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PRODUCTS OF INITIALLY $\mathfrak{m}$-COMPACT SPACES

BY

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ABSTRACT. The main purpose of this paper is to give several theorems and examples which we hope will be of use in the solution of the following problem. For an infinite cardinal number $\mathfrak{m}$, is initial $\mathfrak{m}$-compactness preserved by products? We also give some results concerning properties of Stone-Čech compactifications of discrete spaces.

1. Introduction. The cardinality of a set X will be denoted by $|X|$, and for any cardinal number $\mathfrak{m}$, we will write $\exp(\mathfrak{m})$ instead of $2^{\mathfrak{m}}$. The cofinality of a cardinal number $\mathfrak{m}$ will be denoted by $\mathrm{cf}(\mathfrak{m})$. For a filter base $\mathfrak{F}$ on a topological space X, we denote the set of adherent points of $\mathfrak{F}$ by $\mathrm{ad}_X\mathfrak{F}$ or $\mathrm{ad}\,\mathfrak{F}$. Thus, $\mathrm{ad}_X\mathfrak{F} = \bigcap\{\overline{F} | F \in \mathfrak{F}\}$.

A space X is called *initially* $\mathfrak{m}$-*compact* (where $\mathfrak{m}$ is an infinite cardinal number) provided either of the following equivalent conditions holds:

(i) for every filter base $\mathfrak{F}$ on X, if $|\mathfrak{F}| \leq \mathfrak{m}$, then $\mathrm{ad}_X\mathfrak{F} \neq \emptyset$; or

(ii) for every open cover $\mathfrak{O}$ of X, if $|\mathfrak{O}| \leq \mathfrak{m}$, then $\mathfrak{O}$ has a finite subcover.

Initially $\aleph_0$-compact spaces are called *initially compact* or *countably compact.*

For $\mathfrak{m} = \aleph_0$, the question stated in the Abstract has a negative answer, for it is well known that there exists a countably compact subspace G of βN such that $G \times G$ is not countably compact (see [GJ, 9.15]). If $\mathfrak{m}$ is an infinite singular cardinal, and if one assumes the Generalized Continuum Hypothesis, then it follows from Theorem 1.1 below that the question has an affirmative answer. Theorem 1.1 improves Theorem 4.11 of [SS1] and Theorem 3.1 of [S2].

Theorem 1.1. *Let* $\{X_a | a \in A\}$ *be a family of initially* $\mathfrak{m}$-*compact spaces, where* $\mathfrak{m}$ *is a singular cardinal such that* $\exp(\mathfrak{n}) \leq \mathfrak{m}$ *for every cardinal* $\mathfrak{n} < \mathfrak{m}$. *Then* $X = \Pi\{X_a | a \in A\}$ *is initially* $\mathfrak{m}$-*compact.*

Proof. It is easy to see that for an infinite singular cardinal $\mathfrak{m}$, a space is initially $\mathfrak{m}$-compact if and only if it is initially $\mathfrak{n}$-compact for every infinite car-

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dinal $\mathfrak{n} < \mathfrak{m}$. Let $\mathcal{F}$ be a filter base on X such that $|\mathcal{F}| \leq \mathfrak{n}$. We will show that ad $\mathcal{F} \neq \emptyset$. For each $F \in \mathcal{F}$ choose a point $p_F \in F$, and set $Z = \{p_F | F \in \mathcal{F}\}$. The filter base $\{Z \cap F | F \in \mathcal{F}\}$ is contained in an ultrafilter $\mathfrak{U}$ on Z, and $|\mathfrak{U}| \leq \exp(|Z|) \leq \mathfrak{m}$. Then for each $a \in A$, $\mathrm{pr}_a \mathfrak{U}$ (the projection of $\mathfrak{U}$ onto X_a) is a base for an ultrafilter on the initially $\mathfrak{m}$-compact space X_a, and $|\mathrm{pr}_a \mathfrak{U}| \leq \mathfrak{m}$, so $\mathrm{pr}_a \mathfrak{U}$ converges to a point of X_a. Thus $\mathfrak{U}$ converges to a point of X and hence $\emptyset \neq \mathrm{ad}\, \mathfrak{U} \subset \mathrm{ad}\, \mathcal{F}$.

For an arbitrary infinite cardinal number $\mathfrak{m}$, several authors have succeeded in showing that if a certain number of factor spaces satisfy various natural conditions, only mildly stronger than initial $\mathfrak{m}$-compactness, then the resulting product space is initially $\mathfrak{m}$-compact. In §2 we will state several known results of this type, and we will obtain a new one (Theorem 2.2) concerning the condition $(1)_\mathfrak{m}$ recently studied in [V1]. In §3 we consider examples of spaces which show that for any infinite cardinal $\mathfrak{m}$, $(1)_\mathfrak{m}$ is strictly stronger than initial $\mathfrak{m}$-compactness. In the final section it is shown that for every infinite cardinal $\mathfrak{m}$ there exist an initially $\mathfrak{m}$-compact space and at least a weakly initially $\mathfrak{m}$-compact space (defined in §4) whose product is not initially $\mathfrak{m}$-compact. These examples will all be subspaces of βX for a discrete space X, and their properties will be derived from estimates (obtained in §3) of the cardinalities of the adherences of certain types of filter bases on βX.

2. Initially $\mathfrak{m}$-compact product spaces. In [N2] N. Noble studied properties of $\mathfrak{C}^*$, the family of T_1-spaces in which every infinite subset has infinitely many points in common with some compact subset. He proved that for every $X \in \mathfrak{C}^*$ and countably compact T_1-space Y, the product space $X \times Y$ is countably compact. For completely regular Hausdorff spaces, this theorem was also used by T. Isiwata [I]. In addition to this product theorem, the class $\mathfrak{C}^*$ admits several other nice results (see (d) and (e) below).

Obviously $\mathfrak{C}^*$ contains all sequentially compact spaces as well as all compact spaces. Furthermore, the class of noncompact or non sequentially compact spaces in $\mathfrak{C}^*$ is quite large (e.g., see [F2], [F3], [I], [SS1]), so it is natural to seek analogues of this concept for cardinals $\mathfrak{m} > \aleph_0$. One such analogue was given in [SS1] where the authors defined a T_1-space X to be *strongly* $\mathfrak{m}$*-compact* (for an infinite cardinal number $\mathfrak{m}$) provided that for every filter base $\mathcal{F}$ on X, if $|\mathcal{F}| \leq \mathfrak{m}$, then there exists a compact subset K of X such that $F \cap K \neq \emptyset$ for all $F \in \mathcal{F}$. It was noted in [SS1] that for every strongly $\mathfrak{m}$-compact space X and initially $\mathfrak{m}$-compact T_1-space Y, $X \times Y$ is initially $\mathfrak{m}$-compact. In 4.8–4.9 of [SS1], examples due to Victor Saks were given in order to show that for every regular cardinal $\mathfrak{m}$, there are initially $\mathfrak{m}$-compact completely regular Hausdorff spaces and topological groups that are not strongly $\mathfrak{m}$-compact.

Two other product theorems, both of which are special cases of results appearing in [G, pp. 379–380], are the following: if X and Y are initially $\mathfrak{m}$-compact completely regular Hausdorff spaces, and if X is either locally compact or is of character $\leq \mathfrak{m}$, then $X \times Y$ is initially $\mathfrak{m}$-compact.

Theorem 2.2 below simultaneously extends all of these theorems. First a definition and a lemma are needed.

Definition [V1]. Let $\mathfrak{m}$ be an infinite cardinal number. A space X is said to satisfy condition $(1)_{\mathfrak{m}}$ provided that for every filter base $\mathfrak{F}$ on X, if $|\mathfrak{F}| \leq \mathfrak{m}$, then there exist a compact set $K \subset X$ and a filter base $\mathfrak{G}$ on X such that $|\mathfrak{G}| \leq \mathfrak{m}$, and $\mathfrak{G}$ is finer than both $\mathfrak{F}$ and the filter base of all open sets containing K.

Lemma 2.1. (i) *For a T_1-space X, the following are equivalent*: $X \in \mathfrak{C}^*$; *X is strongly $\aleph_0$-compact; and X satisfies condition $(1)_{\aleph_0}$.*

(ii) *A locally compact initially $\mathfrak{m}$-compact T_1-space is strongly $\mathfrak{m}$-compact.*

(iii) [V1] *If a space X is either* (a) *strongly $\mathfrak{m}$-compact or* (b) *initially $\mathfrak{m}$-compact and of character $\leq \mathfrak{m}$, then X satisfies condition $(1)_{\mathfrak{m}}$.*

(iv) *Any space satisfying $(1)_{\mathfrak{m}}$ is initially $\mathfrak{m}$-compact.*

The proofs are straightforward.

Theorem 2.2. *Let X be a space satisfying condition $(1)_{\mathfrak{m}}$, and suppose that Y is an initially $\mathfrak{m}$-compact space. Then $X \times Y$ is initially $\mathfrak{m}$-compact.*

Proof. Let $\mathfrak{F}$ be a filter base on $X \times Y$ such that $|\mathfrak{F}| \leq \mathfrak{m}$. We wish to show that $\varnothing \neq \operatorname{ad} \mathfrak{F}$, so we may assume that $\mathfrak{F}$ consists of closed sets.

By hypothesis there are a compact set $K \subset X$ and a filter base $\mathfrak{G}$ on X such that $|\mathfrak{G}| \leq \mathfrak{m}$, and $\mathfrak{G}$ is finer than both $\mathrm{pr}_X(\mathfrak{F})$ and the filter base $\mathfrak{V}$ of all open subsets of X containing K. Then

$$\mathfrak{H} = \{F \cap (G \times Y) | F \in \mathfrak{F} \text{ and } G \in \mathfrak{G}\}$$

is a filter base on $X \times Y$ with $|\mathfrak{H}| \leq \mathfrak{m}$, and since Y is initially $\mathfrak{m}$-compact, $\mathrm{pr}_Y(\mathfrak{H})$ has an adherent point y. Let $\mathfrak{W}$ be the family of all open neighborhoods of y. Then for all $H \in \mathfrak{H}$ and $W \in \mathfrak{W}$, we have $H \cap (X \times W) \neq \varnothing$, so for all $F \in \mathfrak{F}$, $G \in \mathfrak{G}$, and $W \in \mathfrak{W}$,

$$\varnothing \neq F \cap (G \times Y) \cap (X \times W) = F \cap (G \times W).$$

Since $\mathfrak{G}$ is finer than $\mathfrak{V}$, it follows that $\varnothing \neq F \cap (V \times W)$ for all $V \in \mathfrak{V}$ and $W \in \mathfrak{W}$. Because $\mathfrak{V} \times \mathfrak{W}$ is a fundamental system of neighborhoods for the compact set $K \times \{y\}$, we may conclude that no open set of the form $(X \times Y) \setminus F$ can contain $K \times \{y\}$, or, equivalently, $\varnothing \neq F \cap (K \times \{y\})$ for all $F \in \mathfrak{F}$. Thus $\varnothing \neq \cap \mathfrak{F} \cap (K \times \{y\}) \subset \operatorname{ad} \mathfrak{F}$.

For products of possibly infinitely many spaces, the condition $(1)_{\mathfrak{m}}$ is also useful as a means to obtain one theorem which covers quite a variety of cases of interest. For instance, consider the following known theorems.

(a) [SS2, Theorem 5.5]. *Every product of at most* $\aleph_1$ *sequentially compact spaces is countably compact.*

(b) [G, p. 379]. *A product of no more than* $\mathfrak{m}$ *completely regular Hausdorff spaces, each initially* $\mathfrak{m}$*-compact and all but one locally compact, is initially* $\mathfrak{m}$*-compact.*

(c) [G, p. 380]. *A product of no more than* $\mathfrak{m}$ *completely regular Hausdorff spaces, each initially* $\mathfrak{m}$*-compact and all but one of character* $\leq \mathfrak{m}$*, is initially* $\mathfrak{m}$*-compact.*

(d) [F3, p. 347]. *If* $\mathcal{C} \subset \mathfrak{C}^*$, *where each member of* $\mathcal{C}$ *is completely regular and Hausdorff, and if* $|\mathcal{C}| \leq \aleph_0$, *then* $\Pi\mathcal{C} \in \mathfrak{C}^*$.

(e) [SS1, Theorem 2.4]. *Every product of at most* $\aleph_1$ *spaces in* $\mathfrak{C}^*$ *is countably compact.*

(f) [N3, Theorem 4.4]. *Every product of at most* $\mathfrak{m}^+$ *spaces, each of which is initially* $\mathfrak{m}$*-compact and of character* $\leq \mathfrak{m}$*, is initially* $\mathfrak{m}$*-compact.*

(a) and (d)–(f) can all be obtained from Theorem 2.3, and (b) and (c) can be obtained from Theorems 2.2 and 2.3.

Theorem 2.3 [V1, Theorems 1.4 and 2.2]. *Let* $X = \Pi\{X_a | a \in A\}$ *where each* X_a *satisfies condition* $(1)_{\mathfrak{m}}$ *for a fixed cardinal number* $\mathfrak{m}$.

(i) *If* $|A| \leq \mathfrak{m}$, *then* X *satisfies* $(1)_{\mathfrak{m}}$.

(ii) *If* $|A| \leq \mathfrak{m}^+$, *then* X *is initially* $\mathfrak{m}$*-compact.*

It is natural to raise the question: Can the restrictions on the cardinality of A in Theorem 2.3 be weakened?

For $\mathfrak{m} = \aleph_0$, this question cannot be answered on the basis of the usual axioms of set theory. Examples due to Frolík [F2, p. 337] show that if $|A| = \exp(\aleph_0)$, then the product space X need not satisfy property $(1)_{\aleph_0}$, and if $|A| = \exp(\exp(\aleph_0))$, then X need not be initially $\aleph_0$-compact (these examples are discussed at the end of §3). It follows that if $\aleph_i = \exp(\aleph_{i-1})$ for $i = 1, 2$, then the restrictions on $|A|$ cannot be weakened. On the other hand, S. H. Hechler [H] has shown that Martin's Axiom implies that if $|A| < \exp(\aleph_0)$ then X satisfies $(1)_{\aleph_0}$, and if $A \leq \exp(\aleph_0)$ then X is initially $\aleph_0$-compact. Thus, Martin's Axiom plus the negation of the Continuum Hypothesis imply that the restrictions on $|A|$ can be weakened.

For $\mathfrak{m} > \aleph_0$, we do not know any answer to the above question except for Theorem 1.1 and the following special case.

Theorem 2.4. *If $\{X_a | a \in A\}$ is a family of spaces, each of which satisfies condition $(1)_\mathfrak{m}$, and if $X = \Pi\{X_a | a \in A\}$ is a normal T_1-space, then X is initially $\mathfrak{m}$-compact (regardless of the cardinality of the index set A).*

Proof. In [N3, Theorem 2.1] Noble proved that if a product space $X = \Pi\{X_a | a \in A\}$ is a normal T_1-space and if $|A| \geq \aleph_0$, then there exists a countable set $B \subset A$ such that $Y = \Pi\{X_a | a \in A\backslash B\}$ is initially $|A|$-compact. Let $Z = \Pi\{X_a | a \in B\}$, and note that by Theorem 2.3 (i), Z satisfies $(1)_\mathfrak{m}$. Thus if $|A| \geq \mathfrak{m}$, Y is initially $\mathfrak{m}$-compact and so $X = Y \times Z$ is initially $\mathfrak{m}$-compact by Theorem 2.2. If $|A| < \mathfrak{m}$, then X is initially $\mathfrak{m}$-compact by Theorem 2.3.

A. H. Stone has asked if every product of sequentially compact spaces is countably compact. A partial answer is provided by the next result.

Corollary 2.5 (P. Kenderov [K]). *Let X be a normal T_1-space which is a product of sequentially compact spaces. Then X is countably compact.*

Since a sequentially compact space satisfies $(1)_{\aleph_0}$, Corollary 2.5 follows at once from Theorem 2.4.

3. **Subspaces of βX not satisfying $(1)_\mathfrak{m}$.** The well-known examples which show that countable compactness is not productive are all based on subspaces of βN. Thus, one is naturally inclined to wonder if there exist a cardinal $\mathfrak{m} > \aleph_0$ and a family $\mathcal{P}$ of initially $\mathfrak{m}$-compact subspaces of βX, for a discrete space X, such that $\Pi\mathcal{P}$ is not initially $\mathfrak{m}$-compact. We have not been able to find such subspaces, but we have found subspaces of βX which are initially $\mathfrak{m}$-compact and which do not satisfy $(1)_\mathfrak{m}$. We will describe some of their properties in this and the next section.

A property of βN that has proved very useful in the construction of product spaces which are not countably compact is the following: if D is any countably infinite discrete subspace of βN, then $\overline{D} = \beta D$ and hence $|\overline{D}| = \exp(\exp(\aleph_0))$. This equality makes possible the construction of a large number of noncompact, countably compact subspaces of βN, since if one arbitrarily selects a subset C of βN such that $|C| < \exp(\exp(\aleph_0))$, then the space $P = \beta N\backslash C$ is countably compact. In case C is chosen so that $C \subset \beta N\backslash N$ and C contains a limit point of each infinite subset of N, then it is easy to see that $P \notin \mathfrak{C}^*$ (and if C is chosen so that $|C| < \exp(\aleph_0)$, then $P \in \mathfrak{C}^*$ by Example 3.10 or by [F3]).

On the other hand, if D is an uncountable discrete subspace of βX for an infinite discrete space X, it is not necessarily true that $|\overline{D}| = \exp(\exp(|D|))$. It is known (e.g. see [N1], [V2], or [SS1, Remark 3.2]) that, under the assumption of the Continuum Hypothesis, there exists a discrete subspace D of βX with $|D| = \aleph_1$ and $|\overline{D}| = \exp(\aleph_1)$. Likewise, if C is a subset of βX with $|C| < \exp(\exp(|X|))$, the space $\beta X \setminus C$ may not be initially $|X|$-compact. In fact, if C consists of a single point x, and if x is a limit point of some countable subset of βX, then the space $\beta X \setminus C$ is obviously not initially $\exp(\aleph_0)$-compact.

In any case, working with certain discrete subspaces D for which the equality $|\overline{D}| = \exp(\exp(|D|))$ holds, we obtain several results below which, combined with a construction due to Saks [SS1], establish the existence of the desired subspaces. First some definitions and conventions are made.

For the remainder of the paper, X will be an infinite discrete space with $|X| = \mathfrak{m}$. For $V \subset X$, we will write V^* for the clopen set $\mathrm{Cl}_{\beta X} V$, and for an arbitrary set $W \subset \beta X$, we sometimes denote $\mathrm{Cl}_{\beta X} W$ by $\overline{W}$.

An infinite subset D of βX will be called *strongly discrete* if there exist sets $V_d \subset X$, $d \in D$, so that $d \in V_d^*$, and for all $d, e \in D$, $d \neq e$ implies that $V_d \cap V_e = \emptyset$.

Given $D \subset \beta X$, we will denote by μD the set of all points $p \in \overline{D}$ such that for every set $E \subset D$, $p \in \overline{E}$ implies $|E| = |D|$. Thus

$$\mu X = \{p \in \beta X \mid \text{for every } V \subset X,\ p \in V^* \text{ implies } |V| = \mathfrak{m}\}.$$

Lemma 3.1. *Let D be a strongly discrete subset of βX. Then the following hold.*

(i) $\overline{D} = \beta D$.

(ii) $|\overline{D}| = |\mu D| = \exp(\exp(|D|))$, *and μD is compact.*

(iii) *If $|D| = \mathfrak{m}$ then $\mu D \subset \mu X$.*

An argument like the one on p. 91 of [GJ] establishes (i), and (ii) follows from (i) and p. 170 of [GJ]. Assertion (iii) is an immediate consequence of the definition of strongly discrete.

We will say that a filter base $\mathfrak{F}$ on a set is of *type* $\mathfrak{m}$ (where $\mathfrak{m}$ is an infinite cardinal number) provided that $|\mathfrak{F}| \leq \mathfrak{m}$ and $|F| = \mathfrak{m}$ for every $F \in \mathfrak{F}$.

Theorem 3.2. *Let $\mathfrak{F}$ be a filter base on X of type $\mathfrak{m}$. Then there exists a strongly discrete subset D of βX with $|D| = \mathfrak{m}$ such that $D \subset \mu X$ and $\overline{D} \subset \mathrm{ad}_{\beta X}\mathfrak{F}$. In particular, $|\mu X \cap \mathrm{ad}_{\beta X}\mathfrak{F}| = \exp(\exp(\mathfrak{m}))$.*

Proof. By Lemma 3.1, it suffices for us to establish the existence of a set D with the above properties.

It follows from a theorem of W. Sierpiński [S1, p. 455] that there exists a family $\mathcal{S}$ of pairwise disjoint subsets of X such that $|\mathcal{S}| = \mathfrak{m}$, and $|S \cap F| = \mathfrak{m}$ for every $S \in \mathcal{S}$ and $F \in \mathfrak{F}$. Thus for each $S \in \mathcal{S}$,

$$\mathcal{H}(S) = \{S \cap F \cap Y | F \in \mathfrak{F},\ Y \subset X, \text{ and } |X \setminus Y| < \mathfrak{m}\}$$

is a filter base and hence has an adherent point $d(S)$. Then, obviously, each $d(S) \in S^* \cap \mu X \cap \operatorname{ad} \mathfrak{F}$, so that set $D = \{d(S) | S \in \mathcal{S}\}$ has the desired properties.

Before stating the next theorem, we need a lemma.

Lemma 3.3. *Let Y be an initially $\mathfrak{m}$-compact space, and let $\mathfrak{F}$ be a filter base on Y such that $|\mathfrak{F}| \leq \mathfrak{m}$. Then the following hold.*

(i) *For any open subset V of Y containing $\operatorname{ad}_Y \mathfrak{F}$, there exists $F \in \mathfrak{F}$ with $V \supset F$.*

(ii) *If $\operatorname{ad}_Y \mathfrak{F}$ is compact and Y is a subspace of a Hausdorff space H, then $\operatorname{ad}_Y \mathfrak{F} = \operatorname{ad}_H \mathfrak{F}$.*

Proof. Obviously (i) is true. In (ii), the compact set $\operatorname{ad}_Y \mathfrak{F} = \bigcap \{\overline{W} | W$ is an open subset of H containing $\operatorname{ad}_Y \mathfrak{F}\}$. By (i), each such W contains a member of $\mathfrak{F}$, so it follows that $\operatorname{ad}_H \mathfrak{F} \subset \operatorname{ad}_Y \mathfrak{F}$. Thus $\operatorname{ad}_H \mathfrak{F} = \operatorname{ad}_Y \mathfrak{F}$.

Theorem 3.4. *Let $P \subset \beta X$ be initially $\mathfrak{m}$-compact, and let E be any discrete subspace of P such that $|E| = \mathfrak{m}$ and $\operatorname{Cl}_{\beta X} E = \beta E$. Suppose that $\mathfrak{F}$ is a filter base of type $\mathfrak{m}$ on E such that $\operatorname{ad}_P \mathfrak{F}$ is compact. Then $|\mu E \cap \operatorname{ad}_P \mathfrak{F}| = \exp(\exp(\mathfrak{m}))$, and there is a strongly discrete subset D of βE such that $|D| = \mathfrak{m}$, $D \subset \mu E$, and $\operatorname{Cl}_{\beta X} D \subset \operatorname{ad}_P \mathfrak{F}$.*

Theorem 3.4 is an immediate consequence of Lemma 3.3 and Theorem 3.2.

Using Theorem 3.4 and the next two lemmas, we will determine the cardinality of the adherences of certain filter bases of type $\mathfrak{m}$ in subspaces of βX that satisfy $(1)_{\mathfrak{m}}$.

Lemma 3.5. *Let Y be a Hausdorff space. The following are equivalent.*

(i) *Y satisfies condition $(1)_{\mathfrak{m}}$.*

(ii) *For every filter base $\mathfrak{F}$ on Y, if $|\mathfrak{F}| \leq \mathfrak{m}$ then there exists a finer filter base $\mathfrak{G}$ on Y such that $|\mathfrak{G}| \leq \mathfrak{m}$ and $\operatorname{ad}_Y \mathfrak{G}$ is nonempty and compact.*

Notation. Given a family $\mathfrak{G}$ of sets, $[\mathfrak{G}]$ will denote the family of all finite intersections of members of $\mathfrak{G}$.

Lemma 3.6. *Let $\mathfrak{F}$ be a filter base on X of type $\mathfrak{m}$. Then there exists a filter base $\mathcal{H}$ of type $\mathfrak{m}$ on X such that $\mathcal{H}$ is finer than $\mathfrak{F}$, and for every subset V of X, if $|V| < \mathfrak{m}$ then there is a set $H \in \mathcal{H}$ with $H \cap V = \varnothing$.*

Proof. Let $\{\mathfrak{r}_t | t \in \operatorname{cf}(\mathfrak{m})\}$ be a collection of regular cardinals such that $\sup\{\mathfrak{r}_t | t \in \operatorname{cf}(\mathfrak{m})\} = \mathfrak{m}$. Then there exists a family of filter bases $\{\mathfrak{F}(t) | t \in \operatorname{cf}(\mathfrak{m})\}$ on X such that

(1) if $V \subset X$ and $|V| < \mathfrak{r}_t$ then there is a set $H \in \mathcal{F}(t)$ with $H \cap V = \emptyset$; and

(2) each $[\mathcal{F} \cup (\bigcup\{\mathcal{F}(s) | s \leq t\})]$ is of type $\mathfrak{m}$.

For suppose that $u \in \mathrm{cf}(\mathfrak{m})$ and filter bases $\mathcal{F}(t)$, $t \in u$, on X of type $\mathfrak{m}$ have already been defined so that (1) and (2) hold for all $t \in u$. We wish to define $\mathcal{F}(u)$.

By (2), $\mathcal{G} = [\mathcal{F} \cup (\bigcup\{\mathcal{F}(s) | s \in u\})]$ is a filter base of type $\mathfrak{m}$, so by the theorem of Sierpiński, there is a family $\mathcal{P}$ of pairwise disjoint subsets of X such that $|\mathcal{P}| = \mathfrak{r}_u$ and $|P \cap G| = \mathfrak{m}$ for all $P \in \mathcal{P}$ and $G \in \mathcal{G}$. Let $\{P_n | n \in \mathfrak{r}_u\}$ be a one-one indexing of the members of $\mathcal{P}$, and define

$$\mathcal{F}(u) = \left\{G \cap \left(\bigcup\{P_n | n \geq q\}\right) \middle| G \in \mathcal{G} \text{ and } q \in \mathfrak{r}_u\right\}.$$

Then, clearly, (2) holds for all $t \leq u$. Furthermore, if $V \subset X$ and $|V| < \mathfrak{r}_u$, it follows from the regularity of the cardinal $\mathfrak{r}_u$ that for some $q \in \mathfrak{r}_u$, $V \cap (\bigcup\{P_n | n \geq q\}) = \emptyset$. Thus (1) holds for all $t \leq u$.

Set $\mathcal{H} = [\mathcal{F} \cup (\bigcup\{\mathcal{F}(t) | t \in \mathrm{cf}(\mathfrak{m})\})]$. Then by (2), $\mathcal{H}$ is a filter base on X of type $\mathfrak{m}$ that is finer than $\mathcal{F}$. Furthermore, if $V \subset X$ and $|V| < \mathfrak{m}$, with, say, $|V| < \mathfrak{r}_t$, then it follows from (1) that there is a set $H \in \mathcal{F}(t) \subset \mathcal{H}$ such that $H \cap V = \emptyset$.

Theorem 3.7. *Let P be a subspace of βX which satisfies condition* $(1)_{\mathfrak{m}}$. *Let E be a discrete subspace of P such that $|E| = \mathfrak{m}$ and $\mathrm{Cl}_{\beta X} E = \beta E$, and suppose that $\mathcal{F}$ is a filter base on E of type $\mathfrak{m}$. Then* $|\mu E \cap \mathrm{ad}_P \mathcal{F}| = \exp(\exp(\mathfrak{m}))$.

Proof. By Lemma 3.6 there is a filter base $\mathcal{H}$ on E of type $\mathfrak{m}$ such that $\mathcal{H}$ is finer than $\mathcal{F}$, and $\mathrm{ad}_{\beta E} \mathcal{H} \subset \mu E$. Since P satisfies $(1)_{\mathfrak{m}}$, there must exist a filter base $\mathcal{G}$ on E finer than $\mathcal{H}$ with $\mathrm{ad}_P \mathcal{G}$ compact and $|\mathcal{G}| \leq \mathfrak{m}$. Then $\mathrm{ad}_{\beta E} \mathcal{G} \subset \mathrm{ad}_{\beta E} \mathcal{H} \subset \mu E$, so $\mathcal{G}$ must be of type $\mathfrak{m}$, and hence $|\mathrm{ad}_P \mathcal{G}| = \exp(\exp(\mathfrak{m}))$ by Theorem 3.4. Since also $\mathrm{ad}_P \mathcal{G} \subset \mathrm{ad}_P \mathcal{H} \subset \mu E \cap \mathrm{ad}_P \mathcal{F}$, it follows that $\exp(\exp(\mathfrak{m})) = |\mathrm{ad}_P \mathcal{G}| \leq |\mu E \cap \mathrm{ad}_P \mathcal{F}|$.

In [SS1] a transfinite induction process due to Saks is used to establish the next result.

Theorem 3.8 [SS1, Theorem 4.5]. *Let S be an initially $\mathfrak{m}$-compact Hausdorff space containing a subspace Y such that $|Y| = \mathfrak{m}$ and $|\overline{Y}| = \exp(\exp(\mathfrak{m}))$. Then there is a set $Y \subset B \subset S$ such that $|B| \leq \exp(\mathfrak{m})$ and B is initially $\mathfrak{m}$-compact. If S is a topological group, then B can also be taken to be a topological group.*

In [SS1], Theorem 3.8 was used to show that for every regular cardinal $\mathfrak{m}$ there exists an initially $\mathfrak{m}$-compact topological group which is not strongly $\mathfrak{m}$-compact. In the example below it is not required that $\mathfrak{m}$ be regular.

Example 3.9. Let S be the topological group $\{f | f: \exp(\mathfrak{m}) \to \{0, 1\}\}$, with the product topology. Then there exists an initially $\mathfrak{m}$-compact subgroup B of S

such that $|B| \leq \exp(\mathfrak{m})$ and B does not satisfy condition $(1)_{\mathfrak{m}}$.

Proof. Since βX is a subspace of S, we can take $Y = X$ and let B be as in Theorem 3.8. Then B is an initially $\mathfrak{m}$-compact subgroup of S and $|B| \leq \exp(\mathfrak{m})$.

If B satisfied $(1)_{\mathfrak{m}}$ then so would its closed subset $P = B \cap \beta X$, and hence it would follow from Theorem 3.7 that the filter base $\mathfrak{F} = \{X\}$ had $|\mathrm{ad}_P \mathfrak{F}| = \exp(\exp(\mathfrak{m}))$, whereas $|P| \leq |B| \leq \exp(\mathfrak{m})$.

For at least $\mathfrak{m} = \aleph_0$, there is a large family of subspaces of βX to which Theorem 3.7 can be applied.

Example 3.10. Let C be a subset of $\beta \mathrm{N}$ such that for any infinite subset D of $P = \beta\mathrm{N}\backslash C$, there is an infinite subset I of D with $|C \cap \mathrm{Cl}_{\beta\mathrm{N}} I| < \exp(\aleph_0)$. Then the space P belongs to $\mathfrak{C}^*$.

Proof. It suffices to prove that if I is any countably infinite discrete subspace of P such that $|C \cap \mathrm{Cl}_{\beta\mathrm{N}} I| < \exp(\aleph_0)$, then for some infinite subset M of I, $\mathrm{Cl}_P M$ is compact.

Let $\mathfrak{M}$ be a family of infinite subsets of I such that $|\mathfrak{M}| = \exp(\aleph_0)$ and the intersection of any two members of $\mathfrak{M}$ is finite (such a family exists by a result of Sierpiński [GJ, 6Q.1]). Then $\mathrm{Cl}_{\beta\mathrm{N}} I = \beta I$, so the sets $M' = \mathrm{Cl}_{\beta I} M \backslash M$, $M \in \mathfrak{M}$, are pairwise disjoint. Hence some $M' \cap C = \emptyset$ and $\mathrm{Cl}_{\beta I} M \subset P$.

The useful examples of Frolík referred to earlier can now be easily described. For each $x \in \beta\mathrm{N}\backslash\mathrm{N}$ let $P_x = \beta\mathrm{N}\backslash\{x\}$ and note that $P_x \in \mathfrak{C}^*$. Let $S = \Pi\{P_x | x \in \beta\mathrm{N}\backslash\mathrm{N}\}$ and $T = R \times F$, where $R = \Pi\{P_x | x \in F\backslash\mathrm{N}\}$, and F is a countably compact subspace of $\beta\mathrm{N}$ such that $\mathrm{N} \subset F$ and $|F| \leq \exp(\aleph_0)$. Then the constant functions are infinite closed discrete subspaces of both S and T, so S and T are not countably compact, and (e.g., by Theorem 2.2), $R \notin \mathfrak{C}^*$. Thus, for $\mathfrak{m} = \aleph_0$, if one assumes the GCH, then the restriction $|A| \leq \mathfrak{m}$ ($|A| \leq \mathfrak{m}^+$) is necessary in Theorem 2.3 (i) (2.3 (ii)).

4. **Products of initially $\mathfrak{m}$-compact subspaces of βX.** In this section we derive further properties of initially $\mathfrak{m}$-compact subspaces of βX. Some of the results will also hold for a larger family of spaces, the weakly initially $\mathfrak{m}$-compact spaces.

In [SS1, Theorem 3.5] it was shown that for an infinite cardinal $\mathfrak{m}$, if (*) $\mathfrak{m}$ is regular and $\exp(\mathfrak{n}) \leq \mathfrak{m}$ for every $\mathfrak{n} < \mathfrak{m}$, then there exist weakly initially $\mathfrak{m}$-compact spaces $X \subset M_i \subset \beta X$, $i = 1, 2$, such that $M_1 \times M_2$ is not weakly initially $\mathfrak{m}$-compact. We will show (Theorem 4.1) that the hypothesis (*) is not needed, and that one can take one of the spaces M_i to be initially $\mathfrak{m}$-compact.

Definition. A space Y is said to be *weakly initially* $\mathfrak{m}$*-compact* (or *almost* $\mathfrak{m}$*-compact* [F1]), where $\mathfrak{m}$ is an infinite cardinal number, provided that one of the following equivalent conditions holds: if $\mathfrak{F}$ is an open filter base on Y such that $|\mathfrak{F}| \leq \mathfrak{m}$ then $\mathrm{ad}\, \mathfrak{F} \neq \emptyset$; if $\mathfrak{O}$ is an open cover of Y such that $|\mathfrak{O}| \leq \mathfrak{m}$, then there is a finite subfamily $\mathcal{W}$ of $\mathcal{V}$ such that $Y = (\bigcup \mathcal{W})^-$.

Weak initial $\aleph_0$-compactness is the same as feeble compactness ([SS2], [S3]) (and, for completely regular spaces, is the same as pseudo-compactness). Obviously every initially $\mathfrak{m}$-compact space is weakly initially $\mathfrak{m}$-compact, and, conversely, it is known that a pseudo-compact normal T_1-space is countably compact [GJ].

Theorem 4.1. *Let P be an initially $\mathfrak{m}$-compact subspace of βX such that P contains X and $|P| < \exp(\exp(\mathfrak{m}))$. Then $M = (\beta X \setminus \mu X) \cup (\mu X \setminus P)$ is weakly initially $\mathfrak{m}$-compact, but $P \times M$ is not weakly initially $\mathfrak{m}$-compact.*

Proof. To see that M is weakly initially $\mathfrak{m}$-compact, consider any open filter base $\mathcal{G}$ on M such that $|\mathcal{G}| \leq \mathfrak{m}$. Since X is dense in M, $\mathcal{F} = \{G \cap X | G \in \mathcal{G}\}$ is a filter base and $|\mathcal{F}| \leq \mathfrak{m}$. If there exists $F \in \mathcal{F}$ with $|F| < \mathfrak{m}$, then M contains the compact set F^*, and $F^* \cap G \neq \emptyset$ for every $G \in \mathcal{G}$, so $\mathrm{ad}_M \mathcal{G} \neq \emptyset$. If $\mathcal{F}$ is of type $\mathfrak{m}$, then $|\mathrm{ad}\, \mathcal{F}| = \exp(\exp(\mathfrak{m}))$ by Theorem 3.2, and hence $\emptyset \neq M \cap \mathrm{ad}\, \mathcal{F} \subset \mathrm{ad}_M \mathcal{G}$.

Now we show that $P \times M$ is not weakly initially $\mathfrak{m}$-compact.

By Lemma 3.6 there exists a filter base $\mathcal{H}$ on X such that $|\mathcal{H}| \leq \mathfrak{m}$ and $\mathrm{ad}\, \mathcal{H} \subset \mu X$. For each $H \in \mathcal{H}$ let $F_H = \{(x, x) | x \in H\}$. Then $\mathcal{F} = \{F_H | H \in \mathcal{H}\}$ is an open filter base on $P \times M$ and $|\mathcal{F}| \leq \mathfrak{m}$, but $\mathrm{ad}_{P \times M} \mathcal{F} = \emptyset$, because any adherent point of $\mathcal{F}$ would have to be of the form (p, p) for some $p \in \mu X$, and we have constructed M so that $P \cap M \cap \mu X = \emptyset$.

The proof above suggests a technique which one might try to apply to βX to produce a not initially $\mathfrak{m}$-compact product of initially $\mathfrak{m}$-compact spaces. Recall [HV, p. 526] that if $\mathcal{P}$ is any closed-hereditary and productive topological property, and if S is any Hausdorff $\mathcal{P}$-space, then for every family $\mathfrak{M}$ of $\mathcal{P}$-subspaces of S, $\bigcap \mathfrak{M}$ is also a $\mathcal{P}$-subspace of S. In particular, for any subset Y of S, there is a smallest $\mathcal{P}$-subspace of S containing Y. Thus, for $\mathcal{P}$ = initially $\mathfrak{m}$-compact, if one wishes to establish the existence of a family of $\mathcal{P}$-spaces whose product is not a $\mathcal{P}$-space, it suffices to prove that there does not exist a smallest $\mathcal{P}$-subspace of βX containing X.

If the cardinal $\mathfrak{m}$ is singular and if $\exp(\mathfrak{n}) \leq \mathfrak{m}$ for every $\mathfrak{n} < \mathfrak{m}$, then for Hausdorff spaces, the property $\mathcal{P}$ = initially $\mathfrak{m}$-compact is productive (by Theorem 1.1) and closed-hereditary, and so βX has a smallest $\mathcal{P}$-subspace P containing X. The construction below provides an explicit description of P.

Example 4.2. Let the cardinal $\mathfrak{m}$ be singular and suppose that $\exp(\mathfrak{n}) \leq \mathfrak{m}$ for all $\mathfrak{n} < \mathfrak{m}$. Let S be an initially $\mathfrak{m}$-compact Hausdorff space and Y a subset of S such that $|Y| \leq \exp(\mathfrak{m})$. Set $P_0 = Y$, and for $i \in \mathfrak{m}^+$ with $i > 0$, define

$$P_i = \bigcup \left\{ \overline{Z} : |Z| < \mathfrak{m} \text{ and } Z \subset \bigcup \{P_n | n \in i\} \right\}.$$

Then the space $P = \bigcup\{P_n | n \in \mathfrak{m}^+\}$ is the smallest initially $\mathfrak{m}$-compact subspace of S that contains Y, and $|P| \leq \exp(\mathfrak{m})$.

Proof. For a singular cardinal $\mathfrak{m}$, a space is initially $\mathfrak{m}$-compact if it is initially $\mathfrak{n}$-compact for every $\mathfrak{n} < \mathfrak{m}$. To show that P is initially $\mathfrak{n}$-compact, where $\mathfrak{n} < \mathfrak{m}$, it suffices to note that for every subset Z of P, if $|Z| \leq \mathfrak{n}$, then, by the regularity of the cardinal $\mathfrak{m}^+$, $Z \subset P_i$ for some i and hence $\mathrm{Cl}_P Z$ is initially $\mathfrak{m}$-compact.

Suppose next that M is an initially $\mathfrak{m}$-compact subspace of S containing Y. We wish to show that $P \subset M$.

Let $i \in \mathfrak{m}^+$ and suppose that $P_n \subset M$ for every $n \in i$. Consider any point $p \in P_i \backslash \bigcup\{P_n | n \in i\}$. There exists a subset Z of $\bigcup\{P_n | n \in i\}$ with $|Z| < \mathfrak{m}$ and $p \in \mathrm{Cl}_S Z$. Since $Z \subset M$ and $\exp(|Z|) \leq \mathfrak{m}$, there is a filter base $\mathcal{F}$ on M such that $|\mathcal{F}| \leq \mathfrak{m}$ and $\mathrm{ad}_S \mathcal{F} = \{p\}$. Thus $p \in M$ and $P_i \subset M$.

To establish the inequality $|P| \leq \exp(\mathfrak{m})$, let $i \in \mathfrak{m}^+$ and suppose that $|P_n| \leq \exp(\mathfrak{m})$ for every $n \in i$. Let $A = \bigcup\{P_n | n \in i\}$ and note that $|A| \leq \mathfrak{m} \cdot \exp(\mathfrak{m}) = \exp(\mathfrak{m})$. Then $|\{Z | Z \subset A \text{ and } |Z| < \mathfrak{m}\}| \leq (\exp(\mathfrak{m}))^{\mathfrak{m}} = \exp(\mathfrak{m})$, and for any Z with $|Z| < \mathfrak{m}$, $|\overline{Z}| \leq \exp(\exp(|Z|)) \leq \exp(\mathfrak{m})$, so $|P_i| \leq \exp(\mathfrak{m}) \cdot \exp(\mathfrak{m}) = \exp(\mathfrak{m})$. Thus $|P| = |\bigcup\{P_i | i \in \mathfrak{m}^+\}| \leq \exp(\mathfrak{m}) \cdot \mathfrak{m}^+ = \exp(\mathfrak{m})$.

Remark 4.3. If the space S is also a topological group, then one can modify the space P by taking each P_i to be the group generated by $\bigcup\{\overline{Z}: |Z| < \mathfrak{m}$ and $Z \subset \bigcup\{P_n | n \in i\}\}$. Then the resulting space P is the smallest initially $\mathfrak{m}$-compact subgroup of S that contains Y, and $|P| \leq \exp(\mathfrak{m})$.

In [SS1], as well as in the examples above, the authors have constructed a number of initially $\mathfrak{m}$-compact spaces and have noted that each one has cardinality less than or equal to $\exp(\mathfrak{m})$. We will conclude the paper by proving that most of these spaces are of cardinality exactly $\exp(\mathfrak{m})$ (for $\mathfrak{m} = \aleph_0$, this result is due to Frolík [F2, p. 337]).

First we obtain a general result in set theory; it may be known to some mathematicians, but we have not been able to find a statement of it in print.

Lemma 4.4. *Let $\mathcal{F}$ be a filter base of type $\mathfrak{m}$ on X. Then there exists a family $\{\mathcal{F}(S) | S \in \mathbf{T}\}$ of filter bases on X of type $\mathfrak{m}$ such that $|\mathbf{T}| = \exp(\mathfrak{m})$, each $\mathcal{F}(S)$ is finer than $\mathcal{F}$, and whenever $S \neq S'$, there exist disjoint sets $F \in \mathcal{F}(S)$ and $F' \in \mathcal{F}(S')$.*

Proof. Let $\{F_p | p \in \mathfrak{m}\}$ be a one-one indexing of the nonempty finite subsets of the set $\mathfrak{m}$, and for each $q \in \mathfrak{m}$, define $Z_q = \{p \in \mathfrak{m} | q \in F_p\}$. Let $\mathbf{T}$ be the set of all subsets of $\mathfrak{m}$, and for each $S \in \mathbf{T}$ define

$$\mathcal{G}(S) = [\{Z_q | q \in S\} \cup \{\mathfrak{m} \backslash Z_q | q \in \mathfrak{m} \backslash S\}].$$

By [GJ, 12E], or by a direct argument, one can show that each $\mathcal{G}(S)$ is a filter base on $\mathfrak{m}$ with $|\mathcal{G}(S)| \leq \mathfrak{m}$, and that if $S, S' \in \mathrm{T}$ with $S \neq S'$, then there exist disjoint sets $G \in \mathcal{G}(S)$ and $G' \in \mathcal{G}(S')$.

By Sierpiński's theorem, there exists a family $\mathcal{P}$ of pairwise disjoint subsets of X such that $|\mathcal{P}| = \mathfrak{m}$ and $|P \cap F| = \mathfrak{m}$ for every $P \in \mathcal{P}$ and $F \in \mathcal{F}$. Let $\{P_n | n \in \mathfrak{m}\}$ be a one-one indexing of the members of $\mathcal{P}$, and for each $S \in \mathrm{T}$ define

$$\mathcal{F}(S) = \left\{F \cap \left(\bigcup\{P_n | n \in G\}\right) \middle| F \in \mathcal{F} \text{ and } G \in \mathcal{G}(S)\right\}.$$

Then it is easily seen that the filter bases $\mathcal{F}(S)$, $S \in \mathrm{T}$, have the desired properties.

Theorem 4.5. *Let P be a weakly initially $\mathfrak{m}$-compact subspace of βX which contains X, and suppose that $\mathcal{F}$ is a filter base on X of type $\mathfrak{m}$. Then* $|\mu X \cap \mathrm{ad}_P \mathcal{F}| \geq \exp(\mathfrak{m})$.

Proof. By Lemma 4.4, there exist filter bases $\mathcal{F}_c$, $c \in \exp(\mathfrak{m})$, of type $\mathfrak{m}$ on X such that each $\mathrm{ad}_{\beta X}\mathcal{F}_c \subset \mathrm{ad}_{\beta X}\mathcal{F}$, and whenever $c \neq d$, $\mathrm{ad}_{\beta X}\mathcal{F}_c \cap \mathrm{ad}_{\beta X}\mathcal{F}_d = \emptyset$. It follows from Lemma 3.6 that for each $c \in \exp(\mathfrak{m})$, there exists a filter base $\mathcal{H}_c$ on X of type $\mathfrak{m}$ such that $\mathrm{ad}_{\beta X}\mathcal{H}_c \subset \mu X \cap \mathrm{ad}_{\beta X}\mathcal{F}_c$. Thus

$$|\mu X \cap \mathrm{ad}_P \mathcal{F}| \geq \sum |\mu X \cap \mathrm{ad}_P \mathcal{F}_c| \geq \sum |\mathrm{ad}_P \mathcal{H}_c| \geq \exp(\mathfrak{m}),$$

where the sums are taken over all $c \in \exp(\mathfrak{m})$.

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