

SUBBUNDLES OF THE TANGENT BUNDLE

BY

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ABSTRACT. This paper studies pairs (M, ξ) where M is a closed manifold and ξ is a k -dimensional subbundle of the tangent bundle of M in terms of cobordism.

1. Introduction. The purpose of this note is to analyze pairs (M, ξ) where M is an n -dimensional manifold and ξ is a k -dimensional subbundle of the tangent bundle of M , $k \leq n$, in terms of cobordism.

In §2, the cobordism class of M is analyzed and the main result is

PROPOSITION. *A class $\alpha \in \mathfrak{N}_n$ is represented by a manifold M^n whose tangent bundle has a k -dimensional subbundle, $k \leq n$, if and only if either*

(a) *k is even, or*

(b) *k is odd and $w_n(\alpha) = 0$.*

In section §3, the case $k = 1$, i.e., ξ a line bundle, will be studied more closely. One defines a homomorphism $\theta: \mathfrak{N}_n(BO_1) \rightarrow Z_2$ as follows. If $\alpha \in \mathfrak{N}_n(BO_1)$, choose a manifold M^n and map $f: M^n \rightarrow BO_1$ representing α . Let $i \in H^1(BO_1; Z_2)$ be the nonzero class, and let $\theta(\alpha)$ be the characteristic number

$$\{w_n(M) + w_{n-1}(M)f^*(i) + \cdots + w_{n-r}(M)(f^*(i))^r + \cdots + (f^*(i))^n\} [M].$$

Letting γ be the universal line bundle over BO_1 , the class α is the class of the pair $(M, f^*(\gamma))$, and interpreting $\mathfrak{N}_n(BO_1)$ as the cobordism classes of n -manifolds with a line bundle, one has

PROPOSITION. *A class $\alpha \in \mathfrak{N}_n(BO_1)$ is represented by a pair (M^n, ξ) where ξ is a sub-line-bundle of the tangent bundle of M if and only if $\theta(\alpha) = 0$.*

Note. In order to make this result seem plausible, one should note that the given characteristic number is the n th Stiefel-Whitney number of $\tau_M - f^*(\gamma)$, which is an $(n-1)$ -plane bundle if $f^*(\gamma)$ is a subbundle of τ_M .

In §4, the problem is stabilized, and the main result is

PROPOSITION. *A class $\alpha = [M, f] \in \mathfrak{N}_n(BO_k)$ is represented by a pair (M', ξ') with $\tau_{M'} \oplus 1 \cong \xi' \oplus \eta' \oplus 1$ where η' is an $n-k$ plane bundle if*

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and only if every Stiefel-Whitney number of α involving a class $w_i(\tau - f^*(\gamma))$ for $i > n - k$ is zero.

In §5, the case $k = 2$ is studied more closely.

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2. The cobordism class of M .

LEMMA 2.1. *If M^n is a closed n -manifold and ξ^k is a subbundle of the tangent bundle of M with k odd, then $w_n[M] = 0$; i.e., M has even Euler characteristic.*

PROOF. If n is odd, $w_n[M] = 0$, so one may assume n even. Let $k = 2p + 1$, $n - k = 2q + 1$ and let η be a complement of ξ in τ , the tangent bundle of M , so that $\xi \oplus \eta = \tau$. Then

$$\begin{aligned} w_n[M] &= w_n(\tau)[M] \\ &= w_{2p+1}(\xi) \cup w_{2q+1}(\eta)[M], \\ &= (Sq^1 w_{2p}(\xi) + w_1(\xi) \cup w_{2p}(\xi)) \cup w_{2q+1}(\eta)[M] \\ &= \{Sq^1 w_{2p}(\xi) \cup w_{2q+1}(\eta) + (w_1(\tau) + w_1(\eta)) \cup w_{2p}(\xi) \cup w_{2q+1}(\eta)\}[M] \\ &= \{Sq^1 w_{2p}(\xi) \cup w_{2q+1}(\eta) + v_1(\tau) \cup w_{2p}(\xi) \cup w_{2q+1}(\eta) \\ &\quad + w_{2p}(\xi) \cup Sq^1 w_{2q+1}(\eta)\}[M] \\ &= \{v_1(\tau) \cup + Sq^1\} \{w_{2p}(\xi) w_{2q+1}(\eta)\}[M] \end{aligned}$$

but cup-product with the Wu class $v_1(\tau) = w_1(\tau)$ gives Sq^1 , and so this vanishes. $\square$

In order to prove the converse, one needs some examples of manifolds. For this, one may use the result of [5, 3.4]:

LEMMA 2.2. *Let $RP(n_1, n_2, \dots, n_t)$, $t > 1$, be the bundle of lines in the fibers of $\lambda_1 \oplus \dots \oplus \lambda_t$ over $RP(n_1) \times \dots \times RP(n_t)$, where λ_i is the pull-back of the canonical bundle over $RP(n_i)$. Then $RP(n_1, \dots, n_t)$ is a closed manifold of dimension $n + t - 1$ where $n = n_1 + \dots + n_t$, and is indecomposable in $\mathfrak{N}_*$ if and only if*

$$\binom{n+t-2}{n_1} + \dots + \binom{n+t-2}{n_t}$$

is odd.

One now defines manifolds X^n of dimension n for $n \neq 2^s - 1$ and $n \neq 2$ as follows:

(a) if $n = 4s$, $s \geq 1$,

$$X^n = RP(1, \underbrace{\dots}_{2s}, 1, 0),$$

(b) if $n = 4s + 2$, $s \geq 1$,

$$X^n = RP(1, \underbrace{\dots}_{2s}, 1, 0, 0, 0),$$

(c) if $n = 2^p(2q + 1) - 1$, $p, q > 0$,

$$X^n = RP(2^p, 1, \underbrace{\dots}_{2^p q - 1}, 1, 0).$$

The above criterion immediately shows that these manifolds are indecomposable in $\mathfrak{N}_*$.

The manifolds X^n have the additional property that, for each integer $k \leq n$, the tangent bundle of X^n has a k -dimensional subbundle. In fact, for $n \neq 5$, the tangent bundle of X^n is a Whitney sum of line bundles.

To see this, let λ be the canonical line bundle over $RP(n_1, \dots, n_r)$ and $\pi: RP(n_1, \dots, n_r) \rightarrow RP(n_1) \times \dots \times RP(n_r)$ the projection. Let λ_i denote $\pi^*(\lambda_i)$ and τ_i the pullback of the tangent bundle of $RP(n_i)$. Then

$$\tau_{RP(n_1, \dots, n_r)} \cong \pi^* \tau_{RP(n_1) \times \dots \times RP(n_r)} \oplus \mu \cong \tau_1 \oplus \dots \oplus \tau_r \oplus \mu$$

where μ is the bundle along the fibers. Then

$$\mu \oplus l \cong (\lambda \otimes \lambda_1) \oplus \dots \oplus (\lambda \otimes \lambda_r) \quad \text{and} \quad \tau_i \oplus l = (n_i + 1)\lambda_i$$

where l is the trivial line bundle. If $n_i = 0$ or 1 , τ_i is trivial, since the tangent bundles of $RP(1) = S^1$ and $RP(0) = \text{point}$ are trivial. Adding the trivial τ_i with $n_i = 1$ to other τ_j or μ represents them as sums of line bundles.

For $n = 5$, $RP(2, 1, 0)$ has tangent bundle $\tau_1 \oplus l \oplus \mu$ which is a line bundle and two 2-plane bundles, while in all other cases there are at least two τ 's and the tangent bundle is a sum of line bundles.

One now has

PROPOSITION 2.3. *A class $\alpha \in \mathfrak{N}_n$ is represented by a manifold M^n whose tangent bundle has a k -dimensional subbundle, $k \leq n$, if either:*

(a) k is even, or

(b) k is odd and $w_n(\alpha) = 0$.

PROOF. Every class $\alpha \in \mathfrak{N}_n$ is represented by the disjoint union of manifolds

$$\underbrace{RP(2) \times \cdots \times RP(2)}_q \times X^{n_1} \times \cdots \times X^{n_s}$$

with $2q + n_1 + \cdots + n_s = n$. For any integer $k \leq n$ of the form $2u + v$ with $u \leq q$, $v \leq n_1 + \cdots + n_s$, this component has a subbundle of its tangent bundle of dimension k . In particular, every even integer can be put in this form, and every odd integer will be of this form except for the component $[RP(2)]^{n/2}$ which has $w_n \neq 0$. $\square$

This completes the proof of the proposition given in the introduction.

REMARK. If ξ is the line bundle over $RP(1)$ and λ is the line bundle over the Klein bottle $RP(\xi \oplus I)$, then the 5-manifold $RP(\lambda \oplus 3)$ is indecomposable in $\mathfrak{N}_*$ and has tangent bundle a sum of line bundles. This manifold could be used in place of X^5 and so five plays no special role.

3. Line bundles.

LEMMA 3.1. *If M^n is a closed n -manifold, ξ a sub-line-bundle of the tangent bundle of M and $f: M \rightarrow BO_1$ classifies ξ , then $\theta([M, f]) = 0$.*

PROOF. Let η be a complement in τ for ξ . Then $w(\eta) = w(\tau)/w(\xi)$, so since η is an $(n-1)$ -plane bundle

$$0 = w_n(\eta) = w_n(\tau) + w_{n-1}(\tau)w_1(\xi) + \cdots + (w_1(\xi))^n.$$

Since $w_1(\xi) = f^*(i)$ and $w_i(\tau) = w_i(M)$, this gives $\theta([M, f]) = 0$. $\square$

In order to prove the converse, one needs to analyze the bordism of BO_1 . Henceforth, classes of $\mathfrak{N}_*(BO_1)$ will be denoted $[M, \xi]$ where M is a closed manifold and ξ is a line bundle over M . There is a homomorphism of $\mathfrak{N}_*$ modules, called the Smith homomorphism,

$$\Delta: \mathfrak{N}_*(BO_1) \rightarrow \mathfrak{N}_*(BO_1)$$

of degree -1 assigning to $[M, \xi]$ the class $[N, \xi|_N]$ where $N \subset M$ is the codimension one submanifold of M dual to ξ . Specifically, if $f: M \rightarrow BO_1 = RP(\infty)$ classifies ξ , f maps M into some $RP(n)$ and may be homotoped in $RP(n)$ to be transverse regular on $RP(n-1)$, with N then taken to be the inverse image of $RP(n-1)$.

Letting $1 = [\text{point}, I] \in \mathfrak{N}_0(BO_1)$, there are unique classes $x_i = [M^i, \xi^i] \in \mathfrak{N}_i(BO_1)$, $i \geq 0$, with

- (1) $x_0 = 1$,
- (2) $\Delta x_i = x_{i-1}$, and
- (3) for $i > 0$, M^i bounds.

These classes form a base for $\mathfrak{N}_*(BO_1)$ as $\mathfrak{N}_*$ module. (A proof of this statement, or more precisely, its complex analogue appears in [2, (5.3)] .)

LEMMA 3.2. For $i > 0$, x_i is the class of the canonical bundle λ over $RP(1, 0, \dots, 0)$ ($i-1$ 0's).

PROOF. In [1, (2.2)], $RP(1, 0, \dots, 0)$ ($i-1$ 0's) is denoted $RP(\xi \oplus (i-1))$, where ξ is the canonical line bundle over $RP(1)$, and is shown to bound. In [4, p. 160] it is shown that for any vector bundle ρ over M , the submanifold dual to λ over $RP(\rho \oplus I)$ is $RP(\rho)$, from which the behaviour of Δ follows. $\square$

For $i > 1$, the tangent bundle of $RP(1, 0, \dots, 0)$ ($i-1$ 0's) is $1 \oplus \mu = \lambda \otimes \pi^*(\xi) \oplus (i-1)\lambda$, which contains a copy of λ , so $\theta(x_i) = 0$ if $i > 1$.

Now if ξ is a line bundle over M , and N is a closed manifold, $\pi_M^*(\xi)$ is a line bundle over $M \times N$, with $[N] \cdot [M, \xi] = [M \times N, \pi_M^*(\xi)]$ giving the module structure of $\mathfrak{N}_*(BO_1)$. If N has dimension n , it is immediate that $\theta([N] \cdot [M, \xi]) = w_n[N] \cdot \theta([M, \xi])$.

Since $\theta(x_0) = \theta(x_1) = 1$, one then has

LEMMA 3.3. $\theta(\sum_i [N^{n-i}] x_i) = w_n(N^n) + w_{n-1}(N^{n-1})$.

PROPOSITION 3.4. If $\alpha \in \mathfrak{N}_n(BO_1)$ with $\theta(\alpha) = 0$, then $\alpha = [M, \xi]$ where ξ is a sub-line-bundle of the tangent bundle of M .

PROOF. Let $\alpha = \sum_{i=1}^n a_i x_i$ with $a_i \in \mathfrak{N}_{n-i}$. Then $w_n(a_0) = 0, w_{n-1}(a_1) = 0$, for if n is odd $w_n(a_0) = 0$ for dimensional reasons while $w_{n-1}(a_1) = \theta(\alpha) = 0$ and if n is even $w_{n-1}(a_1) = 0$ for dimensional reasons while $w_n(a_0) = \theta(\alpha) = 0$. By [1, (4.5)] there are manifolds N^n and N^{n-1} fibered over S^1 , with $[N^{n-i}] = a_i, i = 0, 1$. Choose manifolds N^{n-i} representing a_i for $i > 1$, and let

$$M^n = N^n \cup (N^{n-1} \times RP(1)) \cup \bigcup_{i>1} (N^{n-i} \times \underbrace{RP(1, 0, \dots, 0)}_{i-1})$$

and let ξ be the line bundle over M whose restriction to N^n is trivial, to $N^{n-1} \times RP(1)$ is the pullback of the canonical bundle over $RP(1)$ and to $N^{n-i} \times RP(1, 0, \dots, 0)$ is the pullback of λ . Then $\alpha = [M, \xi]$.

Since N^n fibers over S^1 , the pullback of τ_{S^1} is a trivial line bundle in τ_{N^n} . Since $N^{n-1} \times RP(1)$ fibers over $S^1 \times S^1$, its tangent bundle contains a trivial 2 plane-bundle, but if ξ' is the canonical bundle over $RP(1)$, $2\xi' = 2$ so the tangent bundle contains two copies of the pullback of ξ' . As noted, λ is a subbundle of the tangent bundle of $RP(1, 0, \dots, 0)$ ($i-1$ 0's) if $i > 1$.

Thus ξ is a subbundle of the tangent bundle of M . $\square$

Combining this with Lemma 3.1 gives the second proposition of the introduction.

Now restricting attention to oriented manifolds one has

PROPOSITION 3.5. *A class $\alpha \in \Omega_n$ is represented by an oriented manifold M^n whose tangent bundle contains a line bundle if and only if the Stiefel-Whitney number $w_n(\alpha)$ is zero.*

A class $\alpha \in \Omega_n(RP(\infty))$ is represented by a pair $[M^n, \xi]$ where ξ is a sub-line-bundle of the tangent bundle of the oriented manifold M if and only if the Stiefel-Whitney number $\theta(\alpha)$ is zero.

PROOF. These conditions are clearly necessary. To see that they are sufficient, consider $\alpha \in \Omega_n$ for which $w_n(\alpha) = 0$ and choose a representative M^n for α . Using surgery, one may replace M by the connected sum of its components; i.e., may assume M connected. If n is odd, the tangent bundle has a nonvanishing section, while if n is even, such a section exists if and only if the Euler class of the tangent bundle $X(\tau)$ is zero. Since M is connected, $X(\tau) = \chi(M)\sigma$, where $\chi(M)$ is the Euler characteristic of M and σ is a generator of $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$. Mod 2, $\chi(M)$ is $w_n(\alpha)$ so $\chi(M)$ is even, and by forming the connected sum of M with copies of $S^p \times S^q$ for suitable $p, q > 0$, one obtains a new M with $\chi(M) = 0$ also in α . [Note. If $n = 2$, $\alpha = 0$ and M may be taken empty or $S^1 \times S^1$ while if $n = 2k$, $k > 1$, the connected sum with $S^2 \times S^{n-2}$ increases χ by 2 while that with $S^1 \times S^{n-1}$ decreases it by 2.] Thus every $\alpha \in \Omega_n$ with $w_n(\alpha) = 0$ is represented by a manifold M^n for which τ_M contains a trivial line bundle.

Now turning to $\Omega_*(RP(\infty))$, one has $\Omega_*(RP(\infty)) \cong \Omega_* \oplus \tilde{\Omega}_*(RP(\infty))$ and $\tilde{\Omega}_*(RP(\infty)) \cong \mathfrak{N}_{*-1}$. A class in the Ω_n summand of $\Omega_n(RP(\infty))$ is represented by a manifold M^n with trivial line bundle, and $\theta([M, 1]) = \langle w_n(\tau), [M] \rangle$ so that by the above, a class α in the Ω_* summand is represented by a subbundle if and only if $\theta(\alpha) = 0$. The summand $\mathfrak{N}_{n-1}$ of $\Omega_n(RP(\infty))$ is realized as follows. If $\beta \in \mathfrak{N}_{n-1}$, let N^{n-1} be a manifold in β and let M^n be the real projective space bundle $RP(\xi \oplus 1)$ where ξ is the determinant bundle of the tangent bundle of N and let λ be the canonical line bundle over $RP(\xi \oplus 1)$. Assigning to β the class of $[M, \lambda]$ gives the isomorphism $\mathfrak{N}_{n-1} \cong \tilde{\Omega}_n(RP(\infty))$. Now $\theta([M, \lambda]) = w_{n-1}(\beta)$, and if $\theta([M, \lambda]) = 0$, β is represented by a manifold N whose tangent bundle has a section and so λ is a subbundle of the tangent bundle of $RP(\xi \oplus 1)$. Noting that θ vanishes on the Ω_* summand if n is odd and on the $\tilde{\Omega}_*(RP(\infty))$ summand if n is even, one sees that every class in the kernel of θ is realized by a subbundle of the tangent bundle.

4. Stabilization. One now considers stabilization of the subbundle problem. This permits the use of homotopy theoretic techniques.

One may consider a manifold M^n together with an isomorphism $\tau_M \oplus$

$j \cong \xi^k \oplus \eta^{n-k} \oplus j$ where j denotes a trivial j plane bundle. By stability the existence of an isomorphism is independent of j if $j \geq 1$. The manifold M^n with this structure bounds if $M = \partial V$ where $\tau_V \oplus (j-1) \cong \rho^k \oplus \sigma^{n-k+1} \oplus (j-1)$ is a compatible decomposition; i.e., ρ restricts to ξ and σ to $\eta \oplus 1$. Assuming V has no closed components, V has the homotopy type of an n -dimensional complex, so $\tau_V \cong \rho \oplus \sigma$, but this need not be compatible with the chosen isomorphism along M unless $j > 1$.

Let $\phi_r^k: BO_k \times BO_r \rightarrow BO$ be a map classifying the complement of the Whitney sum $\gamma_k \oplus \gamma_r$ of the universal bundles (converted to a fibration). The structure on M is precisely a lift of the normal map of M to $BO_k \times BO_{n-k}$, while that of V is a lift to $BO_k \times BO_{n-k+1}$.

The techniques of bordism of manifolds with normal structure [3] give that the bordism group of manifolds M^n of the given type is the image of the stable homotopy homomorphism

$$\pi_n^S(T(BO_k \times BO_{n-k})) \rightarrow \pi_n^S(T(BO_k \times BO_{n-k+1}))$$

where $T(BO_k \times BO_r)$ is the Thom spectrum associated with the fibration ϕ_r^k .

Specifically, if one takes the induced fibration

$$\begin{array}{ccc} E & \longrightarrow & BO_k \times BO_r \\ \pi \downarrow & & \downarrow \phi_r^k \\ BO_s & \longrightarrow & BO \end{array}$$

then $\pi_n^S(T(BO_k \times BO_r)) = \lim_{s \rightarrow \infty} \pi_{n+s}(T(\pi^*(\gamma_s)))$. One may also describe these groups as

$$\pi_n^S(T(BO_k \times BO_r)) = \lim_{s, t \rightarrow \infty} \pi_{n+s+t}(T(\gamma_s \oplus \gamma_t))$$

where γ_s, γ_t are the universal s and t plane bundles over the Grassmann manifolds $G_{k,s}$ and $G_{r,t}$.

One may now consider the homomorphism

$$\pi_n^S(T(BO_k \times BO_{n-k})) \rightarrow \lim_{r \rightarrow \infty} \pi_n^S(T(BO_k \times BO_r))$$

$$\parallel$$

$$\pi_n^S(T(BO_k \times BO)).$$

One has $\pi_1 \times \oplus: BO_k \times BO \rightarrow BO_k \times BO$, which is a homotopy equivalence, and induces an equivalence $T(BO_k \times BO) \cong BO_k^+ \wedge MO$ and hence

$$\pi_n^S(T(BO_k \times BO)) \cong \mathfrak{H}_n(BO_k).$$

This describes the forgetful homomorphism assigning to M^n with its structure the bordism class of (M, ξ) .

One now embarks on a program of analyzing the stable homotopy groups involved.

LEMMA 4.1. *Let γ_s be the universal s plane bundle over $G_{r,s}$, $s > r$, and let p be an odd prime. Then $\tilde{H}^i(T(\gamma_s); Z_p) = 0$ for $i < r + s$.*

PROOF. One has the inclusion $G_{r,s} \subset G_{r+1,s}$ with $G_{r+1,s}$ obtained by attaching cells of dimension $(r + 1)$ and higher. This induces an inclusion of Thom spaces $T(\gamma_s|G_{r,s}) \subset T(\gamma_s|G_{r+1,s})$ and the cofiber has cells of dimension $r + 1 + s$ and higher. From the exact cohomology sequence

$$\tilde{H}^i(T(\gamma_s|G_{r,s}); Z_p) \cong \tilde{H}^i(T(\gamma_s|G_{r+1,s}); Z_p) \quad \text{if } i < r + s.$$

Thus

$$\tilde{H}^i(T(\gamma_s|G_{r,s}); Z_p) \cong \tilde{H}^i(T(\gamma_s|G_{r+t,s}); Z_p) \quad \text{if } i < r + s, t \geq 0,$$

but for t large this is $\tilde{H}^i(MO_s; Z_p)$ which is zero. $\square$

LEMMA 4.2. $\pi_i^S(T(BO_k \times BO_r))$ is a 2 group if $i < k + r$.

PROOF. Let γ_s, γ_t be the universal bundles over $G_{k,s}$ and $G_{r,t}$, s and t large. Then $T(\gamma_s \oplus \gamma_t) = T(\gamma_s) \wedge T(\gamma_t)$ and $\tilde{H}^i(T(\gamma_s \oplus \gamma_t); Z_p) = 0$ if $i < k + r + s + t$ if p is odd. By the mod C Hurewicz theorem $\pi_i(T(\gamma_s \oplus \gamma_t))$ is a 2 group if $i < k + r + s + t$. $\square$

Thus, for $r \geq n - k + 1$, $\pi_n^S(T(BO_k \times BO_r))$ is a 2 group, and the problem is entirely a 2 primary problem.

In order to begin the 2 primary analysis, one analyzes a cofibration of spectra

$$T(BO_k \times BO_r) \rightarrow T(BO_k \times BO_{r+1}) \rightarrow X$$

which one realizes by a cofibration $T(\gamma_s \oplus \gamma_t) \rightarrow T(\gamma_s \oplus \gamma'_t) \rightarrow X$ where γ_s, γ_t are universal bundles over $G_{k,s}, G_{r,t}$ and γ'_t is the universal bundle over $G_{r+1,t}$, with s and t being large.

First, consider $G_{r+1,t}$ as the space of $r + 1$ planes in R^{r+1+t} with $\pi: D(\gamma_{r+1}) \rightarrow G_{r+1,t}$ the projection of the disc bundle. Letting $S(\gamma_{r+1})$ be the unit sphere bundle, one has a cofibration

$$\begin{array}{ccccc} \frac{D(\pi^*(\gamma'_t)|S(\gamma_{r+1}))}{S(\pi^*(\gamma'_t)|S(\gamma_{r+1}))} & \rightarrow & \frac{D(\pi^*(\gamma'_t))}{S(\pi^*(\gamma'_t))} & \rightarrow & \frac{D(\pi^*(\gamma'_t))}{D(\pi^*(\gamma'_t)|S(\gamma_{r+1})) \cup S(\pi^*(\gamma'_t))} \\ \parallel & & \parallel & & \parallel \\ A & & B & & C \end{array}$$

Since $D(\pi^*(\gamma'_t))$ is identifiable with $D(\gamma_{r+1} \oplus \gamma'_t)$, C is the Thom space of the trivial bundle $\gamma_{r+1} \oplus \gamma'_t$, and $C \cong \Sigma^{r+t+1}(G_{r+1,t})$ is the $(r+t+1)$ -fold suspension of $G_{r+1,t}$ with a base point adjoined. Since π is a homotopy equivalence, $B \cong T(\gamma'_t)$.

Finally, $S(\gamma_{r+1})$ may be considered as pairs (α, x) with α an $(r+1)$ -plane in R^{r+1+t} and x a unit vector in α . Assigning to (α, x) the point $x \in S^{r+t}$ defines a fibration $p: S(\gamma_{r+1}) \rightarrow S^{r+t}$. The inverse image of $x \in S^{r+t}$ is the space of r planes in R^{r+1+t} orthogonal to x , i.e., $S(\gamma_{r+1})$ is the Grassmann bundle of r planes in the fibers of the tangent bundle of S^{r+t} . The inclusion $G_{r,t} \rightarrow G_{r+1,t}$ may then be considered as factoring via the inclusion as a fiber in $S(\gamma_{r+1})$. The inclusion of the fiber $G_{r,t} \rightarrow S(\gamma_{r+1})$ induces isomorphisms in homotopy and homology in dimensions less than $r+t-1$, and so the inclusion $T(\gamma_t) \rightarrow A$ is a homotopy equivalence (for the prime 2) in dimensions less than $r+2t-1$. Since t is large, one then obtains a cofibration

$$T(\gamma_t) \rightarrow T(\gamma'_t) \rightarrow \Sigma^{r+t+1}(G_{r+1,t}).$$

Smashing with $T(\gamma_s)$ gives a cofibration sequence

$$T(\gamma_s \oplus \gamma_t) \rightarrow T(\gamma_s \oplus \gamma'_t) \rightarrow T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1,t})$$

(i.e., X may be identified with $T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1,t})$ for the prime 2,

having isomorphic mod 2 cohomology up to dimension $s+r+2t-1$ induced by a map of spaces).

One now considers $T(\gamma_s) \wedge \Sigma^{r+t+1}(G_{r+1,t})$ as $\Sigma^{r+t+1} T(\gamma_s) \wedge G_{r+1,t}$ and analyzes the maps

$$G_{k,s} \rightarrow G_{k,s+r+t+1} \rightarrow G_{m,s+r+t+1}$$

inducing

$$\Sigma^{r+t+1} T(\gamma_s) \rightarrow T(\gamma_{s+r+t+1}) \rightarrow MO_{s+r+t+1}$$

(m being large). The maps of Grassmannians induce isomorphisms in mod 2 cohomology in dimensions less than or equal to k and hence the Thom spaces have isomorphic mod 2 cohomology in dimensions less than or equal to $k+s+r+t+1$.

Thus X may be identified with $MO_{s+r+t+1} \wedge (G_{r+1,t})^+$ in dimensions less than or equal to $k+s+r+t+1$ (in mod 2 cohomology). In particular, in dimensions less than or equal to $k+s+r+t+1$ $\tilde{H}^*(X; \mathbb{Z}_2)$ is a free module over the Steenrod algebra and

$$\pi_{i+s+t}(X) \cong \pi_{i+s+t}(MO_{s+r+t+1} \wedge (G_{r+1,t})^+) \cong \mathfrak{N}_{i-r-1}(G_{r+1,t})$$

if $i + s + t \leq k + s + r + t$, $i \leq k + r$ (for 2 primary structure).

Being given a manifold M^i with $\tau_M \oplus j \cong \xi^k \oplus \eta^{r+1} \oplus (i + j - k - r - 1)$ representing a class in $\pi_i^S(T(BO_k \times BO_{r+1}))$, $i \leq k + r$, the class in $\pi_i^S(X) \cong \mathfrak{N}_{i-r-1}(BO_{r+1})$ obtained from the cofibration is represented by the submanifold of M^i dual to η^{r+1} with the $(r + 1)$ -plane bundle obtained by restricting η . The map to X is induced by including $T(\gamma'_t)$ in $T(\gamma'_t \oplus \gamma_{r+1})$ and making the maps transverse regular involves finding the submanifold dual to γ_{r+1} , from which one has the given assertion.

On the other hand, a class in $\pi_i^S(T(BO_k \times BO_{r+1}))$, $i \leq k + r$, is in the image of $\pi_i^S(T(BO_k \times BO_r))$ if and only if it goes to zero in $\pi_i^S(X)$. Since $\tilde{H}^*(X; Z_2)$ is a free module over the Steenrod algebra in dimensions up to $k + s + r + t + 1$, a homotopy element in $\pi_{i+s+r}(X)$ is detected by mod 2 cohomology. Since $T(\gamma_s \oplus \gamma'_t) \rightarrow X$ maps $\tilde{H}^*(X; Z_2)$ isomorphically onto the multiples of $\Phi(w_{r+1})$, the Thom isomorphism image of w_{r+1} , in the $H^*(G_{k,s} \times G_{r+1,t}; Z_2)$ module structure, this asserts that all characteristic numbers involving w_{r+1} should vanish. Thus, one has

LEMMA 4.3. *A manifold M^i with $\tau_M \oplus j \cong \xi^k \oplus \eta^{r+1} \oplus (i + j - k - r - 1)$ representing a class in $\pi_i^S(T(BO_k \times BO_{r+1}))$, $i \leq k + r$, comes from $\pi_i^S(T(BO_k \times BO_r))$ if and only if all characteristic numbers involving $w_{r+1}(\eta)$ are zero.*

For $r \geq n - k$, this determines the image of

$$\pi_n^S(T(BO_k \times BO_r)) \rightarrow \pi_n^S(T(BO_k \times BO_{r+1})).$$

For $r \geq n - k + 1$, this homomorphism is monic, which may be seen as follows.

Consider the homomorphism

$$\pi_{n+1}^S(T(BO_k \times BO_{r+1})) \rightarrow \pi_{n+1}^S(X).$$

Now $\pi_{n+1}^S(X) \cong \mathfrak{N}_{n-r}(BO_{r+1})$ for $n + 1 \leq k + r$, and $\mathfrak{N}_{n-r}(BO_{r+1})$ is generated over Z_2 by the manifolds

$$P = M^m \times RP(\lambda_1 \oplus k_1) \times \cdots \times RP(\lambda_s \oplus k_s) \times (\text{point})$$

where λ_i is the nontrivial bundle over $RP(1)$, $k_i \geq 0$, with $m + (k_1 + 1) + \cdots + (k_s + 1) = n - r$ with bundle

$$\lambda^{(1)} \oplus \cdots \oplus \lambda^{(s)} \oplus (r + 1 - s)$$

where $\lambda^{(i)}$ is the canonical bundle over $RP(\lambda_i \oplus k_i)$. To see this, one notes that the $RP(\lambda \oplus k)$, $k \geq 0$, and the point generate $\mathfrak{N}_*(BO_1)$, over $\mathfrak{N}_*$ and forming the products of $r + 1$ of these gives a $\mathfrak{N}_*$ generating set for $\mathfrak{N}_*(BO_{r+1})$.

One then considers the manifold

$$Q = M^m \times RP(\lambda_1 \oplus k_1 \oplus 1) \times \cdots \times RP(\lambda_s \oplus k_s \oplus 1) \times RP(r+1-s)$$

of dimension $m + (k_1 + 2) + \cdots + (k_s + 2) + r + 1 - s = n - r + s + r + 1 - s = n + 1$ and letting λ be the canonical line bundle over $RP(r+1-s)$, the submanifold dual to $\lambda^{(1)} \oplus \cdots \oplus \lambda^{(s)} \oplus (r+1-s)\lambda = \eta^{r+1}$ is the manifold P given above, with η restricting to the given bundle. Now the tangent bundle of $RP(\lambda_i \oplus k_i \oplus 1)$ is $\lambda^{(i)} \otimes \lambda_i \oplus (k_i + 1)\lambda^{(i)}$ so

$$\begin{aligned} \tau_Q \oplus 1 &\cong [\tau_M \oplus (\lambda^{(1)} \oplus \lambda_1 \oplus k_1 \lambda^{(1)}) \oplus \cdots \\ &\quad \oplus (\lambda^{(s)} \otimes \lambda_s \oplus k_s \lambda^{(s)}) \oplus \lambda] \oplus \eta \\ &= \xi' \oplus \eta \end{aligned}$$

where ξ' is an $m + (k_1 + 1) + \cdots + (k_s + 1) + 1 = n - r + 1 \leq k$ bundle. Thus $\tau_Q \oplus 1 \oplus (k + r - n - 1) \cong [\xi' \oplus (k + r - 1)] \oplus \eta = \xi^k \oplus \eta$ giving a structure on Q mapping to the class of P in $\mathfrak{N}_{n-r}(BO_{r+1})$.

This proves that the forgetful homomorphism $\pi_n^S(T(BO_k \times BO_r)) \rightarrow \mathfrak{N}_n(BO_k)$ is monic for $r \geq n - k + 1$, and that

$$\text{im} \{ \pi_n^S(T(BO_k \times BO_{n-k})) \rightarrow \pi_n^S(T(BO_k \times BO_{n-k+1})) \}$$

is mapped monomorphically into $\mathfrak{N}_n(BO_k)$ with image precisely those classes for which all numbers involving $w_i(\tau - f^*(\gamma_k))$ for $i > n - k$ are zero, or one has

PROPOSITION 4.4. *A class $\alpha = [M, f] \in \mathfrak{N}_n(BO_k)$ is represented by a manifold M^n with $\tau_M \oplus 1 \cong f^*(\gamma_k) \oplus \eta^{n-k} \oplus 1$ if and only if all Stiefel-Whitney numbers of α involving $w_i(\tau - f^*(\gamma_k))$ for $i > n - k$ are zero.*

5. Two plane bundles. The purpose of this section is to prove

PROPOSITION 5.1. *A class $\alpha = [M, f] \in \mathfrak{N}_n(BO_2)$ is represented by a pair $[M, f]$ with $f^*(\gamma_2)$ a subbundle of the tangent bundle of M if and only if all characteristic numbers of α involving $w_i(\tau - f^*(\gamma_2))$ with $i > n - 2$ are zero.*

To begin the proof, one wants manifolds $M_{i,j}$ of dimension $i + 2j$ for each (i, j) and 2 plane bundles $\lambda_{i,j}$ over $M_{i,j}$ for which

$$w_1^p(\lambda_{i,j}) w_2^q(\lambda_{i,j}) [M_{i,j}] = \begin{cases} 0 & \text{if } q > j, \quad p + 2q = i + 2j, \\ 1 & \text{if } q = j, \quad p = i \end{cases}$$

Any collection of such manifolds form a base for $\mathfrak{N}_*(BO_2)$ as $\mathfrak{N}_*$ module. The representatives will be chosen so that $\lambda_{i,j}$ is a subbundle of the tangent bundle of $M_{i,j}$ except for $j = 0$ and $i \leq 3$.

For $j \geq 2$, one lets

$$M_{i,j} = RP(1, \underbrace{0, \dots, 0}_{j-1}) \times RP(1, \underbrace{0, \dots, 0}_{i+j-1})$$

and lets $\lambda_{i,j} = \pi_1^*(\lambda) \oplus \pi_2^*(\lambda)$, where λ is the canonical line bundle over $RP(1, 0, \dots, 0)$.

For $j = 1, i \geq 3$, one lets $M_{i,j} = RP(1) \times RP(3, 0, \dots, 0)$ ($i - 2$ 0's) and lets $\lambda_{i,j} = \pi_1^*(\xi) \oplus \pi_2^*(\lambda)$, ξ being the Hopf bundle over $RP(1)$. The tangent bundle of $RP(3)$ is trivial and so the tangent bundle of $M_{i,j}$ is $3 \oplus \pi_2^*(\lambda) \otimes (\pi_2^*(\xi') \oplus (i - 2))$, where ξ' is the Hopf bundle over $RP(3)$. Since $2\xi = 2$ and $i \geq 3$, $\lambda_{i,j}$ is a subbundle of the tangent bundle.

For $j = 0, i \geq 4$, one lets $M_{i,j} = RP(3, 0, \dots, 0)$ ($i - 3$ 0's) and $\lambda_{i,j} = 1 \oplus \lambda$.

For $j = 1, i = 0$, one lets $M_{i,j} = RP(2)$, and $\lambda_{i,j} = \tau$, the tangent bundle of $RP(2)$.

For $j = 1, i = 1$, one lets $M_{i,j} = RP(1) \times RP(2)$, the tangent bundle being $1 \oplus \pi_2^*(\tau) = 3\pi_2^*(\xi) = (2\pi_1^*(\xi) \otimes \pi_2^*(\xi)) + \pi_2^*(\xi)$ and lets $\lambda_{i,j} = [\pi_1^*(\xi) \otimes \pi_2^*(\xi)] \oplus \pi_2^*(\xi)$.

For $j = 1, i = 2$, let $M_{i,j}$ be the bundle of lines in the fibers of $\lambda \oplus 2$ over $RP(1, 0) = RP(\xi \oplus 1)$ where ξ is the Hopf bundle over $RP(1)$, giving projections

$$\pi: M_{i,j} \rightarrow RP(1, 0), \quad p: RP(1, 0) \rightarrow RP(1).$$

Let θ be the bundle along the fibers of p , η the bundle along the fibers of π , and λ' the canonical line bundle over $M_{i,j}$. Then

$$\tau_{M_{i,j}} = \eta \oplus \pi^*(\tau_{RP(1,0)}) = \eta \oplus \pi^*(\theta) \oplus 1 = (\lambda' \otimes \pi^*(\lambda \oplus 2)) \oplus \pi^*(\theta)$$

which contains a copy of $\lambda_{i,j} = \lambda' \oplus \pi^*(\theta)$.

Finally, let $M_{0,0}$ be a point with $\lambda_{0,0}$ trivial, $M_{1,0} = RP(1)$ with $\lambda_{1,0} = \xi \oplus 1$, and $M_{2,0} = RP(1, 0)$, $M_{3,0} = RP(1, 0, 0)$ with $\lambda_{i,0} = \lambda \oplus 1$.

Note that for $M_{i,0}$, $i \leq 3$, $\lambda_{i,0}$ is a subbundle of $\tau \oplus 2$. In particular, if $\alpha \in \mathfrak{N}_p$ and $w_p(\alpha) = 0$, α is represented by a manifold M^p fibered over $S^1 \times S^1$ [5, Proposition 6.1] and hence τ_M has 2 sections, so $\lambda_{i,0}$ is a subbundle of the tangent bundle of $M \times M_{i,0}$.

Every class in $\mathfrak{N}_n(BO_2)$ is of the form $\sum \alpha_{(i,j)} [M_{i,j}, \lambda_{i,j}]$ with $\alpha_{(i,j)} \in \mathfrak{N}_{n-i-2j}$ and every class $\alpha \in \mathfrak{N}_p$ has the form $\beta + aRP(2)^{p/2}$, $a \in \mathbb{Z}_2, \beta \in \mathfrak{N}_p$ with $w_p(\beta) = 0$. Thus if I is the $\mathfrak{N}_*$ submodule of classes in $\mathfrak{N}_*(BO_2)$

represented by $[M, f]$ with $f^*(\gamma_2)$ a subbundle of the tangent bundle of M , then $\mathfrak{N}_*(BO_2)/I$ is a Z_2 vector space generated by the classes

$$[RP(2)^s] \cdot [M_{i,0}, \lambda_{i,0}] \quad \text{with } i \leq 3.$$

The characteristic numbers $w_n(\tau - f^*(\gamma_2))$ and $w_1(\tau - f^*(\gamma_2)) \cdot w_{n-1}(\tau - f^*(\gamma_2))$ (for $n \geq 2$) may be readily seen to give a homomorphism $\mathfrak{N}_n(BO_2)/I \rightarrow Z_2 \oplus Z_2$ (or to Z_2 if $n \leq 1$) sending the classes $[RP(2)^s] \cdot [M_{i,0}; \lambda_{i,0}]$ of dimension n to linearly independent elements.

This completes the proof of the proposition.

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