ON A CERTAIN SUM IN NUMBER THEORY. II

BY

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ABSTRACT. We derive "exact order" of the function

$$\sum_{k \le \sqrt{x}} k^\rho \min^\beta \left(\frac{\sqrt{x}}{k}, \frac{1}{P_k} \right).$$

Here ρ and β are nonnegative real numbers and, for given real δ_1, δ_2, $\cdots$, δ_r, $P_k = \max_j \langle k \delta_j \rangle$ where $\langle t \rangle$, for real t, denotes distance of t from the nearest integer. Using our results, we obtain the solution of the basic problem in the theory of lattice points with weight in rational many-dimensional ellipsoids.

I. Introduction. Let $Q(u) = Q(u_1, u_2, \cdots, u_r)$ be a positive definite quadratic form in $r \ge 2$ variables with a symmetric integral coefficient matrix and determinant D. Let b_j, M_j be integers, $M_j > 0$ and α_j be real numbers, $j = 1, 2, \cdots, r$. Let us put, for $x > 0$,

$$A(x) = \sum \exp \left(2\pi i \sum_{j=1}^{r} \alpha_j u_j \right), \qquad V(x) = \frac{\pi^{r/2} x^{r/2} \exp(2\pi i \sum_{j=1}^r \alpha_j b_j)}{\sqrt{D} \, M_1 M_2 \cdots M_r \Gamma(r/2 + 1)} \delta,$$

where $\delta = 1$ if all the $\alpha_j M_j$ are integers, and $\delta = 0$ otherwise. Here the summation runs over all systems $u = (u_1, u_2, \cdots, u_r)$ of real numbers such that $Q(u) \le x$ and $u_j \equiv b_j \pmod{M_j}$, $j = 1, 2, \cdots, r$.

For $\rho \ge 0$ put $P_0(x) = P(x) = A(x) - V(x)$ and

$$(1) \qquad P_\rho(x) = \frac{1}{\Gamma(\rho)} \int_0^x P(t)(x - t)^{\rho - 1} dt \quad \text{for } \rho > 0,$$

$$(2) \qquad M_\rho(x) = \int_0^x |P_\rho(t)|^2 dt \quad \text{for } \rho \ge 0.$$

In the papers [3], [5] and [9] it is shown that 0-estimates of these functions can be obtained from 0-estimates of the functions

$$F_1(x) = \sum_{k \le \sqrt{x}} \lg 2k \, \min^{r/2-1}(\sqrt{x}/k, \, 1/P_k),$$

$$F(x) = F_{\rho,\beta}(x; \, \alpha_j) = \sum_{k \le \sqrt{x}} k^\rho \min^\beta(\sqrt{x}/k, \, 1/P_k).$$

Here, ρ and β are nonnegative real numbers, k is a positive integer and $P_k = \max_{j=1,2,\cdots,r} \langle \alpha_j M_j k \rangle$, where $\langle t \rangle$, for real t, denotes the distance of t from the nearest integer, and where we put $\min(A, 1/B) = A$ for $B = 0$.

In fact, we have proved in the aforementioned papers that

(3) $P(x) \ll x^{r/4-1/2} F_1(x)$ for $r > 4,$

(4) $P_\rho(x) \ll x^{r/4-1/2} F_{\rho, r/2-1}(x)$ for $r > 4$ and $\rho > 0$

and

(5) $M_\rho(x) \ll x^{r/2-1/2} F_{2\rho+1, r-1}(x).$

Here, we write $A \ll B$ instead of $|A| \le cB$, where c is a constant depending only on α_j, M_j, b_j, ρ, γ, ϵ and β. In the sequel we always assume that $x > c$ and $\rho \ge 0$, $\beta \ge 0$.

In [6], considered as the first part of this paper, the function $F(x)$ is studied especially in the case when $\alpha_1 = \alpha_2 = \cdots = \alpha_r$. Let us recall some results (see [6, Theorems 1, 5, 6 and 7]).

(A) Always,

$$x^{(\rho+1)/2} \ll F(x) \ll x^{\beta/2} \qquad \text{for } \rho < \beta - 1,$$

$$x^{(\rho+1)/2} \ll F(x) \ll x^{\beta/2} \lg x \quad \text{for } \rho = \beta - 1,$$

$$x^{(\rho+1)/2} \ll F(x) \ll x^{(\rho+1)/2} \quad \text{for } \rho > \beta - 1,$$

and these results generally cannot be improved.

(B) Let $\rho < \beta - 1$, $\gamma > 0$, and let the inequality $P_k \ll k^{-\gamma}$ be fulfilled for infinitely many k. Then $F(x) = \Omega(x^f)$, where

(6) $f = (\beta\gamma + \rho)/2(\gamma + 1).$

(C) Let $\rho < \beta - 1$, $\gamma > 0$, and let the inequality $P_k \gg k^{-\gamma}$ hold for all k. Then

(7) $F(x) \ll x^{f+1/2(\gamma+1)}.$

If moreover $\alpha_1 = \alpha_2 = \cdots = \alpha_r$, then

$$F(x) \ll x^l \qquad\qquad \text{for } \rho < \beta - 2,$$

$$F(x) \ll x^l + x^{(\rho + 1)/2} \lg x \quad \text{for } \rho = \beta - 2,$$

$$F(x) \ll x^l + x^{(\rho + 1)/2} \qquad \text{for } \rho > \beta - 2.$$

The aim of the present paper is to improve the relation (7), actually we are going to prove the following

Main Theorem. *Let $\rho < \beta - 1$, $\gamma > 0$ and let the inequality $P_k \gg k^{-\gamma}$ hold for all k. Then $F(x) \ll x^l$ for $\rho < \beta - 1 - 1/\gamma$, and $F(x) \ll x^{(\rho + 1)/2} \lg^\epsilon x$ otherwise, where we put $\epsilon = 1$ for $\rho = \beta - 1 - 1/\gamma$ or $\beta\gamma = 1$, and $\epsilon = 0$ otherwise.*

II. Proof of the Main Theorem. Let, for $n = 0, 1, \cdots$,

$$T_n = \sum k^\rho \min^\beta\!\left(\frac{\sqrt{x}}{k}, \frac{1}{P_k}\right),$$

where we are summing over all k, $2^n \le k < 2^{n+1}$. Thus

(8) $$F(x) \ll \sum_{2^n \le \sqrt{x}} T_n.$$

First we need some estimates of the number of k, for which P_k is small. Let l and M be positive integers, and let $M \le k_1 < k_2 < \cdots < k_\nu \le 2M$ be numbers of the interval $[M, 2M]$ fulfilling the inequality $2^{-l-1} \le P_k < 2^{-l}$. Denote by K the smallest positive integer such that $P_k < 2 \cdot 2^{-l}$. From the obvious inequality $\langle \xi_1 \pm \xi_2 \rangle \le \langle \xi_1 \rangle + \langle \xi_1 \rangle$, ξ_1 and ξ_2 are real, we obtain $k_1 \ge K$, $k_2 - k_1 \ge K, \cdots, k_\nu - k_{\nu-1} \ge K$, and then $k_\nu \ge \nu K$. Hence, using the assumption $2 \cdot 2^{-l} > P_K \gg K^{-\gamma} \ge (\nu/k_\nu)^\gamma \ge (\nu/2M)^\gamma$, we conclude that

(9) $$\nu \ll 2^{-l/\gamma} M.$$

(A similar method was used in the paper [2, p. 216], the same method in the paper [1, p. 131]). We are going to use this result in the sequel without any further reference. Further, let us note that

$$\sum_{k \le \sqrt{x};\, P_k > c} k^\rho \min^\beta(\sqrt{x}/k, 1/P_k) \ll \sum_{k < \sqrt{x}} k^\rho \ll x^{(\rho + 1)/2}.$$

Thus (cf. the assertion (A) of Introduction) in T_n we can sum only over those k with $P_k < c$. Finally let us recall that it is sufficient to consider only these l, with $2^l \ll M^\gamma$ (see (9)).

Now we successively obtain

$$T_n \ll \sum_{2^l \ll 2^{\gamma n}} 2^n 2^{-l/\gamma} 2^{n(\rho - \beta)} \min^{\beta}(\sqrt{x}, 2^{n+l})$$

$$= 2^{n(\rho - \beta + 1)} \sum_{2^l \ll 2^{\gamma n}} 2^{-l/\gamma} \min^{\beta}(\sqrt{x}, 2^{n+l}).$$

Let us consider two cases. First, let $2^{n(\gamma + 1)} \ll \sqrt{x}$. Hence $2^{n+l} \ll \sqrt{x}$ provided $2^l \ll 2^{\gamma n}$. In this case we obtain

$$T_n \ll 2^{n(\rho - \beta + 1)} \sum_{2^l \ll 2^{\gamma n}} 2^{-l/\gamma + \beta(n+l)} = 2^{n(\rho + 1)} \sum_{2^l \ll 2^{\gamma n}} 2^{l(\beta - 1/\gamma)}.$$

Thus

$$T_n \ll \begin{cases} 2^{n(\gamma \beta + \rho)} & \text{for } \beta\gamma > 1, \\ 2^{n(\rho + 1)}n & \text{for } \beta\gamma = 1, \\ 2^{n(\rho + 1)} & \text{for } \beta\gamma < 1. \end{cases}$$

Summing over all n with $2^{n(\gamma + 1)} \ll \sqrt{x}$, i.e., $2^n \ll x^{1/2(\gamma + 1)}$ we obtain

$$(10) \qquad \sum_{2^n \ll x^{1/2(\gamma + 1)}} T_n \ll \begin{cases} x^J & \text{for } \beta\gamma > 1, \\ x^J \lg x & \text{for } \beta\gamma = 1, \\ x^{(\rho + 1)/2(\gamma + 1)} & \text{for } \beta\gamma < 1. \end{cases}$$

In the sequel we may suppose that $2^{n(\gamma + 1)} \gg \sqrt{x}$ and let us recall that $2^n \ll \sqrt{x}$. We obtain

$$T_n \ll 2^{n(\rho - \beta + 1)} \sum_{2^l \ll 2^{\gamma n}} 2^{-l/\gamma} \min^{\beta}(\sqrt{x}, 2^{n+l})$$

$$\ll 2^{n(\rho - \beta + 1)} \left\{ 2^{n\beta} \sum_{2^{l+n} \ll 2^{n(\gamma+1)}; 2^{l+n} \le \sqrt{x}} 2^{l(\beta - 1/\gamma)} + x^{\beta/2} \sum_{2^l > \sqrt{x}/2^n} 2^{-l/\gamma} \right\}$$

$$\ll 2^{n(\rho + 1)} \sum_{2^l \le \sqrt{x}/2^n} 2^{l(\beta - 1/\gamma)} + x^{(\beta - 1/\gamma)/2} 2^{n(\rho - \beta + 1 + 1/\gamma)}$$

$$\ll x^{(\beta - 1/\gamma)/2} 2^{n(\rho - \beta + 1 + 1/\gamma)} + \begin{cases} 2^{n(\rho + 1)} (\sqrt{x}/2^n)^{\beta - 1/\gamma} & \text{for } \beta\gamma > 1, \\ 2^{n(\rho + 1)} \lg(cx/2^n) & \text{for } \beta\gamma = 1, \\ 2^{n(\rho + 1)} & \text{for } \beta\gamma < 1, \end{cases}$$

i.e.,

(11) $\qquad T_n \ll x^{(\beta - 1/\gamma)/2} 2^{n(\rho - \beta + 1 + 1/\gamma)}$ $\qquad$ for $\beta\gamma > 1$,

(12) $\qquad T_n \ll 2^{n(\rho + 1)} \lg x$ $\qquad$ for $\beta\gamma = 1$,

(13) $\qquad T_n \ll x^{(\beta - 1/\gamma)/2} 2^{n(\rho - \beta + 1 + 1/\gamma)} + 2^{n(\rho + 1)}$ $\quad$ for $\beta\gamma < 1$.

First assume that $\rho - \beta + 1 + 1/\gamma < 0$. Then necessarily $\beta\gamma > 1$, and if

$$S = \sum_{x^{1/2(\gamma + 1)} \ll 2^n \ll \sqrt{x}} T_n,$$

then we find by (11) that

(14) $\qquad S \ll x^{(\beta - 1/\gamma)/2 + (\rho - \beta + 1 + 1/\gamma)/2(\gamma + 1)} = x^f.$

Next, let $\rho - \beta + 1 + 1/\gamma = 0$. Then again $\beta\gamma > 1$, and by (11) we have

(15) $\qquad S \ll x^{(\beta - 1/\gamma)/2} \lg x = x^{(\rho + 1)/2} \lg x.$

Finally, let $\rho - \beta + 1 + 1/\gamma > 0$. In this case we sum (11)–(13) over all n with $2^n \ll \sqrt{x}$ and obtain

(16) $\qquad S \ll x^{(\rho + 1)/2}$ $\qquad$ for $\beta\gamma \neq 1$,

(17) $\qquad S \ll x^{(\rho + 1)/2} \lg x$ $\quad$ for $\beta\gamma = 1$.

From (10), (14)–(17) and (8) we obtain the assertions of the Main Theorem.

From our Main Theorem and assertions (A) and (B) of the Introduction, we are able to state the following corollary (cf. Theorem 8 of [6]).

Corollary 1. *Let γ be the least upper bound of all the numbers $\tau > 0$, for which the inequality $P_k \ll k^{-\tau}$ has infinitely many solutions. Then*

$$\limsup_{x \to +\infty} \frac{\lg F(x)}{\lg x} = \max\left(\frac{\beta\gamma + \rho}{2(\gamma + 1)}, \frac{\rho + 1}{2}\right).$$

(For $\gamma = +\infty$ we put $(\beta\gamma + \rho)/2(\gamma + 1) = \beta/2$.)

Using the inequalities

$$F_{0, \tau/2 - 1}(x) \leq F_1(x) \leq F_{0, \tau/2 - 1}(x) \lg x$$

and the hypothesis of Corollary 1, we obtain

Corollary 2. *Let γ be defined as in Corollary 1. Then*

$$\limsup_{x \to +\infty} \frac{\lg F_1(x)}{\lg x} = \max\left(\left(\frac{r}{4} - \frac{1}{2}\right)\frac{\gamma}{\gamma+1}, \frac{1}{2}\right).$$

Let us recall that moreover

$$F_1(x) \ll x^{r/4-1/2} \sum_{k \leq \sqrt{x}} \frac{\lg 2k}{k^{r/2-1}} \ll \begin{cases} x^{r/4-1/2} & \text{for } r > 4, \\[2mm] \sqrt{x}\,\lg^2 x & \text{for } r = 4, \\[2mm] \sqrt{x}\,\lg x & \text{for } r = 2, 3, \end{cases}$$

and

$$F_1(x) \gg \sum_{k \leq \sqrt{x}} \lg 2k \gg \sqrt{x}\,\lg x.$$

These results together with the results of [6] *thus give full information about "exact order" of the functions* $F(x)$ *and* $F_1(x)$.

III. Applications of the Main Theorem. From the Main Theorem and (3)–(5) we obtain immediately:

Theorem 1. *Let* $r > 4$ *and let* γ *denote the least upper bound of all the numbers* $\tau > 0$, *for which the inequality* $P_k \ll k^{-\tau}$ *has infinitely many solutions. Let* $0 \leq \rho \leq r/2 - 2 - 1/\gamma$ *(for* $\gamma = +\infty$ *we put* $1/\gamma = 0$ *and* $1/2(\gamma+1) = 0$*). Then*

(13) $P_\rho(x) \ll x^{r/2-1-(r/2-1-\rho)/2(\gamma+1)+\epsilon}$

for every $\epsilon > 0$. *Let* $\rho > r/2 - 2 - 1/\gamma$, $\rho \geq 0$. *Then*

$$P_\rho(x) \ll x^{r/2+\rho/2} \lg^\kappa x,$$

where $\kappa = 1$ *for* $\rho = 0$ *and for* $\rho = r/2 - 2$, *else* $\kappa = 0$. *If* $\gamma = +\infty$ *and* $0 \leq \rho < r/2 - 2$ *then moreover,* $P_\rho(x) \ll x^{r/2-1}$.

Theorem 2. *Let* γ *be defined as in Theorem 1. Let* $0 \leq \rho \leq r/2 - 3/2 - 1/2\gamma$. *Then*

$$M_\rho(x) \ll x^{r-1-(r/2-1-\rho)/(\gamma+1)+\epsilon}$$

for every $\epsilon > 0$. *Let* $\rho > r/2 - 3/2 - 1/2\gamma$, $\rho \geq 0$. *Then*

$$M_\rho(x) \ll x^{r/2+\rho+1/2} \lg^\kappa x,$$

where $\kappa = 1$ *for* $\rho = r/2 - 3/2$, *else* $\kappa = 0$. *If* $\gamma = +\infty$ *and* $0 \leq \rho < r/2 - 3/2$ *then moreover* $M_\rho(x) \ll x^{r-1}$.

In papers [5] and [8] the following results are proved (γ is as above):

(D) If $A(x) \not\equiv 0$, then

$$P_\rho(x) = \Omega(x^{(r-1)/4 + \rho/2}) \quad \text{and} \quad M_\rho(x) \gg x^{r/2 + \rho + 1/2}.$$

(E) Let $b_1 = b_2 = \cdots = b_r = 0$, $0 \le \rho \le r/2 - 1$. Then

$$P_\rho(x) = \Omega(x^{r/2 - 1 - (r/2 - 1 - \rho)/2(\gamma + 1) - \epsilon}) \quad \text{and}$$

$$M_\rho(x) = \Omega(x^{r - 1 - (r/2 - 1 - \rho)/(\gamma + 1) - \epsilon})$$

for every $\epsilon > 0$.

Let us note that the assumption $b_1 = b_2 = \cdots = b_r = 0$ cannot be omitted generally. (See [8, p. 271] or [4, p. 393].) These results together with Theorems 1 and 2 give our final results, which partly solve the basic problem in the theory of lattice points with weight in many dimensional ellipsoids.

Theorem 3. *Let γ be defined as in Theorem 1 and $b_1 = b_2 = \cdots = b_r = 0$. Let $0 \le \rho \le r/2 - 2 - 1/\gamma$. Then*

$$\limsup_{x \to +\infty} \frac{\lg |P_\rho(x)|}{\lg x} = r/2 - 1 - \frac{1}{2(\gamma + 1)}(r/2 - 1 - \rho).$$

Theorem 4. *Let γ be defined as in Theorem 1 and κ as in Theorem 2. Let $\rho > r/2 - 3/2 - 1/2\gamma$ and $\rho \ge 0$. Then*

$$x^{r/2 + \rho + 1/2} \ll M_\rho(x) \ll x^{r/2 + \rho + 1/2} \lg^\kappa x.$$

Let $0 \le \rho < r/2 - 3/2 - 1/2\gamma$ and $b_1 = b_2 = \cdots = b_r = 0$. Then

$$\limsup_{x \to +\infty} \frac{\lg M_\rho(x)}{\lg x} = r - 1 - \frac{1}{\gamma + 1}(r/2 - 1 - \rho).$$

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