uparrow i' \\
(B^k \times T^n) - (2D^k \times 2D^n) & \xrightarrow{\quad \tilde{h} \quad} & (B^k \times T^n) - (D^k \times D^n) \\
\uparrow i & & \uparrow i \\
B^k \times (T^n - 2D^n) & \xrightarrow{\quad \hat{h} \quad} & B^k \times (T^n - D^n) \\
\downarrow \alpha & & \downarrow \alpha \\
B^k \times 4B^n & \xrightarrow{\quad h \quad} & B^k \times R^n
\end{array}$$

Let T^n denote the n-fold product of S^1 and identify an n-cell in T^n with $2B^n$. Let $\bar{e}$: $R^n \to T^n$ be a covering projection such that $\bar{e}|2B^n$ is the identity and let e: $B^k \times R^n \to B^k \times T^n$ be equal to $\mathrm{id} \times \bar{e}$. Let D^n, $2D^n$, $3D^n$, $4D^n$ be concentric n-cells in $T^n - 2B^n$ such that $jD^n \subset \mathrm{Int}\,(j+1)D^n$ for $j = 1, 2, 3$. Also let D^k, $2D^k$, $3D^k$, $4D^k$ be concentric k-cells in $\mathrm{Int}\, B^k$ such that $\delta/16B^k \subset D^k$ and $jD^k \subset \mathrm{Int}\,(j+1)D^k$ for $j = 1, 2, 3$. In addition, let $4D^n$ and $4D^k$ be chosen small enough so that the diameter of each component of $e^{-1}(4D^k \times 4D^n)$ is less than $\delta/4$. Then let $\bar{\alpha}$: $T^n - D^n \to \mathrm{Int}\,((a+b)/2)B^n$ be a fixed immersion with the property that $\bar{\alpha}$ restricted to $((3a+b)/4)B^n$ is the identity [17]. We shall choose $\epsilon'_{(a,b,\delta)}$ small enough so that $h(B^k \times aB^n) \subset B^k \times ((3a+b)/4)B^n$ for all $h \in \overline{I}'_{(a,b,\delta)} \cap I$. Let α denote the product immersion $\mathrm{id} \times \bar{\alpha} : B^k \times (T^n - D^n) \to B^k \times \mathrm{Int}\,((a+b)/2)B^n$. If $\epsilon'_{(a,b,\delta)}$ is chosen small enough, for each $h \in \overline{I}'_{(a,b,\delta)} \cap I$ we can canonically choose an embedding $\hat{h}$: $B^k \times (T^n - 2D^n) \to B^k \times (T^n - D^n)$ so that the lower square of the diagram commutes.

We note that $\hat{h}|(B^k - 2D^k) \times (T^n - 2D^n)$ is the identity map, since h is the identity on $[\delta/16, 1]B^k \times 3B^n$ and $\delta/16B^k \subset D^k$. Therefore, to obtain $\tilde{h}$, we extend $\hat{h}$ to be the identity on $(B^k - 2D^k) \times T^n$. If $\epsilon'_{(a,b,\delta)}$ is chosen small enough, then if $x \in (3D^k \times 3D^n) - (2D^k \times 2D^n)$, then $\tilde{h}(x) \in (3 + 1/2)D^k \times (3 + 1/2)D^n$. Consider the restriction of $\tilde{h}$ to $(B^k \times T^n) - (3D^k \times 3D^n)$. By the Schoenflies theorem [5] we can extend this restriction of $\tilde{h}$ to a homeomorphism $\bar{h}$: $B^k \times T^n \to B^k \times T^n$. This extension may not be canonical, i.e., if $\{\tilde{h}_i\}$ is a Cauchy sequence of imbeddings, it does not follow that $\{\bar{h}_i\}$ is a Cauchy sequence of imbeddings.

Until this point the construction of the diagram is independent of i and varies only with the imbedding h. Consider $4D^k \times 4D^n$ to be $\{tx|x \in \partial(4D^k \times 4D^n),\ 0 \leq t \leq 4\}$. We then define the homeomorphism $w_i: B^k \times T^n \to B^k \times T^n$ which takes $(3 + 1/2)D^k \times (3 + 1/2)D^n$ to $(1/i)D^k \times (1/i)D^n$ by

(a) $$w_i|B^k \times T^n - (4D^k \times 4D^n) = 1_{B^k \times T^n - (4D^k \times 4D^n)},$$

(b) $$\begin{aligned} w_i(tx) &= [(t - (3 + ½))(2)(4 - 1/i) + 1/i]x, & 3 + ½ \leq t \leq 4, \\ &= \frac{t}{(3 + ½)i}\, x, & 0 \leq t \leq 3 + ½. \end{aligned}$$

Then $\bar{h}^i: B^k \times T^n \to B^k \times T^n$ is defined by $\bar{h}^i(x) = w_i\bar{h}(x)$. Then $\bar{h}^i$ lifts to the homeomorphism $\tilde{h}^i: B^k \times R^n \to B^k \times R^n$. We note that $\tilde{h}^i$ has the property that for some constant, M, $d(\tilde{h}^i, \mathrm{id}) < M$. Finally, let γ: $\mathrm{Int}(bB^k \times bB^n) \to R^k \times R^n$ be a homeomorphism which is a radial expansion and is the identity on $((3a + b)/4)B^k \times ((3a + b)/4)B^n$. We extend $\tilde{h}^i$ by the identity to a homeomorphism $\tilde{h}^i$: $R^k \times R^n \to R^k \times R^n$ and define $h^i: B^k \times 4B^n \to B^k \times R^n$ by

$$\begin{aligned} h^i(x) &= \gamma^{-1}\tilde{h}^i\gamma(x), & x \in B^k \times bB^n, \\ &= x, & x \in B^k \times (4B^n - bB^n). \end{aligned}$$

Since $d(\tilde{h}^i, \mathrm{id}) < M$, h^i is continuous and therefore is a homeomorphism.

To check property (i) note that $\alpha i^{-1} i'^{-1} w_i^{-1} e\gamma(x) = x$ for all $x \in B^k \times ((3a + b)/4)B^n$, and that ϵ' was chosen small enough so that if $x \in B^k \times aB^n$, then $h(x) \in B^k \times ((3a + b)/4)B^n$. That property (ii) is satisfied was guaranteed by the choice of the homeomorphism γ.

Each stage in the construction of h^i is canonical except the use of the Schoenflies theorem. Therefore, to show that property (iii) is satisfied we must only show that given $\eta' > 0$, there exists $\delta' > 0$ and an integer N such that if $d(\tilde{h}(x), \tilde{g}(x)) < \delta'$ for all $x \in (B^k \times T^n) - (2D^k \times 2D^n)$ and if $i, j > N$, then $\rho(\bar{h}^i, \bar{g}^i) < \eta'$. Let N be chosen so that $2/N < \eta'$ and $\delta' < \eta'/16$. If $x \in 3D^k \times 3D^n$, then $\tilde{h}(x)$ and $\tilde{g}(x)$ are elements of $(3 + 1/2)D^k \times (3 + 1/2)D^n$ and hence $d(\bar{h}^i(x), \bar{g}^i(x)) < 2/N < \eta'$. If $x \notin 3D^k \times 3D^n$, $\bar{h}(x) = \tilde{h}(x)$ and $\bar{g}(x) = \tilde{g}(x)$ and hence $d(\bar{h}^i(x), \bar{g}^j(x)) = d(w_i\bar{h}(x), w_j\bar{g}(x)) < 16\delta' < \eta'$.

Finally, if $h|B^k \times ((a + b)/2)B^n = 1_{B^k \times ((a+b)/2)B^n}$ and i is any positive integer, $\bar{h}^i|(B^k \times T^n) - (4D^k \times 4D^n) = 1_{(B^k \times T^n) - (4D^k \times 4D^n)}$. But, $4D^k$ and $4D^n$ were chosen small enough so that the diameter of each component of $e^{-1}(4D^k \times 4D^n)$ is less than $\delta/4$. Therefore, $\rho(h^i, 1_{B^k \times 4B^n}) < \delta/4$.

Lemma 5.2. *Let* $n+k=2$. *Let positive numbers* a, b, δ *be given with* $1 < a < b \leq 2$. *Then there exists a positive number* $\epsilon_{(a,b,\delta)} \leq \delta$ *so that if*

$$\overline{I}_{(a,b,\delta)} = \{f \in \overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n;\ B^k \times R^n) \mid \rho(f|B^k \times ((a+b)/2)B^n,\ 1_{B^k \times ((a+b)/2)B^n}) < \epsilon_{(a,b,\delta)}\}$$

and $\eta > 0$ *is given then there exists a map* $G_{(a,b,\delta,\eta)}: \overline{I}_{(a,b,\delta)} \to H_\partial(B^k \times 4B^n)$ *such that for all* $f \in \overline{I}_{(a,b,\delta)}$:

(1) $G_{(a,b,\delta,\eta)}(f)|B^k \times (4B^n - bB^n) = 1_{B^k \times (4B^n - bB^n)}$;

(2) $\rho(G_{(a,b,\delta,\eta)}(f)|B^k \times aB^n, f|B^k \times aB^n) < \eta$;

(3) $\rho(G_{(a,b,\delta,\eta)}(f), 1_{B^k \times 4B^n}) < \delta$.

Proof. Let $\epsilon_{(a,b,\delta)}$ be the positive number obtained in Lemma 5.1. By Theorem 3.2 there is a mapping $\theta^1: \overline{H}_\partial(B^k \times bB^n) \to H_\partial(B^k \times 4B^n)$ such that $\theta^1(f)|B^k \times (4B^n - bB^n) = 1_{B^k \times (4B^n - bB^n)}$ and $\rho(\theta^1(f), f) < \min(\delta/2, \eta)$, for all $f \in \overline{H}_\partial(B^k \times bB^n)$. (This follows from applying Theorem 3.2 to the 2-ball $B^k \times bB^n$ and an arbitrary open cover of diameter less than $\min(\delta/2, \eta)$, obtaining a map of $\overline{H}_\partial(B^k \times bB^n)$ into $H_\partial(B^k \times bB^n)$ and then extending the elements of $H_\partial(B^k \times bB^n)$ by the identity on $B^k \times (4B^n - bB^n)$.) Define $G_{(a,b,\delta,\eta)}: \overline{I}_{(a,b,\delta)} \to H_\partial(B^k \times 4B^n)$ by

$$G_{(a,b,\delta,\eta)}(f) = \theta^1(\phi_{(a,b,\delta)}(f)|B^k \times bB^n),$$

where $\phi_{(a,b,\delta)}$ is the mapping obtained in Lemma 5.1.

It is easy to check that this map satisfies properties (1)–(3).

Proposition 5.3. *If* $n+k=2$, *then there exists a neighborhood* Q *of* $1_{B^k \times 4B^n}$ *in* $\overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n;\ B^k \times R^n)$ *and a homotopy*

$$\psi: Q \times [0,1] \to \overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n;\ B^k \times R^n)$$

such that $\psi(f, 0) = f$, $\psi(f,1) \in \overline{I}(B^k \times 4B^n, (\partial B^k \times 4B^n) \cup (B^k \times B^n);\ B^k \times R^n)$, $\psi(f,t)|\partial(B^k \times 4B^n) = f|\partial(B^k \times 4B^n)$, *for all* $f \in Q$ *and* $t \in [0,1]$.

Proof. For each positive integer, i, let $v_i = 1/2^i$. Choose the neighborhood Q small enough so that if $f \in Q$, then $\rho(f, 1_{B^k \times 4B^n}) < \epsilon_{(1+v_2, 1+v_1, v_4)}$.

Assume inductively that mappings $f_1, \cdots, f_j$ have been defined so that for each i, $1 \leq i \leq j$, $f_i \in H_\partial(B^k \times 4B^n)$, f_i depends canonically on f, and

(1) $$f_i|B^k \times (4B^n - (1+v_i)B^n) = 1_{B^k \times (4B^n - (1+v_i)B^n)},$$

(2) $$\rho(f_i, 1_{B^k \times 4B^n}) < v_{i+3},$$

(3) $$\rho(f_i|B^k \times (1+v_{i+1})B^n,\ ff_1^{-1} \cdots f_{i-1}^{-1}|B^k \times (1+v_{i+1})B^n) < \epsilon_{(1+v_{i+2}, 1+v_{i+1}; v_{i+4})},$$

(4) $$ff_1^{-1}\cdots f_i^{-1} \in \overline{T}(B^k\times 4B^n, \partial B^k\times 4B^n; B^k\times R^n),$$

(5) $$\rho\left(ff_1^{-1}\cdots f_i^{-1}|B^k\times\left(\frac{1+v_{i+2}+1+v_{i+1}}{2}\right)B^n,\ 1_{(B^k\times((1+v_{i+2}+1+v_{i+1})/2)B^n)}\right)<\epsilon_{(1+v_{i+2},1+v_{i+1},v_{i+4})}.$$

Now applying Lemma 5.2, for each $f\in Q$, let f_{j+1} be defined by

$$f_{j+1}=G^{(ff_1^{-1}\cdots f_j^{-1})}_{(1+v_{j+1},1+v_{j+1},v_{j+4}),\ \epsilon_{(1+v_{j+3},1+v_{j+2},v_{j+5})}}.$$

[The map G is well defined by Condition (5) of the inductive hypothesis—in the case $j+1=1$, Q was chosen small enough to meet this requirement.]

Conditions (1)–(4) of the inductive hypothesis are guaranteed by the application of Lemma 5.2. Condition (5) follows from (2) and (3): If $x\in B^k\times((1+v_{j+3}+1+v_{j+2})/2)B^n$, then $f_{j+1}(x)\in B^k\times(1+v_{j+2})B^n$, since

$$\rho(f_{j+1}, 1_{B^k\times 4B^n})<v_{j+4}\quad\text{and}\quad \frac{v_{j+3}}{2}+\frac{v_{j+2}}{2}+v_{j+4}=\frac{1}{2^{j+4}}+\frac{1}{2^{j+3}}+\frac{1}{2^{j+4}}=1/2^{j+2}=v_{j+2}.$$

Therefore,

$$d(ff_1^{-1}\cdots f_{j+1}^{-1}(x),x)=d(ff_1^{-1}\cdots f_j^{-1}(f_{j+1}^{-1}(x)), f_{j+1}(f_{j+1}^{-1}(x)))$$
$$<\epsilon_{(1+v_{j+3},1+v_{j+2},v_{j+5})}.$$

Let $g=\lim_{j\to\infty}ff_1^{-1}\cdots f_j^{-1}$. Now, g is continuous since if $x\in B^k\times B^n$, then $g(x)=x$ and if $x\in B^k\times[1+v_{j+1}, 1+v_j]B^n$, then $d(g(x),x)<1/2^j$. (Assume $x\in B^k\times[1\times v_{j+1}, 1+v_j]B^n$. Note that $f_{j-1}^{-1}f_j^{-1}(x)\in B^k\times((1+v_j+1+v_{j-1})/2)B^n$. By property (1), $g(x)=ff_1^{-1}\cdots f_j^{-1}(x)$. Hence

$$\begin{aligned}d(g(x),x)&=d(ff_1^{-1}\cdots f_j^{-1}(x),x)\\&\le d(ff_1^{-1}\cdots f_{j-2}^{-1}(f_{j-1}^{-1}(f_j^{-1}(x))), f_{j-1}^{-1}(f_j^{-1}(x)))+d(f_{j-1}^{-1}(f_j^{-1}(x)),x)\\&\le\epsilon_{(1+v_j,1+v_{j-1},v_{j+2})}+(v_{j+3}+v_{j+2})\\&\le v_{j+2}+v_{j+3}+v_{j+2}=1/2^{j+2}+1/2^{j+3}+1/2^{j+2}<1/2^j.)\end{aligned}$$

For each j, $j=0,1,\cdots$, we can define an Alexander homotopy $\psi_j: Q\times[0,1]\to\overline{T}(B^k\times 4B^n, \partial B^k\times 4B^n; B^k\times R^n)$ by $\psi_j(f,t)=ff_1^{-1}\cdots f_{j-1}^{-1}A(f_j^{-1},1-t)$, where A is the Alexander isotopy defined in §3 with $B^k\times(1+v_j)B^n$ playing the role of B^n. Note that for each $t\in[0,1]$, $\rho(\psi_j(f,t),g)<1/2^{j-1}$. For if $x\in B^k\times 4B^n-(1+v_j)B^n$, $\psi_j(f,t)(x)=g(x)$ and if $x\in B^k\times(1+v_j)B^n$, $d(\psi_j(f,t)(x),g(x))\le$

$d(\psi_j(f, t)(x), x) + d(g(x), x) < 1/2^j + 1/2^j = 1/2^{j-1}$, by the previous argument. Also, $\psi_j(f, 0) = ff_1^{-1} \cdots f_{j-1}^{-1}$ and $\psi_j(f, 1) = ff_1^{-1} \cdots f_j^{-1}$. We obtain the desired map $\psi: Q \times [0, 1] \to \overline{I}(B^k \times 4B^n, \partial B^k \times 4B^n; B^k \times R^n)$ by composing the homotopies $\psi_0, \psi_1, \cdots$ in the following manner:

$$\psi(f, t) = \psi_j\left(f, 2^{j+1}\left(t - \frac{2^j - 1}{2^j}\right)\right), \quad \frac{2^j - 1}{2^j} \le t \le \frac{2^{j+1} - 1}{2^{j+1}},$$

and letting $\psi(f, 1) = g$.

Theorem 5.4. *If M^2 is a compact 2-manifold, then $\overline{H}_\partial(M^2)$ is locally contractible, and hence if $\partial M^2 = \phi$, $\overline{H}(M^2)$ is locally contractible.*

Proof. Since $\overline{H}_\partial(M^2)$ is homogeneous, it suffices to check local contractibility at 1_{M^2}. But this follows from Proposition 5.3 exactly as in [7].

The only place the fact that $n + k = 2$ is used is in the application of Theorem 3.2. Therefore, an affirmative answer to the following question would show that $\overline{H}_\partial(M^m)$ is locally contractible for any compact m-manifold, M^m: Given $\delta > 0$ does there exist a continuous mapping $\theta: \overline{H}_\partial(B^m) \to H_\partial(B^m)$ such that $\rho(f, \theta(f)) < \delta$, for all $f \in \overline{H}_\partial(B^m)$?

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