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A DENSITY PROPERTY AND APPLICATIONS(1)

BY

RICHARD J. O'MALLEY

ABSTRACT. An unexpected metric density property of a certain type of F_σ set is proven. An immediate application is a characterization of monotone functions similar to a well-known result by Zygmund. Several corollaries of this characterization are given as well as a simple proof of a theorem due to Tolstoff.

One of the primary purposes of this paper is the demonstration of an unexpected density property of any F_σ subset A of $(0, 1)$ having left density 1 at all of its points. If B denotes the complement of A relative to $[0, 1)$ we will show there is a point x in B at which A has right density 1. (A similar statement holds if A has right density 1 at all its points.) This property is both intrinsically interesting and useful. As an immediate application we obtain the following characterization of monotone functions, analogous to a well-known result by Zygmund [1, p. 203].

THEOREM 1. *Let f be a function defined on $[0, 1]$ and*

(1) *f is Baire class* 1,

(2) $\operatorname{ap\,lim\,sup}_{x\to x_0^-} f(x) \leqslant f(x_0) \leqslant \operatorname{ap\,lim\,sup}_{x\to x_0^+} f(x)$ *for every* x_0, *and*

(3) $\operatorname{interior}[f(\{x\colon AD^+f(x) \leqslant 0\})] = \varnothing$.

Then f is nondecreasing.

(For this paper we will say that functions satisfying (1) and (2) of Theorem 1 are of type (*). It should be noted that approximately continuous functions are of type (*).)

As with Zygmund's theorem, Theorem 1 is shown to have numerous applications. These deal with the monotonicity and differentiability properties of functions of type (*). Finally we end the paper by considering another classic result by Tolstoff [2]. Through the use of Theorem 1 and an additional lemma, Tolstoff's lengthy proof is simplified.

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(1) This material developed out of research begun at Purdue University where the author was under the supervision of Professor Casper Goffman, and was completed at the University of Wisconsin, Milwaukee.

We will need the following basic definitions. The reader is referred to [1] for further elaboration.

(1) If E is a measurable set and x_0 is any fixed point we define the upper (lower) right density of E at x_0 as

$$\limsup_{x\to x_0^+} \frac{|E \cap [x_0, x]|}{|[x_0, x]|} \left(\liminf_{x\to x_0^+} \frac{|E \cap [x_0, x]|}{|[x_0, x]|}\right),$$

where $|\cdot|$ denotes Lebesgue measure.

The upper and lower left densities of E at x_0 are defined similarly. If all four densities are equal to 1 at x_0 we say x_0 is a point of density of E. We make a similar convention for right or left point of density.

(2) A measurable function $f\colon R \to R$ is said to be approximately continuous if for every x there is a measurable set $E(x)$, having x as a point of density, such that f restricted to $E(x)$ is continuous at x.. We note that an equivalent definition requires that for every open set $U \subset R$, $f^{-1}(U)$ has density 1 at all its points.

(3) A measurable set U is said to be d-open if every point of U is a point of density of U. The collection of d-open sets forms a completely regular topology with respect to which the continuous functions are precisely the approximately continuous functions. (See [3], and [5].) It can easily be seen that a measurable set E has a point x as a d-limit point only if one of the two upper densities of E at x is positive.

For a measurable function f and a fixed point x_0 we can define, relative to the above topology, various limits as in the Euclidean topology.

(4) The upper approximate right limit of f at x_0 is:

$$\operatorname{ap}\limsup_{x\to x_0^+} f(x_0) = \inf[y\colon \{x\colon f(x) > y\} \text{ has upper right density } 0 \text{ at } x_0].$$

The upper approximate left limit of f at x_0 is defined similarly.

(5) Just as any function f has four Dini derivates defined at every point we analogously define the four approximate Dini derivates of f, AD^+f, AD^-f, AD_+f, AD_-f. For example $AD^+f(x_0)$ is the upper right approximate Dini derivate of f at x_0 and equals:

$$\operatorname{ap}\limsup_{x\to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

Finally we remark that in this paper we will use the notation "nearly everywhere" after a property to designate that that property holds except at a countable set of points.

Since the proof of the density property involves the use of two lemmas and is rather tedious while the proof of Theorem 1 is rather elegant, we have chosen, for the moment, to assume the validity of the density property and prove Theorem 1.

Proof of Theorem 1. We assume that f is not nondecreasing. This implies there are points x_1, x_2 with $x_1 < x_2$ and $f(x_1) > f(x_2)$. Consider any $\alpha \in (f(x_2), f(x_1))$ and $E = \{x: f(x) < \alpha\}$. It follows that E is a nonempty F_σ with left density 1 at each of its points. Also since x_1 does not belong to E the complement of E is nonempty. The density property guarantees the existence of x_0, at which E has right density 1, and $x_0 \in [0, 1)\setminus E \subset \{x: f(x) \geqslant \alpha\}$. This, together with the fact that $f(x_0) \leqslant \text{ap}\limsup_{x \to x_0^+} f(x)$ gives that $f(x_0) = \alpha$ and $AD^+f(x_0) \leqslant 0$. Therefore we have

$$(f(x_2), f(x_1)) \subset f(\{x: AD^+f(x) \leqslant 0\}),$$

which contradicts condition (3) of Theorem 1.

We proceed to the proof of the density property through the lemmas:

LEMMA 1. *Let H be a measurable subset of $[0, 1]$ and C a closed subset of H. There is a closed set P such that $C \subset P \subset H$, and for $x \in C$ and for every sequence of intervals I_n such that:*

$$\bigcap_{n=1}^{\infty} I_n = x, \quad |I_n| \to 0, \quad \textit{and} \quad \lim_{n\to\infty} \frac{|H \cap I_n|}{|I_n|} \quad \textit{exists},$$

we have

$$\lim_{n\to\infty} \frac{|P \cap I_n|}{|I_n|} = \lim_{n\to\infty} \frac{|H \cap I_n|}{|I_n|}.$$

This lemma is actually a statement of the Lusin-Menchoff theorem, and the proof is from [3].

PROOF. For each j we define

$$C_j = \left\{x: \frac{1}{j+1} < \rho(x, C) \leqslant \frac{1}{j}\right\} \cap H,$$

where $\rho(x, C)$ denotes the distance from x to C. For each j there is a closed set P_j such that $P_j \subset C_j$ and $|P_j| > |C_j| - 1/2^j$. We set $P = \bigcup_{j=1}^{\infty} P_j \cup C$. It is clear that P is closed and $C \subset P \subset H$. To establish the second part of the lemma let $x \in C$ and let $\{I_n\}$ be a sequence of intervals such that:

$$x = \bigcap_{n=1}^{\infty} I_n, \quad |I_n| \to 0, \text{ and } \lim_{n\to\infty} |H \cap I_n|/|I_n| \text{ exists.}$$

Let α be this limit. If $I_n \subset C$ for infinitely many n it is clear that

$$\lim_{n\to\infty} |I_n \cap P|/|I_n| = \alpha = 1.$$

We can therefore assume that for every n there is a j_n such that $I_n \cap C = \varnothing$, $j < j_n$ and $C_{j_n} \cap I_n \neq \varnothing$. It follows immediately that

$$|I_n \cap (H\backslash P)| \leqslant \sum_{j=j_n}^{\infty} |C_j\backslash P_j| < \frac{1}{2^{j_n-1}}.$$

Thus

$$|H \cap I_n| - \frac{1}{2^{j_n-1}} \leqslant |P \cap I_n| \leqslant |H \cap I_n|.$$

Now since $C_{j_n} \cap I_n \neq \varnothing$ we have $|I_n| \geqslant 1/(j_n + 1)$ and

$$\frac{|H \cap I_n|}{|I_n|} - \frac{j_n + 1}{2^{j_n-1}} \leqslant \frac{|P \cap I_n|}{|I_n|} \leqslant \frac{|H \cap I_n|}{|I_n|}.$$

As $n \to \infty$, $j_n \to \infty$ so

$$\lim_{n\to\infty} \frac{|P \cap I_n|}{|I_n|} = \lim_{n\to\infty} \frac{|H \cap I_n|}{|I_n|} = \alpha.$$

LEMMA 2. *Let H be a measurable subset of $[0, 1]$, $|H| = \gamma > 0$. Let β be a positive number less than 1. Let V be the union of all the open intervals J in $[0, 1]$ such that $|H \cap J| > \beta|J|$. Then V is open, and if I is any component interval of V then $|H \cap I| \geqslant (\beta/2)|I|$, from which it follows that $|H|2/\beta \geqslant |V|$.*

A better conclusion is actually possible (see Denjoy [4, p. 230]), but this will suffice for our purposes.

PROOF. Let I be any component interval of V. Let $\epsilon > 0$. We select a finite collection $J_1, J_2, \cdots, J_n$ of intervals such that

$|H \cap J_i| > \beta|J_i|$, $i = 1, 2, \cdots, n$,
$J_i \subset I$,
$|\bigcup_{i=1}^n J_i| > (1 - \epsilon)|I|$, and
no point of $\bigcup_{i=1}^n J_i$ is in more than two of the J_i.

Then

$$|H \cap I| \geqslant \frac{1}{2} \sum_{i=1}^{n} |H \cap J_i| > \frac{\beta}{2} \sum_{i=1}^{n} |J_i|$$

$$\geqslant \frac{\beta}{2} \left| \bigcup_{i=1}^{n} J_i \right| \geqslant \frac{\beta}{2}(1 - \epsilon)|I|.$$

Therefore $|H \cap I| \geqslant (\beta/2)|I|$. The second part of the conclusion is clear from the fact that V is open and $|H \cap V| = |H|$.

We now restate and prove the density property:

THEOREM 2. *Let A be an F_σ subset of* $(0, 1)$ *which has left density* 1 *at all its points. Then there is an* $x_0 \in [0, 1)\backslash A = B \cup \{0\}$ *such that A has right density* 1 *at* x_0.

PROOF. Using Lemma 1 we may express A as the union of nonempty closed sets, F_n, $n = 1, 2, \cdots$, with the property that, for every n, F_{n+1} has left density 1 at every point of F_n. We will show how to construct a sequence of closed intervals $[a_n, b_n]$ such that for every n

(i) $[a_{n+1}, b_{n+1}] \subset (a_n, b_n)$,

(ii) $|[a_{n+1}, b_{n+1}]| \leqslant \frac{1}{2}|[a_n, b_n]|$,

(iii) $|B \cap [x, y]| \leqslant (1/2^n)|[x, y]|$ for all intervals $[x, y]$ with $a_{n+1} \leqslant x \leqslant b_{n+1} \leqslant y \leqslant b_n$, and

(iv) $(a_n, b_n) \cap F_n = \varnothing$.

From (i) and (ii) there is a point $x_0 = \bigcap_{n=1}^\infty [a_n, b_n]$. From (iii) it follows that A has right density 1 at x_0, and from (iv) that $x_0 \notin A$. Actually we will find $[a_1, b_1]$ and $[a_2, b_2]$ only.

It is clear that if there is an interval $L \subset [0, 1]$ such that $B \cap L \neq \varnothing$ but $|B \cap L| = 0$, any point will serve as x_0. Therefore, we assume that for any interval L either $B \cap L = \varnothing$ or $|B \cap L| > 0$. In particular we assume that $|B \cap (0, 1)| > 0$. From this it will be seen that x_0 can be chosen in $(0, 1)$.

Let $U_1 = (0, 1)\backslash F_1$. U_1 is open and $B \subset U_1$. We pick any component interval of U_1 in which B has positive measure; call it (a, b). We note that $b \in F_1$. Let

$$b_1 = \inf [x\colon B \cap [x, b] = \varnothing].$$

Clearly $a < b_1 \leqslant b$ and if $b_1 < b$ and b_1 does not belong to A it will serve as x_0. Therefore we assume $b_1 \in A$. We find the first index $N_2 > 2$ such that $b_1 \in F_{N_2}$. Let $U_2 = (0, 1)\backslash F_{N_2+1}$. $B \subset U_2$ and U_2 has left density zero at b_1. In fact U_2 has left density zero at all points of F_{N_2}, which contains F_2 (an important fact that we will need later). We pick a_1 so that $a < a_1 < b_1$ and $|U_2 \cap [x, b_1]| \leqslant |[x, b_1]|/8$ for all $a_1 \leqslant x \leqslant b_1$. We now have $[a_1, b_1]$. It is clear that

$$(a_1, b_1) \cap F_1 = \varnothing \quad \text{and} \quad |B \cap (a_1, b_1)| > 0.$$

We now consider U_2 only in $[a_1, b_1]$ and discard from U_2 any component intervals having empty intersection with B. We relabel the remaining open set U_2^*. By Lemma 2, if we let V_2 be the union of all open intervals $J \subset (a_1, b_1)$

such that $|U_2^* \cap J| > \frac{1}{2}|J|$ we have

V_2 is open,

$0 < |V_2| < 4|U_2 \cap (a_1, b_1)| \leqslant \frac{1}{2}|[a_1, b_1]|$, and

$B \cap (a_1, b_1) \subset U_2^* \subset V_2$.

We pick any component interval (c, d) of V_2. By Lemma 2 $|U_2^* \cap (c, d)| \geqslant |(c, d)|/4$ which shows that $d \neq b_1$ because $|U_2^* \cap (c, b_1)| \leqslant |(c, b_1)|/8$. Since U_2^* is open d cannot belong to U_2^* which shows that d belongs to A. We must distinguish two cases in determining $[a_2, b_2] \subset (c, d)$.

Case 1: There is a component of U_2^* in (c, d) which is closest to d. We call this component (a, b). We have then that b belongs to A and by the definition of U_2^*, $|(a, b) \cap B| > 0$. Let $b_2 = \inf[x\colon B \cap [x, b] = \varnothing]$. As before, b_2 must belong to A if it is not to be x_0, and $a < b_2 \leqslant b$. We pick the first index $N_3 > N_2 + 1$ such that $b_2 \in F_{N_3}$. Let $U_3 = (0, 1)\backslash F_{N_3+1}$. U_3 has left density zero at b_2, so we may determine a_2 so that

$$|U_3 \cap [x, b_2]| \leqslant \frac{1}{16}|[x, b_2]|$$

for all $a_2 \leqslant x \leqslant b_2$. In this case we have defined $[a_2, b_2]$. It is easily verified that $[a_2, b_2]$ satisfies conditions (i), (ii), (iii), and (iv) and $|B \cap [a_2, b_2]| > 0$.

Case 2: U_2^* contains a sequence of intervals $\{I_j\}$ converging upward to d. As in case 1 we let N_3 be the first index with $N_3 > N_2 + 1$ and $d \in F_{N_3}$. We let $U_3 = (0, 1)\backslash F_{N_3+1}$, and pick $c*$ so that

$$|U_3 \cap [x, d]| \leqslant \frac{1}{16}|[x, d]|$$

where $c* \leqslant x \leqslant d$ and $|[c*, d]| < |[c, d]|/8$. It is clear that $(c*, d)$ satisfies (i), (ii), and (iii). However, we have no guarantee that $(c*, d) \cap F_2 = \varnothing$. If $F \cap (c*, d) \neq \varnothing$, we relabel $(c*, d)$ as (c_2, d_2), and we begin in (c_2, d_2) as we did in (a_1, b_1) with U_2^* replaced by U_3^*, V_2 by V_3. Hence, we will obtain an interval (c_3, d_3). This interval will satisfy (i), (ii) and (iii) with respect to (a_1, b_1), and we will set $(a_2, b_2) = (c_3, d_3)$ if $F_2 \cap (c_3, d_3) = \varnothing$. Otherwise, we proceed inductively to get (c_k, d_k), $k = 4, 5, \cdots$. We claim that at some finite k we must have $F_2 \cap (c_k, d_k) = \varnothing$. To see this, suppose instead that the process does not terminate. We will have then three infinite sequences $\{[c_k, d_k]\}$, $\{U_k^*\}$, and $\{J_k = [c_k', d_k]\}$ where J_k is the component interval of V_k which we truncate to get $[c_k, d_k]$. We must observe that these sequences have the properties:

(a) $|[c_{k+1}', d_{k+1}]| \leqslant \frac{1}{2}|[c_k, d_k]|$.

(b) $[c_{k+1}', d_{k+1}] \subset (c_k, d_k)$.

(c) $B \subset U_k^* \subset U_2$.

(d) $|U_k^* \cap [c'_k, d_k]| \geqslant |[c'_k, d_k]|/4$.

(e) $|[c_k, d_k]| < |[c'_k, d_k]|/8$.

(f) $(c_k, d_k) \cap F_2 \neq \varnothing$.

These observations allow us to conclude that $\bigcap_{k=1}^{\infty}(c_k, d_k) = x$ and $x \in F_2$. Moreover, consider the interval $[c'_k, x]$. From (d) and (e) it follows that for each k, $|U_k^* \cap [c'_k, x]| \geqslant |[c'_k, x]|/8$. Hence, from (c) we have $|U_2 \cap [c'_k, x]| > |[c'_k, x]|/8$. Since $\lim_{k\to\infty} c_k = x$ this shows that U_2 has positive left density at x. However, U_2 was chosen to have left density zero at each point of F_2. Therefore the process must terminate for some k, and we relabel $[c_k, d_k] = [a_2, b_2]$, $U_{N_k+1} = U_3$, and consider only U_3^*.

A final word may still be in order to indicate how we proceed now that we have $[a_2, b_2]$. We now let V_3 be the union of those intervals J contained in (a_2, b_2) such that $|U_3^* \cap J| > |J|/4$, rather than $|U_3^* \cap J| > |J|/2$ as before. (In each stage we will want $|U_n^* \cap J| > |J|/2^{n-1}$.) This completes the proof of the density property.

At this point the meaning of the property in terms of the density topology on $[0, 1]$ is worth considering. We say that a set A is "left-open" in the Euclidean topology if $x \in A$ implies that A contains an interval of the form $(x - \delta, x]$. The equivalent definition in the density topology is that a measurable set A is "d-left-open" if $x \in A$ implies A has left density 1 at x. It is an elementary fact that if A is "left-open" and its complement B is nonempty, then B contains points which are not limit points of B from the right. The density property says that if A is "d-left-open" and in addition F_σ, its complement B has points which are not d-limit points of B from the right. It is still an open question whether the condition that A be an F_σ set is necessary.

Another immediate application of Theorem 2 is the following.

THEOREM 3. *Let f and g be approximately continuous functions on $[0, 1]$ with $f(0) = g(0)$ and let $E = \{x\colon f(x) = g(x)\}$. If E is nonempty and has positive upper right density at each of its points then $f(x) = g(x)$ for all x in $[0, 1]$.*

PROOF. If $f(x_1) \neq g(x_1)$ for some x_1, we may assume that $f(x_1) - g(x_1) > 0$. Let $A = \{x\colon f(x) - g(x) > 0\}$. Then A satisfies the hypothesis of Theorem 1 and we can find an $x_0 \neq 0$ such that $f(x_0) - g(x_0) \leqslant 0$ and $\{x\colon f(x) - g(x) > 0\}$ has right density 1 at x_0. However, since A has right density 1 at x_0 we must have $f(x_0) - g(x_0) = 0$. So $x_0 \in E$, and E does not have positive upper density from the right at x_0. This is a contradiction, so $f(x) = g(x)$ for all x.

We now prove various corollaries of Theorem 1. It should be noted that the proofs of these results follow very closely those of classic results. (This is deliberately emphasized to show the usefulness of Theorems 1 and 2.) For this reason, where the proof is nearly obvious and would require too much space, we delete it.

The following results deal with the functions of type $(*)$ and to shorten the exposition we state the following simple fact now:

> If f is of type $(*)$ and g is either approximately continuous or nondecreasing, then $f + g$ is of type $(*)$.

The proof is obvious as the various inequalities valid for $\limsup(f + g)$, etc., carry over without alteration to $\operatorname{ap}\limsup(f + g)$.

COROLLARY 1. *Let*: f *be of type* $(*)$, *and* $AD^+f \geqslant 0$ *nearly everywhere. Then* f *is nondecreasing.*

PROOF. Let $\epsilon > 0$ be given. We place $G_\epsilon(x) = f(x) + \epsilon x$. Then $G_\epsilon(x)$ is of type $(*)$ and $AD^+G_\epsilon(x) = AD^+f(x) + \epsilon$ for every x. Therefore at any point where $AD^+f(x) \geqslant 0$ we have $AD^+G_\epsilon(x) \geqslant \epsilon > 0$. Since $AD^+f \geqslant 0$ except for at most countably many points $x_1, x_2, \cdots, x_n, \cdots$ we have $AD^+G_\epsilon > 0$ except at these same points. Hence the interior of the image under G_ϵ of these points must be empty. By Theorem 1 we then have $G_\epsilon(x)$ is a nondecreasing function. So, if $x < y, G_\epsilon(x) \leqslant G_\epsilon(y)$; that is $f(x) + \epsilon x \leqslant f(y) + \epsilon y$. Since ϵ was arbitrary this yields that f itself is nondecreasing. Corollary 1 generalizes the result by Ornstein [6]: If f is approximately continuous and $AD^+f \geqslant 0$ everywhere, then f is nondecreasing and continuous. It was this paper by Ornstein which initially stimulated this research.

It is tempting in light of the interesting result obtained by Bruckner [7, p. 22] to conjecture the following:

> If f is Baire 1, Darboux and $AD^+f \geqslant 0$ everywhere, then f is nondecreasing and continuous.

This is not true, however. Croft [8] has constructed an example of a function f which is upper-semicontinuous, Darboux, zero almost everywhere, but not identically zero. It is clear that $-f$ provides a counterexample. In this sense, Corollary 1 is the best possible.

A more fruitful line of thought is: What additional conditions must be placed on functions of type $(*)$ in order that AD^+f being nonnegative except on a set of measure zero is sufficient to guarantee that f is nondecreasing. The following are examples of such conditions.

COROLLARY 2. *Let*: f *be of type* $(*)$, $AD^+f \geqslant 0$ *a.e., and* $AD^+f > -\infty$ *nearly everywhere. Then* f *is nondecreasing.*

(This is an extension of Gál's result [9].)

PROOF. We need only show $f(0) \leqslant f(1)$. Let $\epsilon > 0$ be given, and $E = \{x: AD^+f(x) < 0\}$, $|E| = 0$. It is a well-known fact (see Titchmarsh [10, p. 369]) that for any set of measure zero, A, there is a function g absolutely continuous, nondecreasing and such that $g'(x) = +\infty$ on A, $g(0) = 0$, and $g(1) < \epsilon$. Let h be such a function for our set E, and consider $f + h$. Then $f + h$ is of type (*) and $AD^+(f + h) \geqslant 0$ except perhaps at the points x where $AD^+f(x) = -\infty$. By Corollary 1, $f + h$ is nondecreasing, and we have

$$f(0) = (f + h)(0) \leqslant (f + h)(1) \leqslant f(1) + \epsilon.$$

This completes the proof.

COROLLARY 3. *Let: f be of type* (*), *and $AD^+f \geqslant M > -\infty$ nearly everywhere. Then f is differentiable a.e.*

PROOF. The function $G(x) = f(x) + Mx$ is nondecreasing. Hence G is differentiable a.e. which implies the same holds for f.

COROLLARY 4. *Let: f be of type* (*), *$AD^+f \geqslant 0$ a.e., and AD^+f a Baire* 1 *function. Then f is nondecreasing.*

(Leonard [11] has a similar theorem using the Dini derivates of f.)

PROOF. We define $E = \{x$: f is not nondecreasing in any neighborhood about $x\}$. We will show E is empty. Suppose on the contrary that E is not empty. It is clear that E is perfect. Let $\epsilon > 0$, and $A = \{x: AD^+f(x) \geqslant 0\}$ and $B = \{x: AD^+f(x) \leqslant -\epsilon\}$. Since AD^+f is a Baire 1 function we cannot have A and B simultaneously dense in any nonempty portion of E. Let I be any interval such that $I \cap E \neq \emptyset$. I must contain a subinterval J where $J \cap E \neq \emptyset$ but either $J \cap E \cap A = \emptyset$ or $J \cap E \cap B = \emptyset$. Suppose first that we have $J \cap E \cap A = \emptyset$. Then $|J \cap E| = 0$ because $AD^+f \geqslant 0$ a.e. We select any component interval (a, b) of $J\backslash E$. The function f is nondecreasing on (a, b). Moreover, since f is of type (*), we have $f(a) \leqslant f(x)$ for all x in (a, b). Hence $AD^+f(a) \geqslant 0$, contradicting $J \cap E \cap A = \emptyset$.

We finish by proving that $J \cap E \cap B = \emptyset$ implies $J \cap E = \emptyset$, contradicting $J \cap E \neq \emptyset$. Suppose $J \cap E \cap B = \emptyset$. For $x \in J \cap E$ we then have $AD^+f(x) > -\epsilon$, and for $x \in J\backslash E$ $AD^+f(x) \geqslant 0$. By Corollary 2, f is nondecreasing on J, so $J \cap E = \emptyset$ Therefore E must be empty.

We also obtain an analogue of Dini's theorem [1, p. 204].

THEOREM 4. *Let: f be of type* (*) *on* $[0, 1]$, $\alpha = \inf$ *of the difference quotients of f on* $[0, 1]$, $\beta = \inf_x [AD^+f(x)] = \inf$ *of AD^+f over* $[0, 1]$. *Then $\alpha = \beta$.*

PROOF. For any function we have $\alpha \leqslant \beta$, so if $\beta = -\infty$ there is nothing to prove. Therefore, suppose $\beta > -\infty$ and let γ be any number less than β. For the function $G(x) = f(x) - \gamma x$ we have $AD^+(G) = AD^+f - \gamma \geqslant \beta - \gamma \geqslant 0$. So, by Corollary 2, if $x < y$, $G(y) \geqslant G(x)$. This gives $\alpha \geqslant \beta$.

The following are corollaries of Theorem 4.

COROLLARY 5. *Let f be approximately continuous on* $[0, 1]$, *let α be the same as in Theorem* 4, *and let δ be the* sup *of the difference quotients of f over* $[0, 1]$. *Then*

$$\alpha = \inf_x [AD^+f] = \inf_x [AD_+f] = \inf_x [AD^-f] = \inf_x [AD_-f]$$

and

$$\delta = \sup_x [AD^+f] = \sup_x [AD_+f] = \sup_x [AD^-f] = \sup_x [AD_-f].$$

PROOF. We need only remark that both f and $-f(1-x)$ are of type (*).

COROLLARY 6. *Let f be approximately continuous. If any of the four approximate Dini derivates are continuous at a point x_0, then all are continuous at x_0 and f has an ordinary derivative at x_0 and is differentiable a.e. in some neighborhood of x_0.*

PROOF. The proof is obvious from Corollary 5 and Corollary 3.

COROLLARY 7. *Let: f be approximately continuous, and AD^+f be Baire class* 1. *Then f is differentiable on a set of second category.*

PROOF. By Corollary 5, f has an ordinary derivative at every point of continuity of AD^+f. Since AD^+f is Baire class 1, its points of continuity are a set of second category.

We mention in passing that other results can be obtained dealing with the approximately continuous Perron integral as defined by Burkill [12, p. 270]. The interested reader is referred to Burkill's paper and [1, pp. 203–204].

We end the paper by considering two classical results.

I. Tonelli-Goldowski [13]. Suppose f is a continuous function, possessing a derivative, f' (possibly infinite), nearly everywhere. In addition suppose $f' \geqslant 0$ a.e. Then f is nondecreasing.

II. Tolstoff [2]. Suppose f is an approximately continuous function, possessing an approximate derivative, f'_{ap} (possibly infinite), nearly everywhere. In addition suppose $f'_{\mathrm{ap}} \geqslant 0$ a.e. Then f is nondecreasing and continuous.

It is clear that II is a more general theorem than I. Further, the before-mentioned result by Bruckner [7] requires only that f be a Baire 1, Darboux function instead of an approximately continuous function.

Viewed in terms of the density topology, II is "the same" as I. The proof of I (see [1, p. 206]) is short and very similar to that of Corollary 4. However, the proof of II is unnecessarily lengthy. With the use of one lemma we show that the "same" proof can be given for II as for I. This proof can also be applied to the result by Leonard [11, p. 759].

LEMMA 3. *Let the hypothesis of* II *hold. For each* x, *let*

$$A(x) = \left\{y: \frac{f(y) - f(x)}{y - x} \geqslant -2\right\}, \quad B(x) = \left\{y: \frac{f(y) - f(x)}{y - x} \leqslant -1\right\},$$

$$A_n = \{x: |A(x) \cap I| > \tfrac{1}{2}|I| \text{ for all intervals } I \text{ containing } x \text{ with } |I| < 1/n\},$$

$$B_n = \{x: |B(x) \cap I| > \tfrac{1}{2}|I| \text{ for all intervals } I \text{ containing } x \text{ with } |I| < 1/n\}.$$

Let A_n^*, B_n^* *denote the closure of* A_n *and* B_n *respectively. Then for* x *belonging to* A_n^*

$$|A(x) \cap [x, x + \delta]| \geqslant \tfrac{1}{2}\delta, \quad 0 < \delta < 1/n,$$

and for x *belonging to* B_n^*

$$|B(x) \cap [x, x + \delta]| \geqslant \tfrac{1}{2}\delta, \quad 0 < \delta < 1/n.$$

We will prove the conclusion for $x_0 \in A_n^*$ only.

PROOF. Let n be fixed and x_0 be a limit point of A_n. Suppose there exists a strictly decreasing sequence of points of A_n, x_k, converging to x_0. We may assume that $x_1 - x_0 < 1/n$. We fix k and show $A(x_k)$ has positive upper density at x_0. For $j = 1, 2, \cdots$ consider the interval $[x_{k+j}, x_k]$:

$$|A(x_k) \cap [x_{k+j}, x_k]| > \tfrac{1}{2}|[x_{k+j}, x_k]|$$

and

$$|A(x_{k+j}) \cap [x_{k+j}, x_k]| > \tfrac{1}{2}|[x_{k+j}, x_k]|$$

which gives that $A(x_{k+j}) \cap A(x_k) \cap [x_{k+j}, x_k] \neq \varnothing$, and consequently

$$f(x_k) - f(x_{k+j}) \geqslant -2(x_k - x_{k+j}).$$

Then for y in $A(x_{k+j}) \cap [x_0, x_{k+j}]$, since we have already $f(x_{k+1}) - f(y) \geqslant -2(x_{k+j} - y)$ we must also have

$$f(x_k) - f(y) \geqslant -2(x_k - y).$$

Therefore $|A(x_k) \cap [x_0, x_{k+j}]| > \tfrac{1}{2}|[x_0, x_{k+j}]|$ for all $j = 1, 2, \cdots$, which, since $x_{k+j} \to x_0$ as $j \to \infty$, shows that $A(x_n)$ has positive upper density at least ½ at x_0. Since f is approximately continuous at x_0 this in turn implies that

$$f(x_k) - f(x_0) \geq -2(x_k - x_0).$$

Now let δ be any positive number less than $1/n$. We select the first index K such that $x_K < x_0 + \delta$. Let $k > K$. For y in $A(x_k) \cap [x_k, x_0 + \delta]$ we have $f(y) - f(x_k) \geq -2(y - x_k)$, and so $f(y) - f(x_0) \geq -2(y - x_0)$, i.e., $y \in A(x_0) \cap [x_k, x_0 + \delta]$. Therefore

$$|A(x_0) \cap [x_k, x_0 + \delta]| > \tfrac{1}{2}|[x_k, x_0 + \delta]|$$

for all $k > K$, giving

$$|A(x_0) \cap [x_0, x_0 + \delta]| \geq \tfrac{1}{2}|[x_0, x_0 + \delta]|.$$

The case when x_k converges to x_0 from below is proven by a similar argument.

More can be said about the behavior of f. It can actually be shown that f is continuous on A_n^* and B_n^* relative to these sets. See [14, p. 499].

We now prove II. We assume that f is defined on [0, 1].

PROOF. Let E be defined as in Corollary 4 and A_n and B_n as in Lemma 3. Again we show that E is empty. Any point x where f has an approximate derivative belongs to A_n or B_n for some n. Therefore $[0,1] = \bigcup_{n=1}^{\infty} A_n^* \cup \bigcup_{n=1}^{\infty} B_n^* \cup C$, where C is the at most countable set at which f does not have an approximate derivative. If E is nonempty, since it is perfect, there must be some interval I and some n such that $I \cap E \neq \varnothing$ and either (a) $I \cap E \subset A_n^*$, or (b) $I \cap E \subset B_n^*$.

Suppose (a) is true. Now for $x \in A_n^*$ we have $AD^+f(x) \geq -2$, and for $x \in I \backslash E$, $AD^+f(x) \geq 0$. If $I \cap E \subset A_n^*$, then $AD^+f \geq 0$ a.e. on I and $AD^+f > -\infty$ everywhere on I. Hence, by Corollary 2, f is nondecreasing on I, contradicting $I \cap E \neq \varnothing$.

Suppose (b) is true. We note that for x in B_n^* we have $AD_+f(x) \leq -1$. Since the approximate derivative of f exists and is nonnegative almost everywhere we must have that $|B_n^* \cap I| = 0$, and so $|I \cap E| = 0$. Let $J = (a, b)$ be any component interval of $I \backslash E$. Then a belongs to E and hence to B_n^*. On (a, b) f must be nondecreasing and, because it is approximately continuous, it follows that f is actually nondecreasing on $[a, b]$. However, this means $f(y) - f(a) \geq 0$ for all y in $[a, b]$, contradicting the fact that a must belong to B_n^*. This completes the proof and the paper.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MILWAUKEE, WISCONSIN 53201