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INVOLUTIONS PRESERVING AN SU STRUCTURE

BY

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ABSTRACT. Bordism theories $SU_*(Z_2, all)$ for SU-manifolds with involution and $SU_*(Z_2, free)$ for SU-manifolds with free involution are defined. The latter is studied by use of the SU-bordism spectral sequence of BZ_2, and the orders of the spheres S^{4n+3} with antipodal action are determined. It is shown that $SU_{2k}(Z_2, free) \to SU_{2k}(Z_2, all)$ is monic, and that an element of $SU_{2k}(Z_2, all)$ bounds as a unitary involution if and only if it is a multiple of the nonzero class $\alpha \in SU_1$.

1. Introduction. Conner and Floyd defined and studied the bordism of unitary manifolds M with smooth involution T preserving the unitary structure ([4], [5], [1]). Suppose M is also an SU-manifold; we think of the SU structure as being given by a trivialization ϕ: $\det \tau(M) \cong M \times \mathbf{C}$ of the (complex) determinant of the tangent bundle of M (see [9, VIII]). Then T *preserves the SU structure* if $\phi(\det dT) = (T \times 1)\phi$.

Two such SU-manifolds with involution, (M_1, T_1) and (M_2, T_2), are *bordant* if there is an SU-manifold N with ∂N the disjoint union of M_1 and $-M_2$, and a structure-preserving involution T' on N with $T'|M_i = T_i$. The set $SU_*(Z_2, all)$ of equivalence classes, under this bordism relation, is then a graded algebra over the bordism ring SU_*, with operations induced by disjoint union and Cartesian product.

One also obtains $SU_*(Z_2, free)$ in the same way, but requiring all involutions to be fixed point free, as well as a relative theory $SU_*(Z_2, rel)$ whose elements are represented by SU-manifolds M with involution free on ∂M. As usual, one obtains a long exact sequence

$$SU_*(Z_2, free) \xrightarrow{r} SU_*(Z_2, all) \xrightarrow{s} SU_*(Z_2, rel) \xrightarrow{\partial} SU_*(Z_2, free) \tag{1.1}$$

of SU_*-modules, where r and s are forgetful and $\partial[M, T] = [\partial M, T|\partial M]$. The reader can easily supply the details (compare [1, §10]).

Also, as usual, a free involution is determined by its quotient space and an element of the relative group is determined by the normal bundle of the fixed point

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set. Unfortunately, the fixed point data is not very amenable to calculation, due to our ignorance of SU-bordism. However, this paper presents several results about the entries in (1.1), including:

(1.2) The kernel of the forgetful map

$$F\colon SU_*(Z_2, \mathit{free}) \longrightarrow U_*(Z_2, \mathit{free})$$

is $\{\beta_0[S^0, A] + \beta_1[S^1, A]\colon \beta_i \in \text{torsion } SU_*\}$.

(1.3) If A represents the antipodal involution on a sphere, $[S^{4n+3}, A] \in SU_{4n+3}(Z_2, \mathit{free})$ has order 2^{2n+2} if n is odd, and 2^{2n+3} if n is even.

(1.4) r is a monomorphism in even dimensions.

(1.5) For $k \geq 1$ there is an exact sequence

$$SU_{2k-1}(Z_2, \mathit{all}) \xrightarrow{t} SU_{2k}(Z_2, \mathit{all}) \xrightarrow{F} U_{2k}(Z_2, \mathit{all}),$$

where t is multiplication by the nonzero class $\alpha \in SU_1$.

Notes. For (1.3), S^{4n+3} is given an SU-structure by means of its usual imbedding in $\mathbf{C}^{2n+2}$. For (1.2) and (1.5), S^1 is given the SU-structure obtained via a trivialization of $\tau(S^1)$; with this structure $[S^1] = \alpha$.

C. B. Thomas [10] has shown that (1.3) also gives the order of $[S^{4n+3}, A]$ in the symplectic group $\mathrm{Sp}_*(Z_2, \mathit{free})$. Theorem (1.5) is definitely not true in odd dimensions; $[S^1, A]$ lies in Ker F but not Im t.

2. **The SU-bordism of BZ_2.** Let (M, T) be an SU-manifold with free involution, and let M/T be the quotient space obtained by identifying Tm with m for each $m \in M$. M/T is a unitary manifold [5]. Furthermore det $\tau(M)$/det dT is identified with det $\tau(M/T)$; hence ϕ defines a trivialization ϕ/T: det $\tau(M/T) \cong (M/T) \times \mathbf{C}$. Thus M/T is an SU-manifold. The double cover $M \longrightarrow M/T$ is classified by a map $f\colon M \to BZ_2$, and we see at once the analog of [3, (19.1)]:

(2.1) **Proposition.** *The assignment* $[M, T] \longrightarrow [M/T, f]$ *defines an isomorphism*

$$SU_*(Z_2, \mathit{free}) \cong SU_*(BZ_2).$$

Since $SU_*(BZ_2) = SU_* \oplus SU_*(BZ_2, *)$ our problem is to study the latter summand. Writing $BZ_2 = \mathbf{R}P(\infty)$, notice that $i^n\colon \mathbf{R}P(n) \subset \mathbf{R}P(\infty)$ is the inclusion of a unitary manifold if n is odd, and of an SU-manifold if $n = 3 \bmod 4$. This defines $r_{2k+1} \in U_{2k+1}(BZ_2, *)$ and $\sigma_{4k+3} \in SU_{4k+3}(Z_2, *)$. Letting $*\colon S^1 \longrightarrow *$ be the point map, there is also

$$\sigma_1 = [S^1, i^1] - [S^1, *] \in SU_1(BZ_2, *).$$

If F forgets SU-structure, $F\sigma_n = r_n$ for $n = 4k + 3$ and also for $n = 1$, since S^1 bounds in U_1.

(2.2) **Proposition** (Conner-Floyd [4], [5]). *$U_*(BZ_2, *)$ is the U_*-module generated by the r_{2k-1}, for all $k \geq 1$, with the relations*

$$2^k r_{2k-1} = 0 \quad and \quad [CP(1)]r_{2k-1} = 2r_{2k+1}.$$

In particular, $U_{2k}(BZ_2, *) = 0$.

For any pair (X, A) there are homomorphisms $d\colon U_n(X, A) \to SU_{n-2}(X, A)$ and $d'\colon U_n(X, A) \to U_{n-4}(X, A)$ which send (M, f) to $(N, f|N)$, where N is the submanifold dual to c_1M and to $(c_1M)^2$, respectively. Notice that $dr_{4k+1} = \sigma_{4k-1}$, but, since r_{4k-1} is represented by an SU-manifold, $d(r_{4k-1}) = 0 = d'(r_{4k-1})$.

Let $t\colon SU_n(X, A) \to SU_{n+1}(X, A)$ be multiplication by α. Combining [6, (15.2)] and (2.2) gives:

(2.3) **Proposition.** *For each $n \geq 0$ there is an exact sequence*

$$0 \to SU_{2m}(BZ_2, *) \xrightarrow{t} SU_{2m+1}(BZ_2, *) \xrightarrow{\cdot F} U_{2m+1}(BZ_2, *)$$
$$\xrightarrow{(d,d')} SU_{2m-1}(BZ_2, *) \oplus U_{2m-3}(BZ_2, *) \xrightarrow{t} SU_{2m}(BZ_2, *) \to 0.$$

(2.4) **Proposition.** *For $1 \leq n \leq 6$, $SU_n(BZ_2, *) = Z_2, Z_2, Z_8, 0, 0, 0$, respectively. The generators are σ_1, $\alpha\sigma_1$, and σ_3. $\alpha^2\sigma_1 = 4\sigma_3$.*

Proof. For $m = 0$, (2.3) gives $F\colon SU_1(BZ_2, *) \cong U_1(BZ_2, *)$. Setting $m = 1$, (2.2) implies $U_3(BZ_2, *) = Z_4$ with generator $r_3 \in \operatorname{Ker}(d, d')$. Thus (2.3) breaks up to show $SU_2(BZ_2, *) = Z_2$ on $\alpha\sigma_1$ and $SU_3(BZ_2, *)$ is a group of order 8.

Define $\beta = 9[CP(1)]^2 - 8[CP(2)] \in U_4$; then $d\beta = \alpha^2 \in SU_2$ [6, p. 33]. Since r_1 is of order 2, $\beta r_1 = [CP(1)]^2 r_1$. Therefore

$$\begin{aligned} 4\sigma_3 &= d(4r_5) = d([CP(1)]^2 r_1), \quad \text{by (2.2)}, \\ &= (d\beta)[S^1, i^1] = (d\beta)\sigma_1, \quad \text{since } (d\beta)[S^1] = 0, \\ &= \alpha^2\sigma_1. \end{aligned}$$

Thus $SU_3(BZ_2, *) = Z_8$ with generator σ_3.

Since $t(\sigma_3) = td(r_5) = 0$, $SU_4(BZ_2, *) = 0$. With $m = 2$ in (2.3), this means (d, d') is onto. But (d, d') connects two groups of order 16, by (2.2) and the previous paragraph. It follows that $F = 0$, so $SU_5(BZ_2, *) = 0$, which also implies $SU_6(BZ_2, *) = 0$. □

We now recall the structure of SU_*, for which [9, X] is a convenient general reference. In particular, SU_n/torsion $= 0$ if n is odd. Moreover, there exist elements $b_i^{8k} \in SU_{8k}$, one for each partition of k, so that torsion SU_* is the Z_2-vector space with generators $\{\alpha b_i^{8k}, \alpha^2 b_i^{8k}\}$.

As in [3, §7] there is a spectral sequence $\{E^r_{p,q}: r \geq 1;\ p,\ q \geq 0\}$ with

$$E^r_{p,q} = \frac{\operatorname{Im} SU_{p+q}(\mathrm{R}P(p), \mathrm{R}P(p-r)) \to SU_{p+q}(\mathrm{R}P(p), \mathrm{R}P(p-1))}{\operatorname{Im} SU_{p+q+1}(\mathrm{R}P(p+r-1), \mathrm{R}P(p)) \to SU_{p+q}(\mathrm{R}P(p), \mathrm{R}P(p-1))}$$

and $E^\infty_{p,q}$ associated to a filtration of $SU_{p+q}(BZ_2, *)$. Observe that

$$E^1_{p,q} = SU_{p+q}(\mathrm{R}P(p), \mathrm{R}P(p-1)) \cong SU_q;$$

$E^1_{p,0}$ is generated by the class of the usual map

$$g_p: (D^p, S^{p-1}) \to (\mathrm{R}P(p), \mathrm{R}P(p-1))$$

attaching the p-cell of $\mathrm{R}P(p)$. Also,

$$E^2_{p,q} \cong \tilde{H}_p(BZ_2; SU_q) = \begin{cases} Z_2 \otimes SU_q & \text{if } p \text{ is odd,} \\ \operatorname{Tor}(Z_2, SU_q) & \text{if } p \text{ is even.} \end{cases}$$

Thus $E^2_{p,q} = 0$ for p even, $q \neq 1, 2 \bmod 8$, and for p odd, $q = 3, 5, 7 \bmod 8$.

This spectral sequence has $E^4 = E^\infty$. We will show something less than this.

(2.5) In $\{E^r_{p,q}\}$ the differentials $d^2_{4k+1,8j}$, $d^2_{4k+1,8j+1}$, $d^2_{4k,8j+1}$, and $d^3_{4k,8j+2}$ are of maximal rank, for all $k \geq 1$, $j \geq 0$.

(2.6) **Corollary.** *Let $p \geq 2$, $q \geq 0$. $E^4_{p,q} = 0$ whenever p is even or q is odd, except for $E^4_{4k+2,8j+1}$.*

The corollary follows without trouble from (2.5) and the structure of torsion SU_*. To show $E^4 = E^\infty$ it suffices to verify that the exceptional entries of (2.6) persist to E^∞.

Proof of (2.5). Begin with $k = 1$. It follows from (2.4) that $d^2_{5,0}$, $d^2_{5,1}$, $d^2_{4,1}$ and $d^3_{4,2}$ must be isomorphisms. SU_* acts on the spectral sequence as in [3, (7.1)]. Since $d^2_{5,0} \neq 0$, it follows that $d^2_{5,8j}(1 \otimes b_i^{8j}) \neq 0$ and hence $d^2_{5,8j}$ and $d^2_{5,8j+1}$ have maximal rank. In the same way, $d^2_{4,1} \neq 0$ implies $d^2_{4,8j+1}$ is an isomorphism for all j.

To use the same reasoning on $d^3_{4,2}$, somewhat more care is required. $d^3_{4,2}$ maps onto $E^2_{1,4}$, which is generated by $1 \otimes \beta'$, where $\beta' \in SU_4$ is the generator. $F\beta' = 2\beta \in U_4$ [6, (19.1)]. Now suppose $2x = \beta' b_i^{8j} \in SU_{8j+4}$; since U_{8j+4} is free abelian, $\beta F(b_i^{8j}) = Fx \in U_{8j+4}$. This cannot be, because $d(\beta F(b_i^{8j})) = \alpha^2 b_i^{8j} \neq 0$. Therefore $1 \otimes b_i^{8j}\beta' \neq 0 \in E^2_{1,8j+4}$ and $d^3_{4,8j+2}$ has maximal rank.

To complete the proof, define *Smith operators* in the spectral sequence as follows. Let $\Delta^1_{p,q}: E^1_{p,q} \to E^1_{p-4,q}$ assign to $[M, f] \in SU_{p+q}(\mathrm{R}P(p), \mathrm{R}P(p-1))$ the class of $[N, f|N] \in SU_{p+q-4}(\mathrm{R}P(p-4), \mathrm{R}P(p-5))$, where f is transverse regular on $\mathrm{R}P(p-4)$ and $N = f^{-1}\mathrm{R}P(p-4)$. This can be done because the normal bundle of $\mathbf{R}P(p-4)$ in $\mathbf{R}P(p)$, being the quotient of the normal bundle of $S^{p-4}\subset S^p$, has a natural SU structure.

This construction commutes with d^1, and we receive $\Delta^r_{p,q}: E^r_{p,q}\to E^r_{p-4,q}$ for each r, commuting with d^r. Clearly, Δ^1 takes $[D^p, g_p]$ to $[D^{p-4}, g_{p-4}]$. Thus $\Delta^1_{p,q}$ is an isomorphism for $p\geq 5$, and so is $\Delta^2_{p,q}$. (2.5) then follows by an easy induction from the case $k=1$. □

From (2.6), $\alpha E^4_{p,q}=0$ for all $p\geq 2$. It follows that $\operatorname{Im} t\subset SU_*(BZ_2, *)$ can contain only multiples of $\alpha\sigma_1$. Together with the exact sequence

$$SU_*(BZ_2)\xrightarrow{t} SU_*(BZ_2)\xrightarrow{F} U_*(BZ_2)$$

of [6, (15.2)], this observation proves Theorem (1.2). In addition

(2.7) **Proposition.** *$SU_{2k}(BZ_2, *)=0$ unless $2k=2 \bmod 8$. $SU_{8k+2}(BZ_2, *)$ is the Z_2-vector space on $\{\alpha b_i^{8k}\sigma_1\}$.*

Proof. It remains only to show that the $\{\alpha b_i^{8k}\sigma_1\}$ are linearly independent. For this, note that the composition

$$SU_{8k+2}(BZ_2, *)\to SU_{8k+2}(Z_2, \mathit{free})\to SU_{8k+2},$$

where the latter map forgets involution, maps the $\alpha b_i^{8k}\sigma_1$ to a basis of torsion SU_{8k+2}. □

Proof of Theorem (1.3). Consider σ_{4n+3}. $F\sigma_{4n+3}$ is of order 2^{2n+2} by (2.2). For odd n, $4n+3=7 \bmod 8$ and F is monic, by (2.3) and (2.7).

Let $\circ$ be the product in U_* introduced by Wall [11]:

$$x\circ y = xy + 2([\mathbf{C}P(1)]^2-[\mathbf{C}P(2)])\,Dx\,Dy$$

where D is the composite $Fd\colon U_*\to U_{*-2}$. Let $x^{(k)}$ be the k-fold product $x\circ x\circ \cdots\circ x$. Then $x^{(k)}\tau_1 = x^k\tau_1$ since τ_1 has order 2. Let $x_1=[\mathbf{C}P(1)]$; then $x_1^{(2)}=\beta$ and $Dx_1=2$. By [9, pp. 265–266], if n is even we can choose $b_n\in SU_{4n}$ such that $\alpha b_n\neq 0$ and $Fb_n = x_1^{(2n)}$. Therefore

$$2^{2n+2}\sigma_{4n+3} = d(2^{2n+2}\tau_{4n+5}) = d(x_1^{2n+2}\tau_1),\ \text{by (2.2)}$$

$$= d(x_1^{(2)}x_1^{(2n)}\tau_1) = (d(x_1^{(2)}))b_n\sigma_1 = \alpha^2 b_n\sigma_1\neq 0,\quad \text{by (2.7) and (2.3).}$$

Thus σ_{4n+3} has order 2^{2n+3}. □

3. **The Smith homomorphism.** The Smith construction appeared abruptly in the proof of (2.5), and it is convenient to reconsider it at this point. Let $[M, T]\in SU_n(Z_2, \mathit{free})$. For large q, there is an equivariant map $g\colon (M, T)\to (S^{4q+3}, A)$. Make g equivariantly transverse regular to S^{4q-1}. Then $N=g^{-1}S^{4q-1}$ is an SU-manifold, and assigning $(N, T|N)$ to (M, T) defines the *Smith homomorphism*

$$\Delta\colon SU_n(Z_2, \mathit{free})\to SU_{n-4}(Z_2, \mathit{free}).$$

Applying the isomorphism (2.1), the Smith operators of (2.5) are induced, not by Δ, but by $\pi\Delta$, where $\pi: SU_*(BZ_2) \to SU_*(BZ_2, *)$ is projection on the summand. But, as it turns out, this makes no difference.

(3.1) **Proposition.** $\operatorname{Im} \Delta \subset SU_*(BZ_2, *)$.

Proof. Consider $i^p_*: SU_*(\mathrm{R}P(p), *) \to SU_*(BZ_2, *)$. The spectral sequence has

$$E^\infty_{p,*-p} = \operatorname{Im} i^p_* / \operatorname{Im} i^{p-1}_*.$$

(3.2) **Claim.** If $\Delta': SU_*(BZ_2, *) \to SU_{*-4}(BZ_2)$ is the restriction of Δ, then $\operatorname{Ker} \Delta' = \operatorname{Im} i^3_*$ and $\operatorname{Ker} \pi\Delta' = \operatorname{Im} i^4_*$.

On the other hand, $E^\infty_{4,q} = 0$ for all q, by (2.6). Hence $\operatorname{Im} \Delta' \cap \operatorname{Ker} \pi = 0$, which is (3.1).

We thus prove (3.2). Suppose $[M, T] \in \operatorname{Ker} \pi\Delta'$. Then there is an SU-manifold P with involution T' such that

$$\partial(P, T') = (N, T|N) - (N \times S^0, 1 \times A).$$

Let $g: N \times S^0 \to S^0 \to S^{4q+3}$ be the obvious equivariant map. Without loss of generality we may assume q is large enough for g to extend to an equivariant $g: (P, T') \to (S^{4q+3}, A)$.

The normal bundle of N in M is clearly $N \times \mathrm{R}^4$ with action $(T|N) \times A$. Thickening P and pasting it to $M \times I$ one can construct a cobordism from (M, T) to (M_0, T_0) where the latter is classified by a map into (S^{4q+3}, A) whose image intersects S^{4q-1}, transversely, in (S^0, A). Thus (M_0, T_0) admits an equivariant map into (S^4, A). Under (2.1) it falls into $\operatorname{Im} i^4_*$. The converse is obvious.

If $[M, T] \in \operatorname{Ker} \Delta'$ the same argument applies, but $\partial(T, T') = (N, T|N)$ and the image of (M_0, T_0) misses S^{4q-1}. Hence (M_0, T_0) admits an equivariant map into (S^3, A). □

It should be noted that $\operatorname{Im} \Delta$ is properly contained in $SU_*(BZ_2, *)$. For example, $\sigma_1 \notin \operatorname{Im} \Delta$ since $SU_5(BZ_2) = 0$.

Let $B: SU_n(Z_2, rel) \to SU_{n+4}(Z_2, rel)$ be multiplication by the 4-disk D^4 with antipodal action.

(3.3) **Proposition.** *There is a commutative diagram:*

$$\begin{array}{ccc} SU_{n+1}(Z_2, rel) & \xrightarrow{\partial} & SU_n(Z_2, free) \\ \Big\downarrow B & & \Big\uparrow \Delta \\ SU_{n+5}(Z_2, rel) & \xrightarrow{\partial} & SU_{n+4}(Z_2, free) \end{array}$$

Proof. Same as the unitary case [1, (10.3)]. Briefly, if $g: (M, T) \to (S^{4q-1}, A)$ is an equivariant map, then by suspending g one obtains an equivariant map $b: \partial(M \times D^4, T \times A) \to (S^{4q+3}, A)$ which is transverse regular on S^{4q-1}. □

Using (2.1), let r' be the restriction of r to SU_*. That is, $r'[M] = [M \times S^0, 1 \times A] \in SU_*(Z_2, all)$.

(3.4) **Proposition.** *r' is a monomorphism.*

Proof. By (3.3) and (1.1), Ker r = Im $\partial \subseteq$ Im Δ. On the other hand, Im Δ is orthogonal to the summand SU_*, by (3.1). □

Proof of Theorem (1.4). By (2.7) every element $x \in SU_{2k}(Z_2, free)$ can be written

$$x = y[S^0, A] + \alpha z[S^1, A], \qquad y \in SU_{2k},\ z \in SU_{2k-2}.$$

Let $\epsilon: SU_*(Z_2, free) \to SU_*$ forget involution. If $r(x) = 0$ then $2y + \alpha^2 z = \epsilon(x) = 0$. Since $\alpha^2 z$ cannot be divisible by 2 we must have $z = 0$. Then $y = 0$ by (3.4). □

4. Complex Wall manifolds. A unitary manifold M has a *complex Wall structure* if there is a map $f: M \to CP(1)$ and an isomorphism ϕ: det $\tau(M) \cong f^*\xi$, where $\xi \to CP(1)$ is the canonical line bundle. There is a bordism theory W_* for such objects, and a homology theory $W_*(X, A)$ for which Stong [9, VIII] is the general reference.

An involution T on M preserves the complex Wall structure if $fT = f$ and $\phi(\det dT) = T'\phi$, where $T': f^*\xi \to f^*\xi$ is induced by $T \times 1: M \times \xi \to M \times \xi$. Without belaboring the details, it should be clear that one has theories $W_*(Z_2, Q)$ for $Q = free, all, rel$, and an exact sequence

(4.1)
$$W_*(Z_2, free) \to W_*(Z_2, all) \to W_*(Z_2, rel) \xrightarrow{\partial} W_*(Z_2, free)$$

Since an SU-structure is a complex Wall structure with f = point map, (1.1) maps into (4.1) via forgetful maps G.

If T is free, f defines $g: M/T \to CP(1)$ and ϕ defines an isomorphism

$$\phi/T: \det \tau(M/T) = \det \tau(M)/\det dT \to g^*\xi = (f^*\xi)/T.$$

Thus M/T is again a complex Wall manifold, so

(4.2)
$$W_*(Z_2, free) \cong W_*(BZ_2).$$

Whether T is free or not, one may choose $x \in CP(1)$ and make f equivariantly transverse regular to x [8, (4.1)]. Since $f^*\xi$ is trivial over $N = f^{-1}(x)$, the assignment of $(N, T|N)$ to (M, T) defines a homomorphism

$$d: W_*(Z_2, Q) \to SU_*(Z_2, Q).$$

Then there are Rohlin-Dold sequences.

(4.3) **Proposition.** *For $Q = free, all, rel$, there are exact sequences*

$$SU_*(Z_2, Q) \xrightarrow{t} SU_*(Z_2, Q) \xrightarrow{G} W_*(Z_2, Q)$$
$$\xleftarrow{\quad\quad d \quad\quad}$$

The proof is a copy of [9, pp. 169–172], using [8, (4.1)] to secure the needed transversalities. For $Q = free$, (4.2) and (2.1) identify (4.3) as the usual Rohlin-Dold triangle in the bordism of BZ_2.

(Perhaps one should note that (4.3) is stronger than the sequence [8, (4.2)] used by the author in the O/SO case. This is because a stronger notion of structure-preserving has been used.)

Unfortunately, $F'\colon W_*(Z_2, all) \longrightarrow U_*(Z_2, all)$ is not monic. Thus the sequence

$$SU_*(Z_2, all) \longrightarrow SU_*(Z_2, all) \longrightarrow U_*(Z_2, all)$$

need not be exact. A similar phenomenon is well known in the case of the SO-bordism theories [2].

Also unfortunately, the product of complex Wall manifolds need not be a complex Wall manifold. Thus $W_*(BZ_2)$ is not a W_*-module under the usual Cartesian product. However, we can circumvent this difficulty.

Recall that for any pair (X, A), $F'\colon W_*(X, A) \longrightarrow U_*(X, A)$ is monic [9, p. 153]. In fact there is a splitting $\Phi\colon U_*(X, A) \longrightarrow W_*(X, A)$ which assigns to (M, f) the submanifold $N \subset M \times CP(1)$ dual to $\det \tau(M) \otimes \xi$. Given $u \in U_*$, $x \in U_*(X, A)$, define the *Wall product* $u \circ x$ by

(4.4) $\quad u \circ x = ux + 2([CP(1)]^2 - [CP(2)])Dx\,Dy,\quad$ where $F = Fd$ as before.

(4.5) **Proposition.** *Under the Wall product the image of $W_*(X, A)$ in $U_*(X, A)$ becomes a W_*-module. In fact, for $w \in W_*$, $x \in W_*(X, A)$, $w \circ x = F'\Phi(wx)$ and $D(wx) = wDx + (Dw)x - [CP(1)]Dw\,Dx$.*

The proof parrots one of Stong's [9]; we indicate the preliminaries. Let $P = CP(\infty)$. $U_*(P)$ is the free U_*-module on $\{a_i = [CP(i) \subset CP(\infty)]\}$ [6, (1.5)].

$$U_*(P \times X, P \times A) = U_*(P) \otimes U_*(X, A) \quad [7, (6.2)].$$

Let $H\colon U_*(P \times X, P \times A) \longrightarrow U_*(P \times X, P \times A)$ send $(M, f \times g)$ to $(N, (f \times g)j)$ where $j\colon N \subset M$ includes the submanifold dual to $f^*\lambda$, $\lambda \longrightarrow CP(\infty)$ being the universal line bundle. Then $H(a_i \otimes y) = a_{i-1} \otimes y$.

Let $\mu\colon U_*(X, A) \longrightarrow U_*(P \times X, P \times A)$ send (M, g) to $(M, f \times g)$, where $f\colon M \longrightarrow P$ classifies $\det \tau(M)$, and let $\pi\colon U_*(P \times X, P \times A) \longrightarrow U_*(X, A)$ be the projection. Then $D = \pi H \mu$ by definition.

If $P \times P \longrightarrow P$ classifies $\lambda \otimes \lambda$, there is induced a product

$$U_*(P) \otimes U_*(P \times X, P \times A) \longrightarrow U_*(P \times X, P \times A).$$

Clearly $\Phi(z) = \pi H(a_1\mu(z))$. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} U_* \otimes U_*(X, A) & \longrightarrow & U_*(X, A) \\ \downarrow \mu\otimes\mu & & \downarrow \mu \\ U_*(P) \otimes U_*(P\times X, P\times A) & \longrightarrow & U_*(P\times X, P\times A) \end{array}$$

in which the top map is the usual U_*-module structure.

Since w, x are represented by Wall manifolds, write $\mu w = a_0 \otimes \alpha_0 + a_1 \otimes \alpha_1$ and $\mu x = a_0 \otimes \beta_0 + a_1 \otimes \beta_1$ for $\alpha_i \in U_*$, $\beta_i \in U_*(X, A)$. One may then compute $D(wx)$ and $\Phi(wx)$ in exactly the same way as in [9, pp. 165–166]. Finally, the identity $(w_1 \circ w_2) \circ x = w_1 \circ (w_2 \circ x)$ is easily obtained from the formula for D. □

Observe that d is natural with respect to maps of pairs, and commutes with $\partial: U_*(X, A) \longrightarrow U_*(A)$. The Wall product inherits these properties. In particular, if (X, A) is a CW-pair then W_* acts, via (4.4), on the W_*-bordism spectral sequence $\{F^r_{p,q}\}$ of (X, A), and the action $F^2_{p,q} \otimes W_s \longrightarrow F^2_{p,q+s}$ is identified with the composite

$$H_p(X, A, W_q) \otimes W_s \longrightarrow H_p(X, A; W_q \otimes W_s) \longrightarrow H_p(X, A; W_{q+s})$$

(compare [3, (7.1)]).

Now set $(X, A) = (BZ_2, *)$. If $n = 1$ or $n = 4k + 3$ let $\omega_n \in W_n(BZ_2, *)$ be $[\mathbf{R}P(n), i^n]$. If $\mu: W_n(BZ_2, *) \longrightarrow H_n(BZ_2, *)$ is the usual evaluation [3, §6], then $\mu(\omega_n)$ is the nonzero class in $H_n(BZ_2, *)$.

(4.6) **Proposition.** *Suppose* $n = 4k + 1$. *Let* $\omega_n \in W_n(BZ_2, *)$ *be represented by* $Y^n = \mathbf{R}P(\xi \oplus (2k-1)\mathbf{C} \longrightarrow \mathbf{C}P(1))$, *and the map* $f: Y^n \longrightarrow BZ_2$ *classifying the double cover* $S(\xi \oplus (2k-1)) \longrightarrow Y^n$. *Then* $\mu(\omega_n) \neq 0 \in H_n(BZ_2, *)$.

Proof. The disk bundle $D(\xi \oplus (2k-1))$ has Chern classes induced from the base $\mathbf{C}P(1)$. Hence it is a Wall manifold and the antipodal involution is structure preserving. Let $\Delta_U: U_q(BZ_2, *) \longrightarrow U_{q-2}(BZ_2, *)$ be the unitary Smith homomormorphism. By [1, (10.3)],

$$\Delta_U^{2k-1}[Y^n, f] = [\mathbf{R}P(3), i^3] = r_3.$$

By [1, (10.2)],

$$[Y^n, f] = r_n + \sum_{j<2k} [X^{n-2j+1}] r_{2j+1} \in U_n(BZ_2, *).$$

Thus $\mu(\omega_n) = \mu(r_n) \neq 0$. □

(4.7) **Corollary.** *Using the Wall product,* $W_*(BZ_2, *)$ *is generated over* W_* *by the* ω_n, $n = 2j + 1$.

Proof. It is clear that $F^2_{p,q} = F^\infty_{p,q}$. Since W_* is torsionfree,

$$F^2_{p,q} \cong H_p(BZ_2, *; W_q) \cong H_p(BZ_2, *) \otimes W_q \cong F^2_{p,0} \otimes W_q.$$

For p even, $F^2_{p,0} = 0$. For p odd, let e_p generate $F^2_{p,0} = Z_2$. If $x \in W_q$, then by (4.4) $x \circ \omega_p$ corresponds to $e_p \otimes x \in F^2_{p,q}$. The rest is entirely standard [3, (18.1)]. □

5. On $SU_*(Z_2, all)$.

(5.1) **Lemma.** *$W_*(Z_2, rel) \cong \Sigma_q W_*(MU(2q))$, where $MU(2q)$ is the Thom space of the universal bundle $\gamma \longrightarrow U(2q)$.*

Proof. Given $[M, T] \in W_n(Z_2, rel)$ let F be a component of the fixed set and let $\nu \longrightarrow F$ be the normal bundle. Since $\det \tau(M)|F = (\det \tau(F) \otimes \det \nu)$, $\det dT$ acts on $\det \tau(M)|F$ as multiplication by $(-1)^{\dim \nu}$ in the fibers.

Imbed $D\nu$ in M as a tubular neighborhood of F and let $f: M \longrightarrow CP(1)$ and $\phi: \det \tau(M) \cong f^*\xi$ give the Wall structure. Via a homotopy, if needed, assume $f|D\nu$ factors through projection on F. Then $\det dT$ must act in the fiber of $\det \tau(M)$ over $x \in F$ as multiplication by the determinant of $(\phi^{-1}\phi)_x = 1$.

Therefore $\dim \nu$ is even, so classifying the fixed set defines a homomorphism

$$W_*(Z_2, rel) \longrightarrow \sum_q W_*(MU(2q)).$$

The rest of the proof is like [8, (3.2)]. □

(5.2) **Lemma.** *Im r: $W_q(Z_2, free) \longrightarrow W_q(Z_2, all)$ is*

$$\{M \times S^0, 1 \times A: [M] \in W_q\} \quad \text{if } q \text{ is even,}$$

$$\{M \times S^1, 1 \times A: [M] \in W_{q-1}\} \quad \text{if } q \text{ is odd.}$$

Proof. Choose $[M] \in W_*$. If $n = 4k+3$, $[M] \circ \omega_n$ corresponds in $W_*(Z_2, free)$ to $[M \times S^{4k+3}, 1 \times A]$, which certainly bounds in $W_*(Z_2, all)$. If $n = 4k+1 \geq 5$, $[M] \circ \omega_n = \partial([M] \circ [D(\xi \oplus (2k-1)), A])$, where (5.1) is used to define the Wall product in $W_*(Z_2, rel)$. Thus (5.2) follows from (4.7) and the fact that $W_{2m+1} = 0$. □

(5.3) **Corollary.** *F': $W_q(Z_2, all) \longrightarrow U_q(Z_2, all)$ is monic if q is even. If q is odd, $W_q(Z_2, all)$ contains only the classes $[M \times S^1, 1 \times A\}$, so $F' = 0$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} W_q(Z_2, free) & \xrightarrow{r} & W_q(Z_2, all) & \longrightarrow & W_q(Z_2, rel) \\ \downarrow a & & \downarrow b & & \downarrow c \\ U_q(Z_2, free) & \longrightarrow & U_q(Z_2, all) & \longrightarrow & U_q(Z_2, rel) \end{array}$$

a and c are monic, by (4.2), (5.1), the results of [1] on the unitary groups, and the knowledge that $W_*(X, A) \subset U_*(X, A)$. Thus Ker $b \subseteq$ Im r. If q is even, the composition $W_* \subset U_* \longrightarrow U_*(Z_2, all)$, sending x to $x[S^0, A]$ is monic [1]. If q is odd,

$U_q(Z_2, all) = 0$, again by [1]. Hence the corollary follows from (5.2) in either case. □

Theorem (1.5) is an obvious corollary of (5.3) and (4.3). We can also prove

(5.4) **Proposition.** *Im* r: $SU_*(Z_2, free) \to SU_*(Z_2, all)$ *is generated by* $[S^0, A]$ *and* $[S^1, A]$.

Proof. Given $x \in SU_n(Z_2, free)$, by the results of §2 we can write $x = y_0[S^0, A] + y_1[S^1, A] + z$, where $z \in \mathrm{Ker}\ t = \mathrm{Im}\ d$. But since $rd = dr$ it follows from (5.2) that Im rd is generated by $[S^0, A]$ and $[S^1, A]$. □

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