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FIELDS OF CONSTANTS OF INTEGRAL DERIVATIONS ON A p-ADIC FIELD

BY

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ABSTRACT. Let K_0 be a p-adic subfield of a p-adic field K with residue fields $k_0 \subset k$. If K_0 is algebraically closed in K and k is finitely generated over k_0 then K_0 is the subfield of constants of an analytic derivation on K or equivalently, K_0 is the invariant subfield of an inertial automorphism of K. If (1) k_0 is separably algebraically closed in k, (2) $[k_0^{p^{-1}} \cap k: k_0] < \infty$, and (3) k is not algebraic over k_0 then there exists a p-adic subfield K_0 over k_0 which is algebraically closed in K. All subfields over k_0 are algebraically closed in K if and only if k_0 is algebraically closed in k. Every derivation on k trivial on k_0 lifts to a derivation on K trivial on K_0 if k is separable over k_0. If k is finitely generated over k_0 the separability condition is necessary. Applications are made to invariant fields of groups of inertial automorphisms on p-adic fields and of their ramification groups.

I. **Introduction.** Let d be a derivation on a field K having characteristic zero. It is well known that the field of constants K_0 of d is algebraically closed in K. Conversely R. Baer showed in 1927 [1] that given any subfield K_0 of K algebraically closed in K there is a derivation on K having K_0 as field of constants.

If K is a p-adic field and d is an analytic derivation on K, that is, d is continuous in the p-adic topology, then the field of constants K_0 of d is a p-adic subfield containing the inertial subfield K^*. Thus, generally, in this paper $K \supset K_0 \supset K^*$ will be p-adic fields with residue fields $k \supset k_0 \supset k^*$, k^* being the maximal perfect subfield of k. The main result of §II states that if K_0 is a p-adic subfield of K, algebraically closed in K and k is finitely generated over k_0 then K_0 is the field of constants of an analytic derivation on K (Theorem 2.6). For convenience of language we will call a field of constants of an analytic derivation on K simply a constant subfield, or constant field.

In §III we investigate which subfields k_0 of k are residue fields of constant subfields of K by determining which k_0 have p-adic fields over them which are algebraically closed in K. The principal result in this direction (Theorems 3.5

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and 3.6) states that if (1) k_0 is separably algebraically closed in k, (2) $[k_0^{p^{-1}} \cap k: k_0] < \infty$, and (3) k is not algebraic over k_0 then there exists a p-adic subfield K_0 algebraically closed in K having k_0 as residue field. All p-adic subfields over k_0 are algebraically closed in K if and only if k_0 is algebraically closed in k. Results of §§II and III combine to indicate, assuming that k is finitely generated over k_0, exactly when there is at least one constant subfield over k_0 and the circumstances under which all p-adic subfields over k_0 are constant subfields. The conditions are, respectively, k_0 separably algebraically closed in k and k_0 algebraically closed in k (Theorem 3.10).

Example 1 of the Appendix exhibits a subfield k_0 algebraically closed in k and two p-adic subfields K_0 and K_0' each having k_0 as residue field such that K_0 is a constant subfield and K_0' is not. Needless to say k is not finitely generated over k_0.

In Example 2 a subfield k_0 of a field k of characteristic p is constructed such that k_0 is algebraically closed in k and the maximal perfect subfield of k is contained in k_0. However, if K is a p-adic field with residue field k and K_0 is any p-adic subfield with residue field k_0 the only analytic derivation on K whose constant field contains K_0 is the zero derivation. Thus, though K_0 is algebraically closed in K and $K^* \subset K_0$, K_0 is not the field of constants of any set of analytic derivations.

§IV extends results of Heerema on lifting derivations on k to K. In [4] it is shown that every derivation on k lifts to a derivation on K. Theorem 4.1 addresses the question: Under what circumstances will all derivations on k trivial on k_0 lift to a derivation on K trivial on K_0? The result is that if k is separable over k_0 every derivation on k over k_0 (k_0 in the field of constants) lifts to a derivation on K over K_0. The converse also holds if k is finitely generated over k_0. An example (Example 3 of the Appendix) is provided which illustrates that the converse need not hold if k is not finitely generated over k_0.

The relationship between the results discussed above and inertial automorphisms on K is treated in §V. The group G_0 of inertial automorphisms of a p-adic field K was first studied in 1939 by S. Mac Lane [9]. He defined the ramification series of G_0, determined the factors of successive terms of the ramification series and showed that K^*, the inertial subfield of K is the invariant subfield of G_0 as well as of each term of the ramification series. Using derivations some of Mac Lane's work has been generalized to ramified extensions of p-adic fields by Heerema [2]–[5], Neggers [11], and Peter [12]. Heerema has noted that the inertial automorphisms of K are all of the form $\exp(pd)$ where d is an integral derivation on K (d maps integers onto integers), and p is the characteristic of the residue field of K [2]. If $p \neq 2$ the field of constants of d is also the invariant subfield of $\exp(pd)$, that is, $d(a) = 0$ if and only if

$\exp(pd)(a) = a$. Thus if K_0 is a subfield of K with residue fields $k \supset k_0$ and k is finitely generated over k_0 then the intermediate subfields of K which are invariant fields for groups of inertial automorphisms are precisely the fields of constants of analytic derivations. These in turn are the intermediate p-adic subfields which are algebraically closed in K. This is contained in Theorem 5.2. Theorem 5.2 also contains an extension of a result of Mac Lane's referred to above: If L is the group of all inertial automorphisms with invariant subfield K_1 then K_1 is the invariant field of each term of the ramification series of L.

II. Subfields of constants. We assume a familiarity with the subject of derivations on fields to the extent covered in Jacobson [7, Chapter IV]. Given a field K and a subfield K_0 algebraically closed in K, R. Baer has shown [1] that a derivation d on K having K_0 as field of constants can be constructed as follows.

Case 1. Assume first that the transcendency degree of K over K_0 is finite and let $\{a_1, \cdots, a_n\}$ be a transcendency basis for K over K_0. The conditions $d(a_1) = 1$, and $d(a_i) = a_{i-1}^{-1}$ for $i = 2, \cdots n$ uniquely determine a derivation on K having K_0 as field of constants.

Case 2. Let $T = \{a_\alpha\}$ be a transcendency basis for K over K_0 well ordered with ordinal γ which we can assume to be a limit ordinal. Let d_0 be the zero derivation on K_0 and let $K_\alpha = K_0(\{a_\beta\}_{\beta \le \alpha})$. For each $\alpha < \gamma$, d_α is defined to be the derivation on K_α into $K_{\alpha+1}$ which extends d_ρ for $\rho < \alpha$ and such that $d_\alpha(a_\alpha) = a_{\alpha+1}$. This defines a derivation on $K_0(T)$ whose unique extension to K is the desired derivation.

We indicate the proof of the following known fact [1, Corollary 1, p. 20] for the reader's convenience.

2.0 **Lemma.** *Let d be a derivation on a field K with field of constants K_0. Let u be separably algebraic over K and let d_1 be the unique extensions of d to $K(u)$. Then u is a d_1 constant if and only if u is separably algebraic over K_0.*

We prove the necessity only. Let $g(X) = \Sigma a_i X^i$ be the minimum polynomial of u over K. With $g^d(X) = \Sigma d(a_i) X^i$ and $g'(X) = \Sigma i a_i X^{i-1}$ we have $g^d(u) + g'(u) d_1(u) = 0$. Since $g'(u) \neq 0$, $d_1(u) = 0$ only if $g^d(u) = 0$ or only if $g^d(X) = 0$, that is only if $g(X) \in K_0[X]$.

Henceforth K will represent a p-adic field with ring of integers R, and residue field k having characteristic p. Also $H: K \to k$ will denote the natural place. A derivation d on K is analytic, or continuous in the p-adic topology if $d(R) \subset p^{-n}R$ for some integer n. If $d(R) \subset R$, d is an integral derivation. Let $\mathrm{Der}(K)$ denote the K space of analytic derivations on K. Thus if $d \in \mathrm{Der}(K)$ then for some n, $p^n d$ is integral.

The field of constants of $d \in \mathrm{Der}(K)$ is clearly closed in the p-adic topology and is thus a p-adic subfield. We have

2.1 Proposition. *The field of constants K_0 of d in* Der(K) *is a p-adic subfield of K algebraically closed in K.*

For the following lemma let K_2 be a field having a p-adic valuation, ring of integers R_2 and residue field k_2. Let K_1 be a subfield of K_2 with residue field k_1. Assume next that U is a set of units in K_2 whose residues are algebraically independent over k_1 (U is a set of units in the valuation ring of K_2).

2.2 Lemma. *If d is a derivation mapping $K_1(U)$ into K_2 such that $d(K_1 \cap R_2) \subset R_2$ and $d(U) \subset R_2$ then $d(K_1(U) \cap R_2) \subset R_2$.*

Proof. An element a in $K_1(U) \cap R_2$ has the rational form $f(u_1 \cdots u_n) g(u_1 \cdots u_n)^{-1}$, the u_i being in U. Also, $f(X_1 \cdots X_n)$ and $g(X_1 \cdots X_n)$ have integer coefficients and at least one coefficient of $g(X_1 \cdots X_n)$ is a unit. Then $g(u_1 \cdots u_n)$ is a unit since $H(u_1), \cdots, H(u_n)$ are algebraically independent over k_1. Now $d(f(u_1 \cdots u_n)) \in R_2$, $d(g(u_1 \cdots u_n)) \in R_2$ and hence $d(f(u_1 \cdots u_n) g(u_1 \cdots u_n)^{-1}) \in R_2$.

We now assume $K_2 \supset K_1$ as above and that K_1 is the field of constants of an integral derivation d on K_2.

2.3 Lemma. *The derivation d extends uniquely to an integral derivation d^* on the completion K_2^* of K_2. The field of constants of d^* is the closure K_1^* of K_1 in K_2^*.*

We show only that if $d^*(b) = 0$ for a unit b in K_2 then $b \in K_1^*$.

Let $b = \Sigma\{p^i b_i \mid b_i \in R_2, i = 0, 1 \cdots\}$. Since $d^*(b) = 0$ we have $d(b_0) \in pR_2$ or $d(b_0) = pa_0$, $a_0 \in R_2$, and $pd(b_1) + p(b_0) \in p^2 R_2$ or $pd(b_1) + pa_0 \in p^2 R_2$. Hence $d(b_1) = -a_0 + pa_1 \in R_2$ and $pd(b_1) + d(b_0) = p^2 a_1$. Proceeding inductively, we have $d(b_i) = -a_{i-1} + pa_i$ with a_{i-1} and a_i in R_2 and $d(p^i b_i + \cdots + b_0) = p^{i+1} a_i$. Since $d(p^{i+1} b_{i+1} + \cdots + b_0) = p^{i+1} d(b_{i+1}) + p^{i+1} a_i \in p^{i+2} R_2$, $d(b_{i+1}) = -a_i + pa_{i+1}$ for a_{i+1} in R_2. If $f_i = p^{-(i+1)} \Sigma\{p^j b_j \mid j = 0, \cdots, i\}$ then $d(f_i) = a_i$ for $i \geq 0$ and hence $c_0 = b_0 - pf_0$ as well as $c_i = b_i + f_{i-1} - pf_i$ are all d constants. That is $c_i \in K_1$ for $i \geq 0$. But $b = \Sigma p^i b_i = c_0 + pf_0 + \Sigma(-f_{i-1} + c_i + pf_i)p^i = \Sigma_i c_i p^i$. Thus $b \in K_1^*$.

For the following theorem we assume K_0 to be a subfield of a p-adic field K the residue fields being $k_0 \subset k$.

2.4 Theorem. *If K_0 is the field of constants of an analytic derivation on K then K_0 is a p-adic subfield algebraically closed in K. Conversely, if K_0 is* *a p-adic subfield of K algebraically closed in K and* (a) *k is finitely generated over* k_0 *then* K_0 *is the constant field of an analytic derivation on K.*

Example 1 in the Appendix shows that condition (a) is not necessary. It also indicates that there are no necessary and sufficient conditions on the residue field only. It also demonstrates that k_0 need not be algebraically closed in k (see Theorem 3.10).

Proof. The first sentence is Proposition 2.1. Proceeding to the converse, let T be a set of representatives in K of a transcendency basis $\overline{T}$ of k over k_0. Define d_1 on $K_0(T)$ having K_0 as field of constants according to the prescription given at the beginning of this section. By Lemma 2.2 d_1 is integral. The extension d_1^* of d_1 to the closure $K_0(T)^*$ of $K_0(T)$ in K also has K_0 as field of constants by Lemma 2.3. Since k is finitely generated over k_0, $[k: k_0(\overline{T})] < \infty$. Hence $[K: K_0(T)^*] < \infty$. Let u be a primitive element for K over $K_0(T)^*$ having minimum function $f(X)$. Denoting the extension of d_1^* to K by d, we have $d(u) = f^d(u)/f'(u)$ which is in $p^{-n}R$ where $n = V(f'(u))$. Hence d is analytic. By Lemma 2.0 the field of constants of d is K_0.

Using the methods of §III one can also prove that if instead of condition (a) it is assumed that k has a transcendency basis $\overline{T}$ over k_0 such that k has bounded exponent over $k_0(\overline{T})$ then, retaining the assumption on K_0, there is a p-adic subfield K_0' with residue field k_0 which is the field of constants of an analytic derivation on K. Since this result is essentially independent of the rest of the paper we do not prove it. It does, however, relate to Example 1 of the Appendix.

For fields of characteristic zero the subfields which are constant fields of higher derivations (of finite or infinite rank) are precisely the fields of constants of derivations. This follows from the fact that if $\{D_i\}$ is a higher derivation on such a field M, then there is a sequence $\{d_i\}$ of derivations on M such that

$$(2.5) \qquad D_j = \sum \{d_{i_1} \cdots, d_{i_r}/r! \mid i_1 + \cdots + i_r = j\}$$

for all j [4, Theorem 3]. One sees directly that if $\{D_i\}$ is an analytic higher derivation on a p-adic field; that is, each D_i is continuous, then the derivations of (2.5) are also analytic and conversely. The fields of constants of $\{D_i\}$ is the intersection of the field of constants of d_i, $i = 1, \cdots, r$, where $r =$ rank D_i ($r \leq \infty$). Since the intersection of any number of p-adic subfields of K algebraically closed in K is a p-adic subfield algebraically closed in K we can restate Theorem 2.4 replacing analytic derivation with analytic higher derivations as follows.

2.6 **Theorem.** *If K_0 is the field of constants of an analytic higher derivation on K then K_0 is a p-adic subfield of K algebraically closed in K. Conversely, if K_0 is a p-adic subfield of K algebraically closed in K and* (a) *k is finitely generated over k_0 then K_0 is the constant field of an analytic higher derivation.*

Proof. To prove the converse we simply take an analytic derivation d on K having K_0 as field of constants. The rank r higher derivation $\{D_i \mid i = 0, \cdots, r\}$ where $D_i = d^i/i!$ has field of constants K_0.

III. **Residue fields of constant subfields.** As the heading suggests we consider in this section those subfields k_0 of the residue field k of a p-adic field K which are residue fields of constant fields, or more generally, residue fields of p-adic subfields algebraically closed in K and containing the inertial subfield K^*.

3.1 **Theorem.** *Every p-adic subfield K_0 with $K^* \subset K_0 \subset K$ is algebraically closed in K if and only if k_0, the residue field of K_0, is algebraically closed in k.*

Proof. If K_0 is not algebraically closed in K, choose u in $K \setminus K_0$, algebraic over K_0. k_0 is not the residue field of $K_0(u)$ [9, Lemma 4, p. 430] so there exists a unit v in $K_0(u)$ with residue $H(v)$ not in k_0. If $\Sigma a_i X^i$ is the minimum polynomial of v over K_0, then $H(v)$ satisfies the polynomial $\Sigma H(a_i)X^i$ in $k_0[X]$ and k_0 is not algebraically closed in k.

Converesly, if k_0 is not algebraically closed in k, the p-adic field $K' \supset K_0$ may be constructed using standard methods [4, pp. 377–378] having k as residue field and such that K_0 is not algebraically closed in K'. There is an analytic isomorphism from K' onto K [9, Corollary 1, p. 431]. Taking K_0' as the image of K_0 under this map, K_0' has residue field k_0 and is not algebraically closed in K.

If $K_0 \subset K$ is algebraically closed in K, Hensel's lemma yields easily that k_0 is separably algebraically closed in k. In the following we develop a partial converse of this statement.

3.2 **Lemma.** *Suppose D is a derivation on a field K of characteristic zero with field of constants K_0. Then the equation $D(y) + (i/m)[D(b)/b]y = 0$, where $b \in K$, $b \neq 0$, and i and m are positive rational integers, has a nontrivial solution for $y \in K$ if and only if b^i is of the form ac^m for some $a \in K_0$ and $c \in K$.*

Proof is by showing that c^{-1} solves the equation in one case and, conversely, if c^{-1} solves the equation, then $D(b^i)/b^i = D(c^m)/c^m$ so b^i and c^m differ by a factor which is a D-constant.

3.3 **Lemma.** *Suppose $K_0 \subset K_1$ are fields of characteristic zero, K_0 algebraically closed in K_1. Let u be algebraic over K_1 with minimum polynomial $f(X) = X^m - b$. Then K_0 is algebraically closed in $K_1(u)$ if and only if b^i is not of the form ac^m with a in K_0, c in K_1, for $i = 1, 2, \cdots, m-1$.*

Proof. By Baer [1, Theorem 5, p. 24], there exists a derivation D on K_1 with field of constants K_0. D extends to a derivation D^* on $K_1(u)$ with $D^*(u) = D(b)/mu^{m-1}$. It suffices to show that K_0 is the field of constants of D^*. Letting $c = \sum_{i=0}^{m-1} c_i u^i$, $c_i \in K_1$, be a D^*-constant, linear independence of $\{1, u, u^2, \cdots, u^{m-1}\}$ yields the equations:

$$D(c_0) = 0 \qquad \text{so } c_0 \in K_0,$$

$$D(c_i)b + (i/m)c_i D(b) = 0, \qquad i = 1, 2, \cdots, m-1.$$

The result follows from Lemma 3.2.

The following simplify the use of Lemma 3.3.

3.4 **Lemma.** *Let $K_0 \subset K_1$ be fields. If $b \in K_1$ is such that $b^i = ac^m$ for some $a \in K_0$, $c \in K_1$, $0 < i < m$, and i is the smallest such integer for b^i to have this form, then i divides m.*

Proof. $b^{\text{g.c.d.}(i,m)}$ has the form ac^m, a in K_0, if b^i does.

3.5 **Corollary.** *Under the hypotheses of Lemma 3.4, if $m = p^n$ where p is a prime, then i is a power of p.*

3.6 **Theorem.** *Let K be a p-adic field with residue field k. Suppose $k_0 \subset k$ is such that:*

(1) *k_0 is separably algebraically closed in k,*

(2) *$[k_0^{p^{-1}} \cap k : k_0] < \infty$, and*

(3) trans deg $k/k_0 \neq 0$.

Then there exists some p-adic subfield $K_0 \subset K$ with residue field k_0 such that K_0 is algebraically closed in K. All such K_0 are algebraically closed in K if and only if k_0 is algebraically closed in k.

Define the function $\phi_{K_0, K}: k_0 \cap k^p \backslash \{0\} \rightarrow k^+/k_0^+$ (the additive groups of k and k_0) as follows: For any $x \in k_0 \cap k^p$, let $y = x^{p^{-1}}$. Let $c \in K_0$ and $b \in K$ be such that $H(c) = x^{-1}$ and $H(b) = y$. Note that $1 \equiv cb^p$, mod pR. Define

$$\phi_{K_0,K}(x) = H((1 - cb^p)/p) + k_0^+.$$

$\phi = \phi_{K_0, K}$ is seen to be well defined. It is easily seen that if $k_0 \subset k_1 \subset k$ with p-adic over fields $K_0 \subset K_1 \subset K$ then $\phi_{K_0, K_1} = \phi_{K_0, K}|_{k_0 \cap k_1^p}$.

3.7 **Lemma.** *If* K_0 *is algebraically closed in* K *and* u *is a root of the irreducible polynomial* $X^p - v(1+pt)$, *with* v *in* $R \cap K_0$ *and* t *a unit in* K *and if* $H(t)$ *is not in* $\phi(k_0 \cap k^p)$, *then* K_0 *is algebraically closed in* $K(u)$.

Proof. K_0 is algebraically closed in $K(u)$ unless $v(1+pt) = b^p/a$, with a in K_0, b in K or, since $v \in K_0$, unless $(1+pt) = b^p/a$ with a in K_0, b in K. This follows from Lemma 3.3 and Corollary 3.5. We can assume b and a to be units. If $a(1+pt) = b^p$ then $H(a) \in k_0 \cap k^p$. Also, by definition of ϕ, $\phi(a) = H(-t)$ and the result follows.

3.8 **Lemma.** *Let* k_1/k_0 *be separable (in the linearly disjoint sense) and let* U *be a subset of an overfield such that* $U^p \subset k_0$. *If* $v \in k_1(U)$ *with* $v^p \in k_0$, *then* $v \in k_0(U)$.

Proof. Note first that $k_1(U) = k_1 \otimes_{k_0} k_0(U)$. Then if $\{y_\alpha\}$ is a linear basis for k_1/k_0 with $1 \in \{y_\alpha\}$ and $\{x_\beta\}$ a linear basis for $k_0(U)/k_0$,

$$(3.9) \quad v = \sum_\alpha \sum_\beta w_{\beta\alpha} z_\beta y_\alpha, \quad w_{\beta\alpha} \in k_0, \quad v^p = \sum_\alpha \Big(\sum_\beta w^p_{\beta\alpha} z^p_\beta\Big) y^p_\alpha \in k_0.$$

If none of the y_α in (3.9) is 1, separability yields $v = 0$. If $y_\gamma = 1$ in (3.9), separability gives:

$$v^p = \sum_\beta w^p_{\beta\gamma} z^p_\beta, \qquad v = \sum_\beta w_{\beta\gamma} z_\beta \in k_0(U).$$

Proof of Theorem 3.6. Let k_1 be the subfield of k obtained by first extending k_0 by a transcendency basis for k/k_0, and then taking the separable algebraic closure of this field in k. Then k_1 is separable over k_0 (in the linearly disjoint sense). k_0 is algebraically closed in k_1, and k is purely inseparable over k_1. Let $U = \{u_1, u_2, \cdots, u_n\}$ be a p-basis for $k_0^{p^{-1}} \cap k$ over k_0. Let $k_2 = k_1(U)$. Let $K_0' \subset K_1$ be p-adic fields having $k_0 \subset k_1$ as residue fields. By Theorem 3.1, K_0' is algebraically closed in K_1. Let t_1 be in $k_1 \setminus k_0$ and choose t in K_1 so that $H(t) = t_1$. Since k_0 is algebraically closed in k_1, ϕ is trivial on $k_0 \cap k_1^p$ so $t_1^{(n+1)!} + k_0^+$ is not in $\phi(k_0 \cap k_1^p)$. Let $V = \{v_1, \cdots, v_n\} \subset K_0'$ be such that $H(v_i) = u_i^p$. Using Mac Lane's construction procedure [8, pp. 377–378] let $K_1 = K_{1,0}$ and $K_{1,i} = K_{1,i-1}(w_i)$ where w_i is a root of $x^p - v_i(1 + pt^{(n-i+2)!})$, and let $K_2 = K_{1,n}$. $K_{1,i}$ has residue field $k_1(u_1, \cdots, u_i)$. We wish to show that K_0' is algebraically closed in K_2 by observing that for $i = 1, \cdots, n$, $t_1^{(n-i+1)!}$ is not in $\phi[k_0 \cap [k_1(u_1, \cdots, u_i)]^p]$ and using Lemma 3.8. Let $x \in k_0$, $y \in k_1(u_1, \cdots, u_i)$ where $y^p = x^{-1}$. By Lemma 3.8, y is in $K_0(u_1, \cdots, u_i)$. Choose $X \in K_0$ and $Y \in K_0(w_1, \cdots, w_i)$ so that $H(X) = x$, $H(Y) = y$. By definition of the w_i's, $H(Y^p) \in k_0(t_1^{(n-i+2)!}, u_1, \cdots, u_i)$ and hence $\phi(x) \in k_0(t_1^{(n-i+2)!}, u_1, \cdots, u_i)$. Thus $t_1^{(n-i+1)!}$ is not in $\phi[k_0 \cap (k_1(u_1, \cdots, u_i))^p]$. It follows that K_0' is algebraically closed in K_2

and $t_1 \notin [k_0 \cap k_2^p] = [k_0 \cap k^p]$.

It follows from this fact and the proof of Lemma 3.7 that if K_3 is any p-adic field with residue field k_3 such that $K_2 \subset K_3$ and $k_2 \subset k_3 \subset k$, then $1 + pt$ is not of the form ac^p for any $a \in K_0'$ and $c \in K_3$. K' is constructed over K_2 by Mac Lane's procedure, well-ordering a set of generators for k/k_2 such that each extension is by a pth root. At each step, if the corresponding polynomial over the p-adic field K_α at that stage is $x^p - b_\alpha$ with b_α of the form ac^p, $a \in K_0'$ and $c \in K_\alpha$, a second choice, $b_\alpha(1 + pt)$ is not of this form and the polynomial $x^p - b_\alpha(1+pt)$ is to be used instead. By transfinite induction K_0' is seen to be algebraically closed in K'. There exists an analytic isomorphism from K' onto K [9, Corollary 1, p. 431]. Letting K_0 be the image of K_0' under this isomorphism establishes the result.

3.9. **Corollary.** *Let K be a p-adic field with residue field k. Suppose $k_0 \subset k$ is such that*

(1) *k is finitely generated over k_0,*

(2) trans deg $k/k_0 \neq 0$.

Then there exists some p-adic subfield $K_0 \subset K$ with residue field k_0 such that K_0 is algebraically closed in K if and only if k_0 is separably algebraically closed in k. All such K_0 are algebraically closed in K if and only if k_0 is algebraically closed in k.

Combining Corollary 3.9 and Theorem 2.4 we have

3.10 **Theorem.** *Let K be a p-adic field with residue field k and let k_0 be a subfield of k such that k is finitely generated over k_0 but is not algebraic over k_0. Then there exists some p-adic subfield $K_0 \subset K$ with residue field k_0 such that K_0 is the constant field of an analytic derivation if and only if k_0 is separably algebraically closed in k. All such K_0 are constant fields of analytic derivations if and only if k_0 is algebraically closed in k.*

Example 1 of the Appendix demonstrates that hypothesis (2) of Theorem 3.6 is not necessary for K_0 is algebraically closed in K whereas $\bar{x}_n = \bar{t}^{p^n} - \bar{u}_n^{p^n}$, for $n \geq 1$, and so (2) fails.

IV. **Lifting derivations.** In this section we extend results of the second author on lifting derivations from k to K. In particular, in 1962 [4] Heerema showed that all derivations on k lift to derivations on K. The following generalization involves field of constant conditions on both fields. Let $\mathrm{Der}(k/k_0)$ be the subspace of those derivations on k which restrict to the zero map on k_0. $\mathrm{Der}(K/K_0)$ is similarly defined.

4.1 Theorem. *Let $K \supset K_0$ be p-adic fields with residue fields $k \supset k_0$. If k is separable over k_0 then every δ in* $\mathrm{Der}(k/k_0)$ *lifts to d in* $\mathrm{Der}(K/K_0)$. *Conversely, if k is finitely generated over k_0 and every δ in* $\mathrm{Der}(k/k_0)$ *lifts to* $\mathrm{Der}(K/K_0)$ *then k is separable over k_0.*

Example 2 of the Appendix shows that the converse need not obtain if k is not finitely generated over k_0.

Proof. Let $\bar{S}$ be a p-basis for k_0. Since k is separable over k_0 a p-basis for k can be constructed by joining to $\bar{S}$ a p-basis $\bar{T}$ for k over k_0. Choose a set of representatives S for $\bar{S}$ from K_0 and a set T of representatives of $\bar{T}$ from K. Each $\delta \in \mathrm{Der}(k/k_0)$ is determined by the set $\{\delta(\bar{t}) | \bar{t} \in \bar{T}\}$. Similarly each d in $\mathrm{Der}(K)$ is determined by its action on a set of representatives of a p-basis for k [6, Theorem 4]. We define d in $\mathrm{Der}(K)$ by the conditions $d(s) = 0$ for $s \in S$ and $d(t) = a_t$, where $H(a_t) = \delta(\bar{t})$, for $t \in T$. Clearly d is trivial on K_0 and $Hd = \delta H$, that is, d induces δ.

Conversely, suppose k to be finitely generated over k_0 but not separably generated. Then $\mathrm{Der}(k/k_0)$ has dimension over k greater than the transcendency degree of k over k_0 [7, Corollary, p. 179]. In constructing K over K_0 by the procedure described in Mac Lane [8] K_0 is extended by an algebraically independent set T of cardinality the transcendency degree of k over k_0. By Lemma 2.3, any integral derivation on K over K_0 is uniquely determined by its action on T since once it is defined on T it extends uniquely to the closure of $K_0(T)$ and then to a separably algebraic extension. Therefore the dimension of the subspace of $\mathrm{Der}(k/k_0)$ induced by $\mathrm{Der}(K/K_0)$ is no more than the cardinality of T which is less than the dimension of $\mathrm{Der}(k/k_0)$. The result follows.

V. Inertial automorphisms and analytic derivations. F. K. Schmidt proved in the early thirties that an automorphism α on a p-adic field K had to be value preserving. Thus every α induces an automorphism on the residue field k. If α induces the identity map on k, α is an inertial automorphism. For d an integral derivation on K, $\exp(pd) = 1 + pd + p^2d^2/2! \cdots$ is an inertial automorphism. Heerema has shown that if $p \neq 2$ every inertial automorphism on K has the form $\exp(pd)$ for d an integral derivation on K [2, Theorem 3.2].

5.1. **Proposition.** *Let d be an integral derivation on a p-adic field K with $p \neq 2$. The invariant subfield of the inertial automorphism* $\exp pd$ *is the constant field of d.*

Proof. Let K_0 be the constant field of d, H the invariant field of $\exp(pd)$. Clearly $K_0 \subset H$. Consider a unit b in H. If $d(b) \neq 0$ then $d(b) = p^r u$ for some $r \geq 0$ and unit u in K. Thus $V(pd(b)) = r + 1$. We need only show that

$V(p^n d^n(b)/n!) > r+1$ for $n > 1$ to conclude that $\exp pd(b) \neq b$ contradicting the choice of b. Clearly, $V(p^n d^n(b)/n!) > r+1$ for $1 < n < p$. By assumption $p \geq 3$. Since $V(n!) \leq (n-1)/(p-1)$ [2, Lemma 2.4, p. 301] we have $V(p^n d^n(b)/n!) \geq n + r - (n-1)/(p-1)$. Also, $n-(n-1)/(p-1) \geq n-(n-1)/2$ and $n-(n-1)/2 > 1$ for $n \geq 3$. Hence $\exp pd(b) = b + pd(b) + \cdots \neq b$, a contradiction, so it must be that $d(b) = 0$. Let G_0 be the group of inertial automorphisms on a p-adic field K. The ramification series on K is the chain $\{G_i\}$ of invariant subgroups where $G_i = \{\alpha \in G_0 \mid \alpha(a) = a, \bmod p^i$, for a an integer in $K\}$. In 1939 Mac Lane showed that, for all $i \geq 0$, the invariant subfield of G_i is the inertial subfield K^* of K [9, Theorem 16]. For any group L of inertial automorphisms on K let $\mathfrak{I}(L)$ be the subfield of L invariants and for a subfield K_0 of K let $\mathfrak{G}(K_0)$ be the group of inertial automorphisms of K leaving K_0 invariant. Define the ramification series of the group L as above i.e., $L_i = \{\alpha \in L \mid \alpha(a) = a, \bmod p^i$, for a an integer in $K\}$.

5.2. **Theorem.** *Let $K \supset K_0$ be p-adic fields with residue fields $k \supset k_0$ and assume k finitely generated over k_0. A field K_1 with $K \supseteq K_1 \supseteq K_0$ is the invariant subfield of a group of inertial automorphisms on K if and only if K_1 is a p-adic subfield algebraically closed in K. If $L = \mathfrak{G}(K')$ for some subfield K' of K then $\mathfrak{I}(L) = \mathfrak{I}(L_i)$ for $i \geq 1$.*

Proof. If $K_1 = \mathfrak{I}(L)$ for a group L of inertial automorphisms then K_1 is the constant field of a set of analytic derivations by Proposition 5.1 and is, thus, the intersection of p-adic subfields each of which is algebraically closed in K. Thus K_1 is a p-adic subfield algebraically closed in K. Conversely, if K_1 satisfies the conditions of the theorem then by Corollary 2.7, K_1 is the constant subfield of an integral derivation. By Proposition 5.1, $K_1 = \mathfrak{I}(\mathfrak{G}(K_1))$.

The last sentence follows immediately from the fact that if $\exp(pd) \in L$ then $\exp(p^i d)$ is in L_i and $\mathfrak{I}(\exp pd) = \mathfrak{I}(\exp p^i d)$.

Appendix

Example 1.

Summary. Two p-adic fields K_0 and K_0' are constructed in the same p-adic field K. They are both algebraically closed in K and have the same residue field. K_0 is the field of constants of an integral derivation on K while K_0' is not. Their common residue field k_0 is not algebraically closed in the residue field k of K.

Construction. Let k^* be a perfect field of characteristic $p \neq 0$. Let $\{\bar{t}, \bar{x}_1, \bar{x}_2, \cdots\}$ be a denumerable set of quantities algebraically independent over k^*. For $n \geq 1$ let $\bar{u}_n$ be a root of $\bar{f}_n(z) = z^{p^n} - (\bar{x}_n + \bar{t}^{p^n})$ over $k^*(\bar{t}, \bar{x}_1, \bar{x}_2, \cdots)$. Define k_0 to be $k^*(\bar{x}_1, \bar{x}_2, \cdots)$ and $k = k_0(\bar{t}, \bar{u}_1, \bar{u}_2, \cdots)$. Let K_0 be a

p-adic field having k_0 as its residue field with representatives x_i of $\bar{x}_i$. Extend K_0 by a transcendental t, representing $\bar{t}$, and complete $K_0(t)$ to obtain a p-adic field K_1. K_0 is algebraically closed in K_1.

Extensions K_1' and K_1'' are obtained by extending K_1 by roots u_n of polynomials $f_n(z) = z^{p^n} - (x_n + t^{p^n})$, $n \geq 1$, and u_n' of polynomials $g_n(z) = f_n(z) - pt$, $n \geq 1$, respectively. Finally, let K and K' be the completions of K_1' and K_1'' respectively. Since K and K' have the same residue fields there is an analytically isomorphic $K \to K'$ which leaves residues fixed [9, Corollary 1, p. 431]. Let K_0' be the image of K_0.

A.1 Proposition. *K_0 is algebraically closed in K and in K'.*

Proof. By Lemmas 3.3, 3.4 and Corollary 3.5 it will suffice to show that for any n, $\bar{b}_n = \bar{x}_n + \bar{t}^{p^n}$ does not satisfy the condition

A.2 $$\bar{b}_n^{p^n} = (\bar{a}\bar{c}_n)^{p^n}$$

with $\bar{a} \in k_0$, $\bar{c}_n \in k_0(\bar{t}, \bar{u}_1, \cdots, \bar{u}_{n-1})$ and $0 \leq m < n$. For if $\bar{a}$ and $\bar{c}_m$ exist fulfilling A.2 then $\bar{c}_n^{p^n}$ is a rational function over k^* in $\bar{t}^{p^n}$, the $\bar{x}_i^{p^n}$ and $\bar{u}_1^{p^n}, \cdots, \bar{u}_{n+1}^{p^n}$ whereas $\bar{a}$ is a rational expression in the $\bar{x}_i$ over k^*. Since $\bar{u}_i^{p^n} = \bar{x}_i^{p^{n-i}} + (\bar{t}^{p^i})^{p^{n-i}}$, $\bar{c}_n^{p^n}$ is a rational expression in $\bar{t}^{p^n}$ and the $\bar{x}_i$. Since $\bar{b}^{p^n} = \bar{x}_n^{p^n} + \bar{t}^{p^{m+n}} = \bar{a}^{p^n}\bar{c}_n^{p^n}$ we may assume $\bar{a}^{p^n}$ is a polynomial in the $\bar{x}_i$ over k^* and $\bar{c}_n^{p^n}$ is a polynomial $\bar{t}^{p^n}$ and the $\bar{x}_i$. Comparing coefficients of $\bar{t}^{p^{m+n}}$ we conclude that $\bar{a}^{p^n} \in k^*$. We note next that $\bar{x}_n$ appears in $\bar{c}_n$ only in powers of p^n. Thus the equality A.2 cannot be obtained.

A.3 Proposition. *K_0 is the field of constants of an integral derivation.*

Proof. Define $d \in \mathrm{Der}(K/K_0)$ by the condition $d(t) = 1$. Then

$$d(u_n) = -f_n^d(u)/f_n'(u) = p^n t^{p^n - 1}/p^n u^{p^n - 1}.$$

Since each integer in K_1' is a polynomial in the u_i with coefficients which are integers in K_0 it follows that $d|_{K_1}$ is integral. Hence d is integral. The field of constants of $d|_{K_1'}$ is K_0 since $K_1 = K_0(t)$ and $d(t) = 1$. Thus, since K_0 is algebraically closed in K, K_0 is the field of constants of d by Lemma 2.0.

A.4 Proposition. *There does not exist an integral derivation on K with field of constants K_0'.*

Proof. It will suffice to prove there is no integral derivation d' on K' with field of constants K_0 in view of the analytic isomorphism used to define K_0'. Suppose there is such a derivation d'. Then $d'(t) \neq 0$ and $d'(x_n) = 0$ for all $n > 0$. Let $V(d'(t)) = m$. Then

$V(d'(u_{m+2})) = V(g^{d'}_{m+2}(u_{m+2})/g'_{m+2}(u_{m+2})) = V(p^{m+2}d'(t) + pd'(t)) - V(p^{m+2}) = -1$
contradicting the assumption that d' is integral.

We observe finally that since $\bar{u}_n^{p^n} = \bar{x}_n + \bar{t}^{p^n}$, $\bar{u}_n - \bar{t}$ is a p^nth root of $\bar{x}_n$ and hence k_0 is not algebraically closed in k.

Example 2.

Summary. Fields $k_0 \subset k$ of characteristic p are constructed such that if $K_0 \subset K$ are p-adic fields with these as residue fields there is no nonzero analytic derivation on K with field of constants containing K_0. k_0 is algebraically closed in k and contains the maximal perfect subfield of k. This is an adaptation of an example due to Mac Lane [10, pp. 36–37].

Construction. Let k^* be a perfect field of characteristic $p \neq 0$. Let $\bar{X} = \{\bar{x}_1, \bar{x}_2, \cdots\}$ be a denumerable set and $\bar{t}$ be such that $\bar{X} \cup \{\bar{t}\}$ is algebraically independent over k^*. Let $\bar{U} = \{u_i\}_{i \geq 1}$ where $\bar{u}$ is a root of $\bar{f}_1(z) = z^p - (\bar{x}_1 + \bar{t})$ and for $n > 1$, u_n is a root of $\bar{f}_n(z) = z^p - (\bar{x}_n + \bar{u}_{n-1})$. Mac Lane has shown that $k_0 = k^*(\bar{X})$ is algebraically closed in $k = k_0(\bar{t}, \bar{U})$, that $\bar{X}$ is a p-basis for k, and that k^* is the maximal perfect subfield of k.

A.5 **Proposition.** *Let $K_0 \subset K$ be p-adic fields with residue fields $k_0 \subset k$ respectively. There is no analytic derivation on K with K_0 in its field of constants.*

Proof. If $d \in \mathrm{Der}(K)$ has K_0 in its field of constants then multiplying by a suitable power of p we can assume that d induces a nontrivial derivation on k which is constant on k_0. However, since k_0 contains a p-basis for k, $\mathrm{Der}(k/k_0) = \{0\}$. Hence $\mathrm{Der}(K/K_0) = \{0\}$.

Example 3.

Summary. We construct p-adic fields $K_0 \subset K$ with residue fields $k_0 \subset k$ such that every δ in $\mathrm{Der}(k/k_0)$ is induced by some d in $\mathrm{Der}(K/K_0)$ yet k is not separable over k_0. k_0 contains k^* the maximal perfect subfield of k.

Construction. Let k^* be a perfect field of characteristic $p \neq 0$. Let $\bar{X} = \{\bar{x}_0, \bar{x}_1, \bar{x}_2, \cdots\}$ and $\bar{Y} = \{\bar{y}_1, \bar{y}_2, \cdots\}$ be denumerable sets and let $\bar{t}$ be such that $\bar{X} \cup \bar{Y} \cup \{\bar{t}\}$ is algebraically independent over k^*. Let $\bar{u}_0$ be a root of $\bar{f}_0(z) = z^p - (\bar{x}_0 + \bar{t}^p)$ and, for $n > 0$, let $\bar{u}_n$ be a root of $\bar{f}_n(z) = z^p - (\bar{x}_n + \bar{u}_{n-1})$. Also, $\bar{v}_1$ is a root of $\bar{g}_1(z) = z^p - (\bar{y}_1 + \bar{t})$ and for $n > 1$, $\bar{v}_n$ is a root of $\bar{g}_n(z) = z^p - (\bar{y}_n + v_{n-1})$. Define k_0 to be $k^*(\bar{X}, \bar{Y})$ and $k = k_0(\bar{t}, \bar{U}, \bar{V})$ where $\bar{U} = \{u_i \mid i = 0, 1, \cdots\}$ and $\bar{V} = \{v_i \mid i = 1, 2, \cdots\}$.

A.6 **Contention.** $k = k_0(k^p)$ *and hence* $\mathrm{Der}(k/k_0) = \{0\}$.

Proof. The following equalities establish that $\{\bar{t}\} \cup \bar{U} \cup \bar{V} \subset k_0(k^p)$: $\bar{t} = \bar{v}_1^p - \bar{y}_1$, $\bar{u}_n = \bar{u}^p_{n+1} - \bar{x}_n$ for $n \geq 0$, and $\bar{v}_n = \bar{v}^p_{n+1} - \bar{y}_n$ for $n \geq 1$.

We show next that k^* is in fact the maximal perfect subfield as follows.

A.7 Contention. *For each* $n \geq 0$ *the set* $W_n = \{\bar{t}\} \cup \{\bar{x}_m \mid m \geq n\} \cup \{\bar{y}_m \mid m \geq n\} \cup \{\bar{u}_m \mid m < n\} \cup \{\bar{v}_m \mid m < n\}$ *is algebraically independent over* k^*.

Proof. $\bar{W}_0 = \bar{X} \cup \bar{Y} \cup \{\bar{t}\}$ which is algebraically independent by construction. Assume, $\bar{W}_n$ algebraically independent over k^* and consider any nontrivial algebraic relation in $\bar{W}_{n+1}$. Taking the pth power followed by the substitutions $\bar{u}_n^p = \bar{x}_n + \bar{u}_{n-1}$, $\bar{v}_n^p = \bar{y}_n + \bar{v}_{n-1}$ yields a nontrivial algebraic relation in $\bar{W}_n$, a contradiction.

By Contention A.7 we see that $\bar{U} \cup \bar{V} \cup \{\bar{t}\}$ is algebraically independent over k^*. Since $\bar{x}_0 = \bar{u}_0^p - \bar{t}^p$; $\bar{x}_n = \bar{u}_n^p - \bar{u}_{n-1}$, for $n \geq 1$ it follows that $k = k^*(\bar{U}, \bar{V}, \bar{t})$. It follows that k^* is the maximal perfect subfield of k.

Note finally that by definition of $\bar{u}_0$ we have $\bar{x}_0 = (\bar{u}_0 - \bar{t})^p$. Thus $\bar{x}_0$ is p-independent in k_0 but is not p-independent in k. It follows that k is not separable over k_0.

BIBLIOGRAPHY

1. R. Baer, *Algebraische Theorie der differentierbaren Funktionenkörper*, S.-B. Heidelberger Akad. Wiss. Abh. 8 (1927), 15 – 32.

2. N. Heerema, *Exponential automorphisms of complete local rings*, Math. Z. 122 (1971), 299 – 306.

3. ———, *Derivations and embeddings of a field in its power series ring*, Proc. Amer. Math. Soc. 11 (1960), 188 – 194. MR 23 #A893.

4. ———, *Derivations on p-adic fields*, Trans. Amer. Math. Soc. 102 (1962), 346 – 351. MR 26 #1311.

5. ———, *Inertial automorphisms of a class of wildly ramified v-rings*, Trans. Amer. Math. Soc. 132 (1968), 45 – 54. MR 36 #6407.

6. ———, *Convergent higher derivations on local rings*, Trans. Amer. Math. Soc. 132 (1968), 31 – 44. MR 36 #6406.

7. N. Jacobson, *Lectures in abstract algebra*. Vol. III: *Theory of fields and Galois theory*, Van Nostrand, Princeton, N. J., 1964. MR 30 #3087.

8. S. Mac Lane, *The uniqueness of the power series representation of certain field with valuations*, Ann. of Math. (2) 39 (1938), 370 – 382.

9. ———, *Subfields and automorphisms groups of p-adic fields*, Ann. of Math. (2) 40 (1939), 423 – 442.

10. ———, *Steinitz field towers for modular fields*, Trans. Amer. Math. Soc. 46 (1939), 23 – 45. MR 1, 3.

11. J. Neggers, *Derivations on $\bar{p}$-adic fields*, Trans. Amer. Math. Soc. 115 (1965), 496 – 504. MR 33 #5610.

12. T. Peter, *An intrinsic characterization of those derivations on the residue field of a ramified v-ring which lift*, Dissertation, Florida State University, Tallahassee, Fla., 1973.

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