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AN ASYMPTOTIC FORMULA IN ADELE DIOPHANTINE APPROXIMATIONS

BY

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ABSTRACT. In this paper an asymptotic formula is found for the number of solutions of a system of linear Diophantine inequalities defined over the ring of adeles of an algebraic number field. The theorem proved is a generalization of results of S. Lang and W. Adams.

1. **Introduction.** Serge Lang [5] defines a number to have type $\leq g$ if g is a positive increasing function for which $|qb - p| \geq 1/qg(q)$ for all q sufficiently large. Lang then shows that the number $\lambda(N, b)$ of solutions of $|qb - p| \leq \psi(q)$ with $q \leq N$ is asymptotic to $S_N = \Sigma_{q=1}^{N} 2\psi(q)$ if b has type $\leq g$ and ψ decreases so slowly that $\psi(q)qg(q)^{-1}$ increases to infinity with q. W. Adams [1] has extended this result of Lang to the simultaneous approximation of real numbers by rationals. I have also shown in [8] how these results may be extended to linear forms. The purpose of this paper is to show that the Lang-Adams theorem holds for the approximation of linear forms in the ring of adeles over a number field k. A p-adic theorem, as well as some of the results in [8], could be stated as corollaries to the theorem proved here. The theorem proved is probably not the best possible such theorem. This is suggested by a metric example I will give later.

Diophantine approximations over the adeles have previously been considered by David Cantor in [2]. In his paper Cantor shows adele analogues of some of the basic theorems. To some extent, I have followed Cantor in notation and setting up the problem in the ring of adeles.

I wish to thank Professor W. Adams for his help and encouragement in my work.

2. **Notation.** We use k to denote an algebraic number field of degree n with ring of integers $\mathfrak{O}$. Let P be the set of all primes of k. We write P_∞ for the set of all infinite primes, and P_0 for the set of all finite primes. When P_0 and P_∞ are used as subscripts, we will replace them by 0 and ∞ respectively. For $\mathfrak{p} \in P$, we let $k_\mathfrak{p} \supseteq k$ denote the completion of k with respect to $\mathfrak{p}$.

We may assume P_0 is the set of all prime ideals of $\mathfrak{O}$. For $\mathfrak{p} \in P_0$, $x \in k$, let $\nu = \nu_\mathfrak{p}(x)$ be the $\mathfrak{p}$-order of x. We normalize the absolute value $|\ |_\mathfrak{p}$ associated with $\mathfrak{p}$ so that $|x|_\mathfrak{p} = N\mathfrak{p}^{-\nu}$, where $N\mathfrak{p}$ is the norm of the ideal $\mathfrak{p}$.

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Let $x \longrightarrow x^{(i)}$, $i = 1, \cdots, n$, be the embeddings of k into $\mathbf{C}$, the complex numbers. We arrange the notation so that the first R_1 embeddings map into the real numbers $\mathbf{R}$ and the remaining maps consist of R_2 pairs of complex conjugate mappings listed so that

$$x^{(R_1+R_2+i)} = \overline{x^{(R_1+i)}} \quad \text{for } i = 1, \cdots, R_2.$$

The infinite primes of k can be identified with the first $R = R_1 + R_2$ of these mappings. We use $|\ |$ to stand for the ordinary absolute value on $\mathbf{C}$. If $\mathfrak{p}$ is the infinite prime corresponding to $x \longrightarrow x^{(i)}$, then we set $|x|_{\mathfrak{p}} = |x^{(i)}|$ if $k^{(i)}$ is real, otherwise we set $|x|_{\mathfrak{p}} = |x^{(i)}|^2$. The infinite prime $\mathfrak{p}$ is called real when $k^{(i)} \subseteq \mathbf{R}$ and complex otherwise. If $\mathfrak{p}$ is real, then $k_{\mathfrak{p}} = \mathbf{R}$ and we will often identify k with a subfield of $\mathbf{R}$ by means of $x \longrightarrow x^{(i)}$. A similar statement can be made when $\mathfrak{p}$ is complex, in which case $k_{\mathfrak{p}} = \mathbf{C}$. Hence, if we write $|\ |_{\mathfrak{p}}$ for the extension of $|\ |_{\mathfrak{p}}$ to $k_{\mathfrak{p}}$, we may think of $|\ |_{\mathfrak{p}}$ as the ordinary absolute value when $k_{\mathfrak{p}} = \mathbf{R}$ and the square of the ordinary absolute value when $k_{\mathfrak{p}} = \mathbf{C}$.

For $\mathfrak{p} \in P_0$, the set $\mathfrak{o}_{\mathfrak{p}}$ of all x in $k_{\mathfrak{p}}$ for which $|x|_{\mathfrak{p}} \leq 1$ is the ring of $\mathfrak{p}$-adic integers of $k_{\mathfrak{p}}$. For $\mathfrak{p} \in P_\infty$, we set $\mathfrak{o}_{\mathfrak{p}} = k_{\mathfrak{p}}$.

Let S be any subset of P. Consider the product $\Pi k_{\mathfrak{p}}$ over all $\mathfrak{p} \in S$, with componentwise algebraic operations. For any a in this product we use $a_{\mathfrak{p}}$ to stand for the $\mathfrak{p}$th component of a. We define the ring k_S of S-adeles to be the subset of this product consisting of all a with $a_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$ for all but a finite number of $\mathfrak{p}$. Note that this is not the ring usually referred to as the S-adele ring. We embed k in k_S by identifying $a \in k$ with the element in k_S, also denoted by a, for which $a_{\mathfrak{p}} = a \in k$ for all $\mathfrak{p} \in S$. We let $S_\infty = S \cap P_\infty$ and $S_0 = S \cap P_0$. Then we can write $k_S = k_{S_\infty} \times k_{S_0}$. For $a \in k_S$ we write a^∞ for the k_{S_∞} component of a, and we write a^0 for the k_{S_0} component of a.

We denote the multiplicative group of units of k_S by k_S^*, and call this the group of S-ideles. Clearly, $a \in k_S$ is an idele if and only if $a_{\mathfrak{p}}$ is nonzero for all $\mathfrak{p}$ in S and $|a_{\mathfrak{p}}|_{\mathfrak{p}} = 1$ for all but a finite number of $\mathfrak{p} \in S$.

We extend $|\ |_{\mathfrak{p}}$ to k_S by defining $|a|_{\mathfrak{p}} = |a_{\mathfrak{p}}|_{\mathfrak{p}}$ for a in k_S. For $T \subseteq S$ and $a \in k_S$, put $|a|_T = \Pi_{\mathfrak{p} \in T} |a|_{\mathfrak{p}}$, if this product converges; and otherwise set $|a|_T = 0$. So, if $a \in k_S^*$, then $|a|_T \neq 0$. For $a, b \in k_S$, write $a \leq b$ if $|a|_{\mathfrak{p}} \leq |b|_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$, and write $a < b$ if $a \leq b$ and $|a|_{\mathfrak{p}} < |b|_{\mathfrak{p}}$ for all infinite primes in S. If $S \supseteq P_\infty$ and $x = (x_1, \cdots, x_m) \in k_S^m$ we write $\boxed{x} = \max |x_i|_{\mathfrak{p}}^{1/n_{\mathfrak{p}}}$ where $n_{\mathfrak{p}}$ is the local degree of $\mathfrak{p}$ and the max is taken over all $\mathfrak{p} \in P_\infty$ and all i satisfying $1 \leq i \leq m$.

We topologize k_S in the usual way by requiring that the sets $\{x \in k_S\colon x - b \leq a\}$, $a \in k_S^*$, form a neighborhood basis at b in k_S. This makes k_S into a locally compact additive topological group.

It is well known that $\mathfrak{o}$ is a discrete subset of k_∞ and $k_\infty/\mathfrak{o}$ is compact. If $S \subseteq P_0$, by the strong approximation theorem, k is dense in k_S.

We now define some measures, all of which will be denoted by μ when there is no ambiguity. Let $\mu_\mathfrak{p}$ be the Haar measure on $k_\mathfrak{p}$ normalized so that $\mu_\mathfrak{p}(\mathfrak{o}_\mathfrak{p}) = 1$ when $\mathfrak{p} \in P_0$, and so that $\mu_\mathfrak{p}$ is ordinary Lebesgue measure when $\mathfrak{p} \in P_\infty$. The Haar measure μ_S on k_S is normalized by requiring that this measure agree with the product measure on

$$k_S(T) = \prod_{\mathfrak{p}\in T} k_\mathfrak{p} \times \prod_{\mathfrak{p}\in S-T} \mathfrak{o}_\mathfrak{p}$$

where T is any finite subset of S. So

$$\mu_S\{x \in k_S: x \leq a\} = 2^{R_1}\pi^{R_2}|a|_S.$$

Whenever we talk about a measure on k_S^m we mean the product measure μ_S^m. If G is a discrete subgroup of k_S^m we will always take the counting measure. Furthermore, if k_S^m/G is compact we normalize the measure μ on this group so that the measure of the group is just the μ_S^m measure of any measurable set of representatives in k_S^m of the cosets of G. So $\mu(k_\infty^m/\mathfrak{o}^m) = 2^{-mR_2}|d|^{m/2}$ where d will always stand for the discriminant of k.

If σ is a topological automorphism of k_S^m the modulus of σ is defined by $\operatorname{mod}\sigma = \mu(\sigma X)/\mu(X)$ where X is any measurable set in k_S. If σ is a k_S module automorphism of k_S^m with determinant $\det\sigma$, then $\operatorname{mod}\sigma = |\det\sigma|_S$.

3. **Statement of the theorem.** Let L be the system

$$L_i(x) = \sum_{j=1}^{s} a_{ij}x_j, \qquad i = 1, \cdots, r,$$

of linear forms with coefficients in k_S. Set $m = r + s$. We will suppose $z = (z_1, \cdots, z_m)$, $x = (x_1, \cdots, x_s)$, and $y = (y_1, \cdots, y_r)$ are related by $z = (x, y)$. Suppose $A_\mathfrak{p}$ is the $\mathfrak{p}$th component of the coefficient matrix of the system

(1)
$$L_i^0(z) = \sum_{j=1}^{s} a_{ij}^0 z_j - z_{i+s}, \qquad 1 \leq i \leq r.$$

Write $\delta_\mathfrak{p}$ for the determinant of the $r \times r$ submatrix of $A_\mathfrak{p}$ with the $\mathfrak{p}$-adic absolute value of its determinant maximal. We define $\delta = \delta(L) = (\delta_\mathfrak{p}) \in k_S$. For simplicity, we will assume that $S \supseteq P_\infty$, except when we specifically state otherwise.

We let ψ be a mapping from the positive reals $\mathbf{R}_+$ to k_S^*. We would like to count the number $\lambda(N)$ of solutions $x \in \mathfrak{o}^s$, $y \in \mathfrak{o}^r$ of

(2)
$$L_i(x) - y_i < \psi(\lceil x \rceil), \qquad 1 \leq i \leq r,$$
$$\lceil x \rceil \leq N.$$

We will show how to do this when $|\psi(t)|_S$ does not decrease too fast. Note, there are only finitely many $x \in \mathfrak{O}^s$ with $\lceil x \rceil \leq N$, because $\lceil x \rceil \leq N$ defines a bounded region in $k_\infty^s = \mathbf{R}^{sn}$ which therefore contains a finite number of points of the lattice $\mathfrak{O}^s$. Also, in the same way the number of y corresponding to a given x in (2) is finite. In fact, if $|\psi(t)|_\infty < 2^{-n}$, then y is uniquely determined; for, if y' and y'' both correspond to the same x, then

$$y_i = y_i' - y_i'' \leq H \max\{L_i(x) - y_i',\ L_i(x) - y_i''\} \leq H\psi(\lceil x \rceil),$$

so

$$|\text{Norm } y_i| = |y_i' - y_i''|_\infty \leq |H\psi(\lceil x \rceil)|_\infty < 1$$

and thus $y = 0$.

We use M to denote the transpose system

$$M_j(y) = \sum_{i=1}^{r} a_{ij} y_i, \qquad 1 \leq j \leq s.$$

Let $g\colon \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ be an increasing function. We say L has *type* $\leq g$ if

$$\max_j |M_j(y) - x_j|_S \leq g(\lceil z \rceil)^{-1} \lceil z \rceil^{-rn/s} \tag{3}$$

has only finitely many solutions $z = (x, y) \in \mathfrak{O}^m$. The motivation for the right-hand side of (3) is the following version of Dirichlet's theorem.

Proposition. *If $S \supseteq P_\infty$ then there are infinitely many $z = (x, y) \in \mathfrak{O}^m$ such that*

$$|L_i(x) - y_i|_S \leq c\lceil z \rceil^{-sn/r}, \qquad 1 \leq i \leq r.$$

If $S \subseteq P_0$ then there are infinitely many $z \in \mathfrak{O}^m$ such that

$$|L_i(x) - y_i|_S \leq c\lceil z \rceil^{-mn/r}, \qquad 1 \leq i \leq r.$$

Here c is some constant depending on k and L.

A proof of a slightly different version of this adele theorem may be found in [2, Theorem 2.3].

We prove the following theorem.

Theorem. *Assume the following*:

(i) *L has type $\leq g$.*

(ii) *$\psi(t)$ is decreasing.*

(iii) $F(t)^{n(r+2s)} = |\psi(t)|_S^r t^{sn} g(t^{s/r})^{-s}$ *increases to* ∞.

(iv) $\psi^0(t) \leq 1$, *i.e.*, $|\psi(t)|_\mathfrak{p} \leq 1$ *for all finite primes* $\mathfrak{p} \in S$.

(v) $|\psi(t)|_{\mathfrak{p}_1} |\psi(t)|_{\mathfrak{p}_2}^{-1} \leq C$ *for all pairs of infinite primes* $\mathfrak{p}_1, \mathfrak{p}_2$, *where* C *is a constant independent of* t.

Then the number $\lambda(N)$ *of solutions of* (2) *is*

$$\lambda(N) = \gamma \int_1^N t^{sn-1} |\psi(t)|_S^r \, dt + O\left(\int_1^N \frac{t^{sn-1} |\psi(t)|_S^r}{F(t)} \, dt \right) \tag{4}$$

with $\gamma = ns 2^{Rm} \pi^{mR_2} |\delta(L)|_{s_0}^{-1} |d|^{-m/2}$.

Remark. If we specialize the type theorem to the case $k = Q$, $S = P_\infty$, we get the homogeneous version of the theorem in [8].

Remark. If we assume $S \subseteq P_0$, delete condition (v), and replace the right-hand side in condition (iii) by $|\psi(t)|_S^r t^{mn} g(t^{s/r})^{-s}$, then we can show, by making only minor changes in the proof of the above theorem, that the number of solutions of (2) and $\overline{|y|} \leq N$ satisfies

$$\lambda(N) \sim \gamma \int_1^N t^{mn-1} |\psi(t)|_S^r \, dt$$

for some constant γ. This specializes to a p-adic theorem when $k = Q$. A similar result may be proved when S includes some but not all primes of P_∞.

In §4 I develop some results from the geometry of numbers which I will need when I prove the above theorem in §5. In §6 I will show how a metric result follows from this theorem.

4. **The geometry of numbers over k.** We call Λ an m-dimensional $\mathfrak{o}$-lattice if Λ is a discrete $\mathfrak{o}$ submodule of k_∞^m and k_∞^m/Λ is compact; this last condition is the same as requiring that Λ contain m k-independent elements. We call $\mu(k_\infty^m/\Lambda)$ the determinant of Λ and denote this by det Λ. Note that $\mathfrak{o}^m$ is a lattice with det $= 2^{-mR_2} |d|^{m/2}$. From our identification of k_∞ with $\mathbf{R}^{R_1} \times \mathbf{C}^{R_2} \cong \mathbf{R}^n$, it is clear that an $\mathfrak{o}$-lattice is just an ordinary $\mathbf{R}^{nm}$ lattice with the same determinant. Note that not every lattice in $\mathbf{R}^{nm}$ is an $\mathfrak{o}$-lattice.

If $\mathfrak{a}$ is an ideal of k, we let $\mathfrak{a}\Lambda$ be the set of all sums $\Sigma\, a_i x_i$ with a_i in $\mathfrak{a}$ and x_i in Λ. It has been shown by K. Rogers and H. P. F. Swinnerton-Dyer [7, Theorem 1] that

Proposition 1. *If* Λ *is an* $\mathfrak{o}$-*lattice in* k_∞^m, *there exist* m k_∞ *independent points* $P_1, \cdots, P_m$ *in* Λ *and an ideal* $\mathfrak{b} \supseteq \mathfrak{o}$ *in* k *such that*

$$\Lambda = \mathfrak{o}P_1 + \cdots + \mathfrak{o}P_{m-1} + \mathfrak{b}P_m$$

where the ideal class of $\mathfrak{b}$ *depends only on* Λ.

We may now state the following:

Proposition 2. $\mathfrak{a}\Lambda$ *is an* $\mathfrak{o}$*-lattice with* $\det \mathfrak{a}\Lambda = N\mathfrak{a}^m \det \Lambda$.

Proof. The first assertion follows from the expression

$$\mathfrak{a}\Lambda = \mathfrak{a}P_1 + \cdots + \mathfrak{a}P_{m-1} + \mathfrak{a}\mathfrak{b}P_m.$$

To prove the second assertion we may suppose $\mathfrak{a}$ is integral. Then $\mathfrak{a}\Lambda \subseteq \Lambda$ and

$$\Lambda/\mathfrak{a}\Lambda = \frac{\mathfrak{o}P_1 + \cdots + \mathfrak{o}P_{m-1} + \mathfrak{b}P_m}{\mathfrak{a}P_1 + \cdots + \mathfrak{a}P_{m-1} + \mathfrak{a}\mathfrak{b}P_m}$$
$$\cong (\mathfrak{o}/\mathfrak{a})^{m-1} \times \mathfrak{b}/\mathfrak{a}\mathfrak{b} \cong (\mathfrak{o}/\mathfrak{a})^m$$

so the order of $\Lambda/\mathfrak{a}\Lambda$ is $(N\mathfrak{a})^m$. The proposition now follows.

For $x = (x_1, \cdots, x_m)$, $y = (y_1, \cdots, y_m)$ we denote the dot product by $x \cdot y = \Sigma_i x_i y_i$. Also, let Tr denote the trace function extended to k_∞. We define

$$\Lambda^{-1} = \{x \in k_\infty^m : x \cdot y \in \mathfrak{o} \text{ for all } y \in \Lambda\},$$
$$\Lambda^* = \{x \in k_\infty^m : \mathrm{Tr}(x \cdot y) \in \mathbf{Z} \text{ for all } y \in \Lambda\}.$$

It is straightforward to show

Proposition 3. $\Lambda^* = \mathfrak{D}^{-1}\Lambda^{-1}$, *where* $\mathfrak{D}$ *is the different of* k, *i.e.* $\mathfrak{D}^{-1}$ *is the fractional ideal consisting of all* $x \in k$ *such that* $\mathrm{Tr}(ax) \in \mathbf{Z}$ *for all* $a \in \mathfrak{o}$.

If $P_1, \cdots, P_m$ are the independent points in Proposition 1, we can find points $P_1', \cdots, P_m'$ such that

$$P_i \cdot P_j' = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

So, $\Lambda^{-1} = \mathfrak{o}P_1' + \cdots + \mathfrak{o}P_{m-1}' + \mathfrak{b}^{-1}P_m'$, and hence Λ^{-1}, and therefore, also, Λ^*, is an $\mathfrak{o}$-lattice. We call Λ^* the polar lattice of Λ. This is just the ordinary polar lattice in $\mathbf{R}^{nm}$ with respect to the bilinear form $\langle x, y\rangle = \mathrm{Tr}(x \cdot y)$.

We now give some examples of $\mathfrak{o}$-lattices we will need latter.

Example 1. Let L be the independent system $L_i(z) = \Sigma_{j=1}^m a_{ij}z_j$, $1 \le i \le m$, with $a_{ij} \in k_\infty$. The coefficient matrix A of this system has determinant in k_∞^*. So, L determines an automorphism $L: z \to L(z) = zA$ of k_∞^m with mod $L = |\det A|_\infty$. If Λ is an $\mathfrak{o}$-lattice, then so is $L(\Lambda)$. It is clear that

$$\det L(\Lambda) = \mathrm{mod}\, L \det \Lambda = |\det A|_\infty \det \Lambda.$$

Now, let M be the system with coefficient matrix ${}^tA^{-1}$ (the t stands for transpose). Then

$$L(z)\cdot M(w)=(zA)\cdot(w^tA^{-1})=zAA^{-1}({}^tw)=z\cdot w.$$

Hence $L(\Lambda)^{-1}=M(\Lambda^{-1})$ and therefore also

$$L(\Lambda)^*=\mathfrak{D}^{-1}L(\Lambda)^{-1}=\mathfrak{D}^{-1}M(\Lambda^{-1})=M(\mathfrak{D}^{-1}\Lambda^{-1})=M(\Lambda^*).$$

Example 2. Assume $S\subseteq P_0$, and let L be the system of independent linear forms $L_i(z)=\sum_{j=1}^m a_{ij}z_j$, $1\le i\le r\le m$, with coefficients $a_{ij}\in k_S$. Let ϵ be an idele ≤ 1 in k_S and define

$$\Lambda=\Lambda_{L,\epsilon}=\{z\in\mathfrak{o}^m\subseteq k_\infty^m:\ L_i(z)\le\epsilon,\ 1\le i\le r\}.$$

Since all $\mathfrak{p}\in S$ are nonarchimedean the set $\Lambda_{L,\epsilon}$ is an $\mathfrak{o}$-module. The set is discrete because $\Lambda_{L,\epsilon}\subseteq\mathfrak{o}^m$. Also, it contains the m k_∞-independent elements ae_i where a is an appropriately chosen element of $\mathfrak{o}$ and e_i is the m-tuple with 1 in the ith position and 0 elsewhere. Hence $\Lambda_{L,\epsilon}$ is an $\mathfrak{o}$-lattice.

We compute the determinant of $\Lambda_{L,\epsilon}$. Let A be the $r\times m$ coefficient matrix of the system L and let $A_\mathfrak{p}$ be the $\mathfrak{p}$th component of this matrix. Write $\delta_\mathfrak{p}$ for the determinant of the $r\times r$ submatrix of $A_\mathfrak{p}$ with the $\mathfrak{p}$-adic absolute value of its determinant maximal. Also, write $\delta'_\mathfrak{p}$ for the determinant of the submatrix of $A_\mathfrak{p}$ with the $\mathfrak{p}$-adic absolute value of its determinant maximum; this last submatrix may be of any size $i\times i$ with $0\le i\le r$, and by convention we take the determinant of a 0×0 matrix to be 1. We define $\delta=(\delta_\mathfrak{p})\in k_S$, $\delta'=(\delta'_\mathfrak{p})\in k_S$.

Proposition 4. *If δ,δ' are ideles and $\epsilon\le\delta/\delta'$, then*

$$\det\Lambda_{L,\epsilon}=2^{-mR_2}|d|^{m/2}|\epsilon^{-r}\delta|_S.$$

Proof. It suffices to prove the order of $\mathfrak{o}^m/\Lambda$ is $|\epsilon^{-r}\delta|_S$. Set

$$E=\{z\in k_S^m:\ z_i\le 1,\ 1\le i\le m\},$$
$$E'=\{z\in k_S^m:\ L_i(z)\le\epsilon,\ z_j\le 1,\ 1\le i\le r,\ 1\le j\le m\}.$$

Since all $\mathfrak{p}$ in S are nonarchimedean, E and E' are groups with $E'\subseteq E$. Because $\mathfrak{o}^m$ is dense in E, each coset of E' in E contains an element of $\mathfrak{o}^m$ and therefore the injection $\mathfrak{o}^m\to E$ induces an isomorphism $\mathfrak{o}^m/\Lambda\cong E/E'$. Thus, we need to find the order $\#(E/E')$ of E/E'. But $\mu(E)=1$. So $\#(E/E')=\mu(E')^{-1}$, and therefore it suffices to prove $\mu(E')=|\epsilon^r\delta^{-1}|_S$.

Consider the inequalities

$$(5)\qquad \epsilon^{-1}L_i(z)\le 1,\quad 1\le i\le r,\qquad z_j\le 1,\quad 1\le j\le m.$$

Let B be the coefficient matrix of the left-hand side of (5), and let $B_{\mathfrak{p}}$ be the $\mathfrak{p}$th component of B. Let $C_{\mathfrak{p}}$ denote the $m \times m$ submatrix of $B_{\mathfrak{p}}$ with the $\mathfrak{p}$-adic absolute value of its determinant maximum. Clearly, $\det C_{\mathfrak{p}} = \epsilon_{\mathfrak{p}}^{-j} \det D_{\mathfrak{p}}$ where $D_{\mathfrak{p}}$ is a $j \times j$ submatrix of $A_{\mathfrak{p}}$. I claim that $j = r$, and therefore, clearly, $\det D_{\mathfrak{p}} = \delta_{\mathfrak{p}}$. Suppose that $j < r$. The submatrix of $A_{\mathfrak{p}}$ with determinant $\delta_{\mathfrak{p}}$ yields a submatrix of $B_{\mathfrak{p}}$ with determinant $\epsilon_{\mathfrak{p}}^{-r}\delta_{\mathfrak{p}}$; so $|\epsilon^{-r}\delta|_{\mathfrak{p}} < |\epsilon_{\mathfrak{p}}^{-j} \det D_{\mathfrak{p}}|_{\mathfrak{p}}$ and therefore

$$|\epsilon|_{\mathfrak{p}} \geq |\epsilon|_{\mathfrak{p}}^{r-j} > |\delta_{\mathfrak{p}}/\det D_{\mathfrak{p}}|_{\mathfrak{p}} \geq |\delta/\delta'|_{\mathfrak{p}}$$

which is a contradiction.

We may assume $C_{\mathfrak{p}}$ appears in the same rows of $B_{\mathfrak{p}}$ for each $\mathfrak{p}$. We denote the submatrix of B in these m rows by C. The other rows of B may be represented as linear combinations of the m rows of C. By Cramer's rule, the coefficients in these combinations will be of the form $\det C'/\det C$ where C' is some submatrix of B. But, by the choice of C, $\det C'/\det C \leq 1$. Hence, because all $\mathfrak{p} \in S$ are nonarchimedean, the inequalities (5) hold if and only if the inequalities hold for the rows of C. Hence, $E' = C^{-1}E$ and therefore

$$\mu(E') = \mu(C^{-1}E) = (\operatorname{mod} C^{-1})\mu(E) = |\det C^{-1}|_S = |\epsilon^r\delta^{-1}|_S.$$

This proves the proposition.

A theorem similar to Proposition 4 may be found in [6].

Suppose L has the form

$$L_i(z) = \sum_{j=1}^{s} a_{ij}z_j - z_{s+i}, \qquad 1 \leq i \leq r, \tag{6}$$

with $m = r + s$. Then $\delta = \delta'$ and both are ideles.

We now compute the polar lattice of $\Lambda_{L,\epsilon}$ when L has the special form (6). Let M be the transposed system

$$M_j(w) = w_j + \sum_{i=1}^{r} a_{ij}w_{i+s}, \qquad 1 \leq j \leq s,$$

so that

$$z \cdot w = -\sum_{i=1}^{r} L_i(z)w_{i+s} + \sum_{j=1}^{s} M_j(w)z_j. \tag{7}$$

Define $\mathfrak{a}_L = \mathfrak{a}$ to be the integral ideal of k consisting of all a in $\mathfrak{o}$ for which $aa_{ij} \leq 1$, $1 \leq i \leq r$, $1 \leq j \leq s$. Also, define $\mathfrak{b} = \mathfrak{b}_\epsilon$ to be the ideal $\mathfrak{b}_\epsilon = \prod_{\mathfrak{p} \in S} \mathfrak{p}^{\nu_{\mathfrak{p}}(\epsilon)}$; so $a \in \mathfrak{o}$ is such that $a \leq \epsilon$ if and only if $a \in \mathfrak{b}$. We now prove

Proposition 5. $\mathfrak{b}_\epsilon \mathfrak{a}_L \Lambda_{L,\epsilon}^{-1} \subseteq \Lambda_{M,\epsilon}$. *If all the a_{ij} satisfy $a_{ij} \leq 1$, then equality holds.*

Proof. Let e_i be the m-tuple with 1 in the ith position and 0 elsewhere. It is clear $\mathfrak{b}\mathfrak{a}e_i \subseteq \Lambda_L$. So $(\mathfrak{b}\mathfrak{a}e_i)\cdot\Lambda_L^{-1}\subseteq \mathfrak{o}$, and therefore $\mathfrak{b}\mathfrak{a}\Lambda_L^{-1}\subseteq \mathfrak{o}^m$. Since k is dense in k_S, we can replace the a_{ij} by elements of k and still get the same lattices $\Lambda_{L,\epsilon}, \Lambda_{M,\epsilon}$. So assume $a_{ij}\in k$ and set

$$a_j = (0, \cdots, 0, 1, 0, \cdots, 0, a_{1j}, \cdots, a_{rj}) \in k^m$$

where the 1 is in the jth position. Because $\mathfrak{a}a_j \in \mathfrak{o}^m$ and $L_i(\mathfrak{a}a_j) = 0$, then $\mathfrak{a}a_j \subseteq \Lambda_L$, so $(\mathfrak{a}a_j)\cdot\Lambda_L^{-1}\subseteq\mathfrak{o}$. By (7), with $w\in\mathfrak{a}\mathfrak{b}\Lambda_L^{-1}$ and $z\in\mathfrak{a}a_j$, we get

$$M_j(\mathfrak{a}\mathfrak{b}\Lambda_L^{-1})\mathfrak{a} = (\mathfrak{a}\mathfrak{b}\Lambda_L^{-1})\cdot \mathfrak{a}a_j = \mathfrak{a}\mathfrak{b}(\Lambda_L^{-1}\cdot\mathfrak{a}a_j)\subseteq\mathfrak{a}\mathfrak{b}$$

so canceling the $\mathfrak{a}$'s we have $\mathfrak{a}\mathfrak{b}\Lambda_L^{-1}\subseteq\Lambda_M$, as desired.

Now assume $a_{ij}\le 1$ for all i and j. So $\mathfrak{a}=\mathfrak{o}$, and we can assume $a_{ij}\in\mathfrak{o}$. Let $w\in\Lambda_M$ and $z\in\Lambda_L$. Then $M_j(w)\in\mathfrak{b}$ and $L_i(z)\in\mathfrak{b}$. So, by equation (7), we see that $z\cdot w\in\mathfrak{b}$, and therefore $z\cdot(\mathfrak{b}^{-1}w)\in\mathfrak{o}$. This shows that $\mathfrak{b}^{-1}\Lambda_M\subseteq\Lambda_L^{-1}$, as desired.

It is easy to produce an example to show that equality does not in general hold in Proposition 5.

5. **Proof of the theorem.** Let ϵ be an idele with $\psi(0)\ge\epsilon\ge\psi(N)$ and satisfying (v′) $|\epsilon|_{\mathfrak{p}_1}|\epsilon|_{\mathfrak{p}_2}^{-1}\le C$ for all infinite primes $\mathfrak{p}_1, \mathfrak{p}_2$ where C is the constant of condition (v). Set $l_N = N/F(N)$ and note $1\le l_N\le N$ if N is sufficiently large. We first find an estimate of the number $\alpha(N,\epsilon)$ of solutions $x\in\mathfrak{o}^s$, and $y\in\mathfrak{o}^r$ of the inequalities

$$L_i(x) - y_i \le \epsilon, \qquad N - l_N \le \lceil x\rceil \le N.$$

Define systems $\overline{L}$ and $\overline{M}$ by the formulas

$$\overline{L}_i(z) = \begin{cases} z_i & \text{for } 1\le i\le s,\\[2ex] \dfrac{-l_N}{\epsilon^\infty}\left(\displaystyle\sum_{j=1}^{s} a^\infty_{i-s\,j}z_j - z_i\right) & \text{for } s+1\le i\le m,\end{cases}$$

$$\overline{M}_j(z) = \begin{cases} z_j + \displaystyle\sum_{i=1}^{r} a^\infty_{ij}z_{s+i} & \text{for } i\le j\le s,\\[2ex] \dfrac{\epsilon^\infty}{l_N} z_j & \text{for } s+1\le j\le m,\end{cases}$$

where for $a\in k_S$, as usual, a^∞ denotes the k_∞ component of a. Note, we are assuming real numbers such as l_N are embedded along the diagonal in k_∞. Let

L^0 be as in (1) and define M^0 by

$$M_j^0(z) = z_j + \sum_{i=1}^{r} a_{ij}^0 z_{s+i}, \qquad 1 \le j \le s.$$

In Example 2 of §4 we used L^0 and ϵ^0 to define an $\mathfrak{D}$-lattice $\Lambda_{L^0} = \Lambda_{L^0,\epsilon^0}$ with determinant

$$\det \Lambda_{L^0} = 2^{-mR_2} |d|^{m/2} |\epsilon^{-r}\delta|_{S_0}.$$

Then, by Example 1 of §4, $\Lambda = \overline{L}(\Lambda_{L^0})$ is an $\mathfrak{D}$-lattice with determinant

(8) $$\det \Lambda = \left| \left(\frac{l_N}{\epsilon^\infty} \right)^r \right|_\infty \det \Lambda_{L^0} = \frac{\gamma_1 l_N^{rn}}{|\epsilon|_S^r}$$

with $\gamma_1 = 2^{-mR_2} |d|^{m/2} |\delta(L)|_{S_0}$. We see $\alpha(N, \epsilon)$ is just the number of points of Λ in the region T of k_∞^m consisting of all $z \in k_\infty^m$ satisfying

$$N - l_N \le \overline{|x|} \le N, \qquad x = (z_1, \cdots, z_s),$$
$$z_i \le l_N, \qquad i = s+1, \cdots, m.$$

Let B_b be the boundary of T expanded by the diameter b of some fundamental parallelepiped of $\Lambda \subseteq \mathbf{R}^{nm}$. Then, if μ is Lebesgue measure on $\mathbf{R}^{nm}$, we see that

(9) $$\alpha(N, \epsilon) = \frac{\mu T}{\det \Lambda} + O\left(\frac{\mu B_b}{\det \Lambda} \right).$$

We have

$$\mu T = ((2^{R_1} \pi^{R_2} N^n)^s - (2^{R_1} \pi^{R_2} (N - l_N)^n)^s)(2^{R_1} \pi^{R_2} l_N^m)^r$$
$$= (2^{R_1} \pi^{R_2})^m l_N^{nr} \int_{N-l_N}^{N} ns t^{ns-1}\, dt$$

and

$$\mu B_b = O(N^{ns-1} l_N^{rn} b)$$

if

(10) $$b \ll l_N.$$

Using in (9) the value for $\det \Lambda$ given in (8), we get

(11) $$\alpha(N, \epsilon) = \gamma |\epsilon|_S^r \int_{N-l_N}^{N} t^{ns-1}\, dt + O(N^{ns-1} |\epsilon|_S^r b)$$

provided that (10) holds, where

$$\gamma = ns(2^{R_1}\pi^{R_2})^m/\gamma_1 = 2^{mR}\pi^{mR_2}|\delta(L)|_{S_0}^{-1}|d|^{-m/2}ns.$$

We now find an upper bound for b. Let $\mu_1, \cdots, \mu_{mn}$ be the successive minimum of Λ with respect to the distance function f^* polar to the distance function $f\colon k_\infty^m \to \mathbf{R}_+$, $f(z) = \lceil z \rceil$. It can be shown (see [3, Chapter V, Lemma 8]) there is a basis $c_1, \cdots, c_n$ of Λ satisfying $f^*(c_i) \leq \frac{1}{2}nm\mu_i$. So, if we choose b to be the diameter of the fundamental parallelepiped determined by this basis, we see that

$$b \leq \sum |c_i| \ll \sum f^*(c_i) \ll \mu_{nm}.$$

By Mahler's theorem (see [3]), if μ_1^* is the first minimum of Λ^* with respect to f, then

$$\mu_1^* \mu_{nm} \ll 1. \tag{12}$$

So, we can find an upper bound for μ_{nm} and hence for b by finding a lower bound for μ_1^*. This is where we use the type condition.

If $\mathfrak{a} = \mathfrak{a}_{L^0}$, $\mathfrak{b} = \mathfrak{b}_{\epsilon^0}$, and $\Lambda_{M^0} = \Lambda_{M^0, \epsilon^0}$ are defined as in §4, we know

$$\Lambda^*_{L^0} \subseteq \mathfrak{c}^{-1}\Lambda_{M^0} \tag{13}$$

where $\mathfrak{c} = \mathfrak{D}\mathfrak{b}\mathfrak{a} \subseteq \mathfrak{o}$. Now $\overline{M}$ and $\overline{L}$ are such that

$$\overline{M}(z') \cdot \overline{L}(z'') = z' \cdot z'';$$

so, as in Example 1 of §4, the lattices Λ and $\Lambda^* = \overline{M}(\Lambda^*_{L^0})$ are polar. Define

$$\breve{\Lambda} = \overline{M}(\mathfrak{c}^{-1}\Lambda_{M^0}).$$

Then, by (13), $\Lambda^* \subseteq \breve{\Lambda}$. So, if $\breve{\mu}_1$ is the first minimum of $\breve{\Lambda}$ with respect to f, then $\breve{\mu}_1 \leq \mu_1^*$. Hence, we will find a lower bound for $\breve{\mu}_1$.

Choose $z' \in \mathfrak{c}^{-1}\Lambda_{M^0}$ such that $f(z') = \lceil z \rceil = \breve{\mu}_1$. By a simple application of Minkowski's theorem, there is $c \in \mathfrak{c}$ such that

$$\lceil c \rceil \leq (2^{R_2}\pi^{-R_2}|d|^{1/2} \text{ Norm } \mathfrak{c})^{1/n}.$$

By the definition of $\mathfrak{b}$ given in §4, we have

$$\text{Norm } \mathfrak{b} = \prod_{\mathfrak{p} \in S_0} N\mathfrak{p}^{\nu_\mathfrak{p}(\epsilon)} = |\epsilon|_{S_0}^{-1},$$

so $\lceil c \rceil \ll |\epsilon|_{S_0}^{-1/n}$ and therefore, also, $|\text{Norm } c| \ll |\epsilon|_{S_0}^{-1}$ where the constants implied by $\ll$ do not depend on N.

We have $z = cz' \in \Lambda_{M^0} \subseteq \mathfrak{o}^m$. Hence, with this $z = (x, y)$, we have

$$x_j + \sum_{i=1}^{r} a_{ij}^0 y_i \leq \epsilon^0, \qquad 1 \leq j \leq s.$$

From the definition of f and $\overline{\Lambda}$ we see that

$$(14) \qquad \begin{aligned} x_j + \sum_{i=1}^{r} a_{ij}^\infty y_i &\leq c\bar{\mu}_1, & 1 \leq j \leq s, \\ y_i &\leq \frac{l_N}{\epsilon^\infty} c\bar{\mu}_1, & 1 \leq i \leq r. \end{aligned}$$

Hence $\max_j |x_j + M_j(y)|_S \leq |\text{Norm } c|\bar{\mu}_1^n|\epsilon|_{S_0} \ll \bar{\mu}_1^n$. By the type condition, this implies

$$(15) \qquad g(\lceil z \rceil)^{-1} \lceil z \rceil^{-rn/s} \ll \bar{\mu}_1^n.$$

By (14) and condition (v′) for ϵ,

$$\lceil y \rceil \leq l_N \bar{\mu}_1 \lceil c \rceil \lceil \epsilon^{-1} \rceil \ll l_N \bar{\mu}_1 |\epsilon|_{S_0}^{-1/n} |\epsilon|_\infty^{-1/n} = l_N \bar{\mu}_1 |\epsilon|_S^{-1/n}.$$

We also have $\lceil x \rceil \ll l_N \bar{\mu}_1 |\epsilon|_S^{-1/n}$ from (14), since $\epsilon \leq \psi(0)$ implies that $c\bar{\mu}_1 \leq (l_N/\epsilon^\infty) c\bar{\mu}_1$ for large N. Therefore $\lceil z \rceil \ll l_N \bar{\mu}_1 |\epsilon|_S^{-1/n}$, and then by (15)

$$|\epsilon|_S^{r/s} l_N^{-rn/s} \bar{\mu}_1^{-rn/s} g(\lceil z \rceil)^{-1} \ll \bar{\mu}_1^n.$$

Solving for $\bar{\mu}_1$ we get

$$(16) \qquad (|\epsilon|_S^r l_N^{-rn} g(\lceil z \rceil)^{-s})^{1/mn} \ll \bar{\mu}_1.$$

Minkowski's convex body theorem says $\bar{\mu}_1^{nm} \leq 2^{nm} \det(\overline{\Lambda})/V_f$ where V_f is the volume of the region defined by $f(z) \leq 1$. It is easy to see (in the same way we got (8)) that

$$\bar{\mu}_1^{nm} \ll \det(\overline{\Lambda}) = \text{Norm } c^{-m} (|\epsilon|_\infty^r l_N^{-rn}) (2^{-mR_2} |d|^{m/2} |\epsilon^{-s} \delta|_{S_0}) \ll |\epsilon|_S^r l_N^{-rn}.$$

So, by our bound for $\lceil z \rceil$, we have

$$\lceil z \rceil^{mn} \ll l_N^{mn} |\epsilon|_S^{-m} \bar{\mu}_1^{mn} \ll l_N^{sn} |\epsilon|_S^{-s} = N^{sn}/F(N)^{sn} |\epsilon|_S^s.$$

From condition (iii), it is now easy to see that $\lceil z \rceil \leq N^{s/r}$ if N is large. Hence by (16)

$$(|\epsilon|_S^r l_N^{-rn} g(N^{s/r})^{-s})^{1/mn} \ll \bar{\mu}_1 \leq \mu_1^*,$$

and therefore from (12) and condition (iii)

$$b \ll \mu_{nm} \ll (g(N^{s/r})^s l_N^{rn} |\epsilon|_S^{-r})^{1/mn} \ll l_N F(N)^{-1}.$$

Now (10) is clearly satisfied, so (11) now reads

$$\alpha(N, \epsilon) = \gamma|\epsilon|_S^r \int_{N-l_N}^N t^{ns-1}\,dt + O(N^{ns-1}|\epsilon|_S^r l_N F(N)^{-1}). \tag{17}$$

The rest of the proof follows Lang [5]. We apply formula (17) to $\epsilon = \psi(N)$ and $\epsilon = \psi(N - l_N)$ to get the theorem. Since ψ is decreasing we see

$$\alpha(N, \psi(N)) \leq \lambda(N) - \lambda(N - l_N) \leq \alpha(N, \psi(N - l_N)).$$

Then, by (17) with $\epsilon = \psi(N)$ and $\epsilon = \psi(N - l_N)$,

$$\begin{aligned}\lambda(N) - \lambda(N - l_N) &= \gamma|\psi(N)|_S^r \int_{N-l_N}^N t^{ns-1}\,dt \\ &\quad + O\left((|\psi(N-l_N)|_S^r - |\psi(N)|_S^r)N^{ns-1}l_N + \frac{|\psi(N-l_N)|_S^r N^{sn-1} l_N}{F(N)}\right).\end{aligned} \tag{18}$$

Note, F increasing implies $|\psi(t)|_S^r t^{sn}$ is also increasing. Hence

$$|\psi(N - l_N)|_S^r (N - l_N)^{sn} \leq |\psi(N)|_S^r N^{sn} \leq |\psi(N)|_S^r ((N - l_N)^{sn} + snN^{sn-1} l_N),$$

so

$$|\psi(N - l_N)|_S^r - |\psi(N)|_S^r \leq \frac{snN^{sn-1}|\psi(N)|_S^r l_N}{(N - l_N)^{sn}} \ll \frac{l_N|\psi(N)|_S^r}{N} = \frac{|\psi(N)|_S^r}{F(N)},$$

and therefore also $|\psi(N - l_N)|_S^r \ll |\psi(N)|_S^r$. Using these estimates in (18) we get

$$\lambda(N) - \lambda(N - l_N) = \gamma|\psi(N)|_S^r \int_{N-l_N}^N t^{sn-1}\,dt + O\left(\frac{|\psi(N)|_S^r N^{sn-1} l_N}{F(N)}\right). \tag{19}$$

Now $F(t) \to \infty$. So if N is large enough $N - l_N \geq N(1 - 1/F(N)) \geq N/2$ and therefore, because $\psi(t)$ and $1/F(t)$ are both decreasing,

$$\frac{|\psi(N)|_S^r N^{sn-1} l_N}{F(N)} \ll \frac{|\psi(N)|_S^r (N - l_N)^{sn-1} l_N}{F(N)} \leq \int_{N-l_N}^N \frac{|\psi(t)|_S^r t^{sn-1}}{F(t)}\,dt. \tag{20}$$

Also, because ψ is decreasing, we get

$$\begin{aligned}|\psi(N)|_S^r \int_{N-l_N}^N t^{sn-1}\,dt &= \int_{N-l_N}^N |\psi(t)|_S^r t^{sn-1}\,dt \\ &\quad + O((|\psi(N - l_N)|_S^r - |\psi(N)|_S^r)N^{ns-1} l_N).\end{aligned} \tag{21}$$

We have already estimated the error term in this last expression. Hence (19), (20), and (21) yield

$$\lambda(N) - \lambda(N - l_N) = \gamma \int_{N-l_N}^{N} |\psi(t)|_S^r t^{sn-1}\,dt + O\left(\int_{N-l_N}^{N} \frac{|\psi(t)|_S^r t^{sn-1}}{F(t)}\,dt\right).$$

Equation (4) now follows by induction.

6. **A metric theorem.** We put a measure on the space of all systems L of r linear forms in s variables by identifying the form L with an rs-tuple in k_S^{rs} made up of the coefficients of L. We will determine a type for almost all systems L. For simplicity, we restrict ourselves to the case when $S \supseteq P_\infty$. As preparation we state the following adele version of the convergence theorem:

Proposition 6. *Let* ϵ: $\mathbf{R}_+ \to k_S^*$. *If* $\sum_{x \in \mathfrak{o}^s} |\epsilon(\lceil x \rceil)|_S^r < \infty$ *then, for almost all systems* L*, there are only finitely many solutions* $x \in \mathfrak{o}^s$, $y \in \mathfrak{o}^r$ *of*

$$L_i(x) - y_i \leq \epsilon(\lceil x \rceil), \quad 1 \leq i \leq r. \tag{22}$$

This is the easy part of the Khinchin metric theorem; the other part asserts that, if the above sum diverges, then, under certain conditions, for almost all systems L (22) will have infinitely many solutions. A proof of this theorem for the adeles, in the case $s = 1$, may be found in [2].

If $k = \mathbf{Q}$ and $S = P_\infty$, the above proposition gives a type for almost all systems L. However, in the general case, type is defined in terms of an inequality on the volume $|\ |_S$ and not by simultaneous inequalities such as in (22), so the proposition does not apply directly. By modifying the proof of a theorem in [4, p. 96] we can get what we need, if the set of primes S is finite.

Proposition 7. *Let* $S \supseteq P_\infty$ *be a finite set of primes, and let* ϵ: $\mathbf{R}_+ \to \mathbf{R}_+$. *If*

$$\epsilon(t) < 1 \quad \textit{and} \quad \int_1^\infty t^{nm-1}\epsilon(t)^{r(1-\eta)}\,dt < \infty, \quad 1 > \eta > 0,$$

then for almost all L*, there are only finitely many* $x \in \mathfrak{o}^s$, $y \in \mathfrak{o}^r$ *satisfying*

$$\max_i |L_i(x) - y_i|_S \leq \epsilon(\lceil z \rceil), \qquad z = (x, y). \tag{23}$$

Proof. It is easy to see, if we replace (22) by

$$\inf\{1, L_i(x) - y_i\} \leq \epsilon(\lceil z \rceil), \qquad z = (x, y),\ 1 \leq i \leq r, \tag{24}$$

then the proof of Proposition 6 shows that for almost all systems L the inequalities (24) have only a finite number of solutions when $\int_1^\infty t^{nm-1}|\epsilon(t)|_S^r\,dt < \infty$ (the ϵ in (24) is as in Proposition 6, i.e., ϵ: $\mathbf{R}_+ \to k_S^*$).

For the proof of Proposition 7, we assume, for the sake of simplicity, that $r = 1$. Let F be the set of all L for which (23) has infinitely many solutions. Suppose (23) holds for $z = (x, y)$. If we put

$$\inf\{1, |L_1(x) - y_1|_{\mathfrak{p}}\} = \epsilon(\lceil z \rceil)^{\tau_{\mathfrak{p}}(z)}, \tag{25}$$

then $r_{\mathfrak{p}} = r_{\mathfrak{p}}(z) \geq 0$ and $\Sigma_{\mathfrak{p}}\, r_{\mathfrak{p}} \geq 1$. Let v be the number of elements in S, and choose a positive integer A so large that $v/A < \eta$. We have $A \leq [\Sigma_{\mathfrak{p}}\, Ar_{\mathfrak{p}}] \leq \Sigma_{\mathfrak{p}}\, [Ar_{\mathfrak{p}}] + v$; and therefore, if $B = A - v > 0$, then $B \leq \Sigma_{\mathfrak{p}}\, [Ar_{\mathfrak{p}}(z)]$. So there exists $b_{\mathfrak{p}} = b_{\mathfrak{p}}(z)$ such that $b_{\mathfrak{p}}$ is an integer and

$$0 \leq b_{\mathfrak{p}} \leq [Ar_{\mathfrak{p}}(z)] \leq Ar_{\mathfrak{p}}(z), \qquad \sum_{\mathfrak{p}} b_{\mathfrak{p}} = B. \tag{26}$$

There are only a finite number of possibilities for each $b_{\mathfrak{p}}$. So, if $L \in F$, we may assume, for each $\mathfrak{p} \in S$, $b_{\mathfrak{p}} = b_{\mathfrak{p}}(z)$ takes on the same value for infinitely many solutions $z = (x, y)$ of (23); i.e., we may assume $b_{\mathfrak{p}}$ takes on a value depending only on L and not on z. By (26), if we set $l_{\mathfrak{p}} = b_{\mathfrak{p}}/A$, then

$$0 \leq l_{\mathfrak{p}} \leq r_{\mathfrak{p}}, \qquad \sum_{\mathfrak{p}} l_{\mathfrak{p}} = B/A = (A - v)/A > 1 - \eta.$$

Then (25) implies there are infinitely many solutions of

$$\inf\{1, |L_1 - y_1|_{\mathfrak{p}}\} \leq \epsilon(\lceil z \rceil)^{l_{\mathfrak{p}}}. \tag{27}$$

Now $\Pi_{\mathfrak{p}}\, \epsilon(t)^{l_{\mathfrak{p}}} \leq \epsilon(t)^{1-\eta}$. Therefore, since $\int_1^\infty t^{nm-1}\epsilon(t)^{1-\eta}\,dt$ converges, we see that the set $E(b)$, $b = (b_{\mathfrak{p}})_{\mathfrak{p}\in S}$, for which (27) has infinitely many solutions, has measure zero. But $F \subseteq \bigcup E(b)$ where the union is over all tuples $b = (b_{\mathfrak{p}})$ with $b_{\mathfrak{p}} \geq 0$ and $\Sigma_{\mathfrak{p}}\, b_{\mathfrak{p}} = B$. So the measure of F is also zero. This proves Proposition 7.

If we apply Propositon 7 to the transposed system M of s forms in r variables, we find that by taking $g(t)$ so that

$$\int_1^\infty \frac{t^{nm-1}}{(g(t)t^{rn/s})^{s(1-\eta)}}\,dt \text{ converges,} \tag{28}$$

then almost all L have type $\leq g$. So, for a g satisfying (28) and a ψ satisfying conditions (ii)–(v), we have that formula (4) holds for almost all L.

It may be possible that Proposition 7 can be refined, and therefore a better metric theorem would result. For example, in the case $k = \mathbf{Q}$, $S = P_\infty$, almost all systems have type $\leq \log^{1+\eta} t$, while Proposition 7 can never give a type any better than $O(t^\alpha)$. Also, in the case $k = \mathbf{Q}$ and S consists of one p-adic prime, one can show almost all systems L have type $\leq \log^{1+\eta} t$ (see the Khinchin metric theorem in [6] where it is shown that almost all p-adic systems

$$|L_i(x) - y_i|_p \leq \epsilon(t), \qquad t = \max_{i,j}\{|x_j|, |y_i|\}$$

have only a finite number of solutions, if $t\epsilon(t)$ is decreasing and $\Sigma\, t^{m-1}\epsilon(t)^r < \infty$). However, if S contains more than one infinite prime it seems unlikely the integral in Proposition 7 can be improved to anything better than

$$\int_1^\infty t^{ns-1}\epsilon(t)^r \log \epsilon(t)^{-1}\, dt$$

since, for example, the measure of the set

$$\{(a, b) \in \mathbf{R}^2 \colon \inf\{1, |a|\} \inf\{1, |b|\} \le \epsilon\}$$

is of the form $2\epsilon(1 + 2\log \epsilon^{-1})$.

In the case $s < r$ our theorem will still hold if we replace the definition of type with the following definition of ψ-type:

Definition. Let $g\colon \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing function, and let $\psi\colon \mathbf{R}_+ \to k_S^*$. Define $\epsilon(t)$ by the formulas

$$\epsilon_{\mathfrak{p}}(t) = \psi_{\mathfrak{p}}(t) \quad \text{for } \mathfrak{p} \in S_0,$$
$$\epsilon_{\mathfrak{p}}(t) = (g(t)t^{rn/s}|\psi(t)|_{S_0})^{-1/n} \quad \text{for } \mathfrak{p} \in P_\infty,$$

Then we say the system L has ψ-type $\le g$, if $M_j(y) - x_j \le \epsilon(\lceil y \rceil)$, $1 \le j \le s$, has only finitely many solutions $y \in \mathfrak{O}^r$ and $x \in \mathfrak{O}^s$.

In this case we may apply the Khinchin convergence theorem (Proposition 6) directly to obtain the following metric corollary to the type theorem:

Proposition 8. *Assume* $s < r$. *If* $\int_1^\infty g(t)^{-s}t^{-1}\,dt$ *converges and conditions* (ii) *through* (v) *of the type theorem hold, then*

$$\lambda(N) \sim \gamma \int_1^N t^{sn-1}|\psi(t)|_S^r\, dt$$

for almost all systems L.

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