

THE p -CLASS IN A DUAL B^* -ALGEBRA

BY

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ABSTRACT. In this paper, we introduce and study the class A_p ($0 < p < \infty$) in a dual B^* -algebra A . We show that, for $1 \leq p < \infty$, A_p is a dual A^* -algebra which is a dense two-sided ideal of A . If $1 < p < \infty$, we obtain that A_p is uniformly convex and hence reflexive. We also identify the conjugate space of A_p ($1 \leq p < \infty$). This is a generalization of the class C_p of compact operators on a Hilbert space.

1. Introduction. Let H be a Hilbert space and $LC(H)$ the algebra of all compact operators on H . Then $LC(H)$ is a simple dual B^* -algebra and every simple dual B^* -algebra is of this form. The class C_p of compact operators in $LC(H)$ has many interesting properties and has been studied in various articles (e.g., see [2], [3] and [4]). The present work is an attempt to introduce a similar class of spaces in an arbitrary dual B^* -algebra.

Let A be a dual B^* -algebra. The class A_p ($0 < p \leq \infty$) is introduced in §3. After establishing some crucial inequalities, we show that A_p ($1 \leq p \leq \infty$) is a dual A^* -algebra which is a dense two-sided ideal of A . In §4, we study the algebras A_1 and A_2 . We obtain that every proper H^* -algebra is of the form A_2 and $A_1 = \{xy: x, y \in A_2\}$. §5 is devoted to showing the uniform convexity in A_p ($1 < p < \infty$). Finally we identify the conjugate space of A_p ($1 \leq p < \infty$) in §6.

In this paper, our approach is elementary and the techniques are not new. In fact, they are borrowed from [3], [4], [10] and [11]. The author is grateful for these invaluable references.

2. Notation and preliminaries. Definitions not explicitly given are taken from Rickart's book [7].

For any set E in a Banach algebra A , let $l(E)$ and $r(E)$ denote the left and right annihilators of E , respectively. Then A is called a dual algebra

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if for every closed right ideal R and every closed left ideal I , we have $r(l(R)) = R$ and $l(r(I)) = I$. See [5] and [7] for some of its properties.

An idempotent e in a Banach algebra A is said to be minimal if eAe is a division algebra. In case A is semisimple, this is equivalent to saying that Ae (eA) is a minimal left (right) ideal of A .

Let A be a Banach algebra. A bounded linear operator S on A is called a right centralizer if $S(xy) = (Sx)y$ for all x, y in A . For each a in A , the operator $L_a: x \rightarrow ax$ ($x \in A$) is a right centralizer on A .

Let H be a Hilbert space with an inner product $(\cdot, \cdot)$. If x and y are elements in H , then $x \otimes y$ will denote the operator on H defined by $(x \otimes y)(h) = (h, y)x$ for all h in H .

In this paper, all algebras and linear spaces under consideration are over the field of complex numbers.

NOTATION. In this paper, A will denote a dual B^* -algebra with norm $\|\cdot\|$.

We shall often use, without explicitly mentioning, the following fact: For any orthogonal family $\{e_\alpha\}$ of hermitian idempotents of A , $\sum_\alpha e_\alpha x$ is summable in A , and especially when $\{e_\alpha\}$ is a maximal family, $x = \sum_\alpha e_\alpha x$ for all x in A (see [5, p. 30, Theorem 16]).

Let B be a closed commutative $*$ -subalgebra of A and e a minimal idempotent in B . It follows easily from [7, p. 261, Lemma (4.10.1)] that e is hermitian. Also if f is any other minimal idempotent in B , then $fe = ef = 0$. If B is maximal, then e is a minimal idempotent in A .

LEMMA 2.1. *Let e be a hermitian minimal idempotent in A , $a \in A$, and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A . Then $\|ae\|^2 = \sum_\beta \|f_\beta ae\|^2$.*

PROOF. Since A is a dual B^* -algebra, it follows from [7, p. 259, Theorem (4.9.24)] and [7, p. 269, Corollary (4.10.20)] that $A = (\sum_\lambda LC(H_\lambda))_0$, where $LC(H_\lambda)$ is the algebra of all compact operators on a Hilbert space H_λ . It is easy to see that $e \in LC(H_{\lambda_0})$ for some λ_0 . Let $\{f_\gamma\} = \{f_\beta\} \cap LC(H_{\lambda_0})$. Then we can write $f_\gamma = x_\gamma \otimes x_\gamma$ with $x_\gamma \in H_{\lambda_0}$ and $\|x_\gamma\| = 1$. Similarly $e = y \otimes y$ with $y \in H_{\lambda_0}$ and $\|y\| = 1$. Since $\{f_\gamma\}$ is a maximal orthogonal family of hermitian minimal idempotents in $LC(H_{\lambda_0})$, it follows easily that $\{x_\gamma\}$ is a complete orthonormal set in H_{λ_0} . Put $b = ae$. Then $b \in LC(H_{\lambda_0})$ and $be = ae$. Hence

$$\|ae\|^2 = \|eb^*be\| = \|(y \otimes y)b^*b(y \otimes y)\| = \|by\|^2.$$

Similarly $\|f_\gamma be\| = |(by, x_\gamma)|$. Since $f_\beta ae = 0$ if $f_\beta \notin \{f_\gamma\}$, by Parseval's identity we have

$$\sum_{\beta} \|f_{\beta} a e\|^2 = \sum_{\gamma} \|f_{\gamma} b e\|^2 = \sum_{\gamma} |(b y, x_{\gamma})|^2 = \|b y\|^2 = \|a e\|^2.$$

This completes the proof.

The following lemma is useful in this paper and it is similar to [10, p. 29, Lemma 1].

LEMMA 2.2. *Let $a \in A$ and $\{e_{\alpha}\}, \{f_{\beta}\}$ any two maximal orthogonal families of hermitian minimal idempotents in A . Then*

$$\sum_{\alpha} \|a e_{\alpha}\|^2 = \sum_{\beta} \|a f_{\beta}\|^2 = \sum_{\beta} \|a^* f_{\beta}\|^2.$$

PROOF. We note first that $\|f_{\beta} a e_{\alpha}\| = \|e_{\alpha} a^* f_{\beta}\|$. If $\sum_{\alpha} \|a e_{\alpha}\|^2$ is summable, then by Lemma 2.1, we have

$$(2.1) \quad \sum_{\alpha} \|a e_{\alpha}\|^2 = \sum_{\alpha} \sum_{\beta} \|f_{\beta} a e_{\alpha}\|^2 = \sum_{\beta} \sum_{\alpha} \|e_{\alpha} a^* f_{\beta}\|^2 = \sum_{\beta} \|a^* f_{\beta}\|^2.$$

Hence, in particular, $\sum_{\beta} \|a f_{\beta}\|^2 = \sum_{\beta} \|a^* f_{\beta}\|^2$. The lemma now follows from (2.1).

Suppose b is a normal element in A . Let B (resp. B') be a maximal commutative $*$ -subalgebra of A containing b and $\{e_{\alpha}\}$ (resp. $\{e_{\omega}\}$) the maximal orthogonal family of hermitian minimal idempotents in B (resp. B'). Then $b e_{\alpha} = e_{\alpha} b e_{\alpha} = k_{\alpha} e_{\alpha}$ for some constant k_{α} . Similarly $b e_{\omega} = k_{\omega} e_{\omega}$ for some constant k_{ω} . Let K (resp. K') be the set of all nonzero k_{α} (resp. k_{ω}). We note that k_{α_1} may be equal to k_{α_2} for some $\alpha_1 \neq \alpha_2$. However we consider them as different elements in K .

LEMMA 2.3. *The set K is either finite or countable and $K = K'$. The set of all distinct constants in K is precisely the set of all nonzero constants in the spectrum of b .*

PROOF. Let B_0 be the intersection of all maximal commutative $*$ -subalgebras of A containing b . Let $\{f_{\beta}\}$ be the maximal orthogonal family of hermitian minimal idempotents in B_0 . Since B_0 is a dual B^* -algebra, $b = \sum_{\beta} b f_{\beta} = \sum_{\beta} \lambda_{\beta} f_{\beta}$, where λ_{β} are constants. Therefore there exists only a countable number of f_{β} for which $b f_{\beta} \neq 0$. Also, for each nonzero λ_{β_0} , the set $\{\lambda_{\beta} : \lambda_{\beta} = \lambda_{\beta_0}\}$ is finite. It is now easy for us to write $b = \sum_{n=1}^{\infty} \lambda_n f_n$, where λ_n are distinct nonzero constants and $\{f_n\}$ is an orthogonal family of hermitian idempotents in B_0 such that $\lambda_n f_n = b f_n$. Note that f_n is not necessarily minimal. Since B is dual and $f_n \in B$, it is well known that

$$f_n = e_{\alpha_{n_1}} + \cdots + e_{\alpha_{n_p}}, \text{ where } e_{\alpha_{n_i}} \in \{e_{\alpha}\} \text{ (} i = 1, 2, \dots, p \text{)}.$$

Considering the right ideal $f_n A$ of A , by [1, p. 497, Theorem 2.2], the number n_p is independent of the choice of B . Since $b f_n = \lambda_n f_n$, we see easily that

$$(2.2) \quad b e_{\alpha_{n_i}} = \lambda_n e_{\alpha_{n_i}} \quad (i = 1, 2, \dots, p).$$

If $f_m = e_{\alpha_{m_1}} + \dots + e_{\alpha_{m_q}}$ ($m \neq n$), then it follows from (2.2) that

$$\{e_{\alpha_{m_1}}, \dots, e_{\alpha_{m_q}}\} \cap \{e_{\alpha_{n_1}}, \dots, e_{\alpha_{n_p}}\} = \emptyset,$$

because $\lambda_n \neq \lambda_m$. Also

$$b = \sum_n \lambda_n f_n = \sum \lambda_n (e_{\alpha_{n_1}} + \dots + e_{\alpha_{n_p}}).$$

Let E be the set of all such $e_{\alpha_{n_p}}$. Then E is countable. For simplicity, we write $E = \{e_1, e_2, \dots\}$ and $b = \sum_{n=1}^{\infty} k_n e_n$, where $k_n e_n = b e_n$ and $k_n \neq 0$ (because $\lambda_n \neq 0$). Let $\{e_\gamma\} = \{e_\alpha\} - E$. We show that $b e_\gamma = 0$ for all γ . In fact, since $b = \sum_\alpha b e_\alpha = \sum_n b e_n$, it follows that $\sum_\gamma b e_\gamma = 0$. Let F_α be the multiplicative linear functional on B corresponding to the maximal modular ideal $B(1 - e_\alpha)$ of B . For any fixed γ_0 , we have $b e_{\gamma_0} = k_{\gamma_0} e_{\gamma_0}$, for some constant k_{γ_0} . Then

$$k_{\gamma_0} = F_{\gamma_0}(b e_{\gamma_0}) = \sum_\gamma F_{\gamma_0}(b e_\gamma) = F_{\gamma_0} \left(\sum_\gamma b e_\gamma \right) = 0.$$

Hence it follows that $b e_\gamma = 0$ for all γ . Consequently $K = \{k_n\}$. Similarly we can show that $K' = \{k_n\}$. Therefore $K = K'$. Now the last part of the lemma follows easily from [7, p. 111, Theorem (3.1.6)]. This completes the proof.

Let b , $\{e_\alpha\}$ and $\{e_n\}$ be as in the proof of Lemma 2.3. Then $b = \sum_\alpha k_\alpha e_\alpha = \sum_n k_n e_n$ is called a spectral representation of b . By Lemma 2.3, $\{k_n\}$ is independent of $\{e_n\}$. Also if $k_\alpha \neq k_n$ for all n , then $k_\alpha = 0$.

Suppose a is a nonzero element in A . Let $a^* a = \sum_n r_n e_n$ be a spectral representation of $a^* a$. We claim that

$$(2.3) \quad a = \sum_n a e_n.$$

In fact, since $\sum_n a e_n$ is summable and $a^* a = \sum_n a^* a e_n = \sum_n e_n a^* a e_n = \sum_n r_n e_n$, it follows that $(a - \sum_n a e_n)^* (a - \sum_n a e_n) = 0$. Hence $a = \sum_n a e_n$. We note that $a e_n \neq 0$; for otherwise $r_n e_n = a^* a e_n = 0$.

Since $a^* a$ is a positive element, $r_n > 0$ for all n . Put $k_n = \sqrt{r_n} > 0$. We show that $\sum_n k_n e_n$ is summable in A . In fact, for any two positive integers m, n ($m < n$), $\|\sum_{i=m}^n k_i e_i\|^2 = \|\sum_{i=m}^n r_i e_i\|$. Since $\sum_n r_n e_n$ is summable, so

is $\sum_n k_n e_n$. Put

$$(2.4) \quad [a] = \sum_n k_n e_n.$$

Then $[a]^* = [a]$ and $[a]^2 = a^*a$. Hence $[a] = (a^*a)^{1/2}$. For each x in A ,

$$(2.5) \quad \left\| \sum_{i=m}^n k_i^{-1} a e_i x \right\|^2 = \left\| \left(\sum_{i=m}^n k_i^{-1} a e_i x \right)^* \left(\sum_{i=m}^n k_i^{-1} a e_i x \right) \right\| \\ = \left\| \sum_{i=m}^n x^* e_i x \right\| = \left\| \sum_{i=m}^n e_i x \right\|^2 \leq \|x\|^2.$$

Since $\sum_n e_n x$ is summable in A , so is $\sum_n k_n^{-1} a e_n x$. Define a mapping W on A by

$$(2.6) \quad Wx = \sum_n k_n^{-1} a e_n x \quad (x \in A).$$

Then it follows from (2.3), (2.4) and (2.5) that $W[a] = a$ and $\|W\| = 1$.

We note that $a e_n a^* \neq 0$; for otherwise $r_n^2 e_n = a^* a e_n a^* a = 0$. Put $f_n = k_n^{-2} a e_n a^*$. Since $(0) \neq f_n A \subset a e_n A$ and $a e_n \neq 0$, it follows from [7, p. 45, Lemma (2.1.8)] that $f_n A = a e_n A$ is a minimal right ideal of A . Hence we see that $\{f_n\}$ is an orthogonal family of hermitian minimal idempotents in A . By (2.3), $aa^* = \sum_n a e_n a^* = \sum_n k_n^2 f_n$ and so it is a spectral representation of aa^* by the proof of Lemma 2.3. For each x in A , by a similar argument in (2.5), we have

$$(2.7) \quad \left\| \sum_{i=m}^n k_n^{-1} e_n a^* x \right\|^2 = \left\| \sum_{i=m}^n f_i x \right\|^2 \leq \|x\|^2.$$

Since $\sum_n f_n x$ is summable, so is $\sum_n k_n^{-1} e_n a^* x$. Therefore we can define a mapping W^* on A by

$$(2.8) \quad W^*x = \sum_n k_n^{-1} e_n a^* x \quad (x \in A).$$

It follows easily from (2.4) and (2.7) that $W^*a = [a]$ and $\|W^*\| = 1$. Also both W and W^* are right centralizers on A . We shall refer to the operator W as the partial isometry associated with a .

We remark that similar concepts were introduced in [9].

3. The p -class in A . As before, A will be a dual B^* -algebra with norm $\|\cdot\|$. Suppose a is a nonzero element in A . Let $a^*a = \sum_n r_n e_n$ be a spectral representation of a^*a and $k_n = \sqrt{r_n}$. Since a^*a is a positive element in A , $r_n > 0$ and so $k_n > 0$. We define

$$(3.1) \quad |a|_p = \left(\sum_n k_n^p \right)^{1/p} \quad (0 < p < \infty),$$

$$|a|_\infty = \max \{k_n : n = 1, 2, \dots\}.$$

For $a = 0$, we define $|a|_p = 0$ ($0 < p \leq \infty$).

REMARK. By Lemma 2.3, $|a|_p$ is well defined.

DEFINITION. For $0 < p \leq \infty$, let $A_p = \{a \in A : |a|_p < \infty\}$.

REMARK. For $0 < p \leq \infty$, $|a|_p \geq 0$ and $|a|_p = 0$ if and only if $a = 0$. Also $|ka|_p = |k| |a|_p$ for any constant k .

We now have some elementary properties of $|a|_p$.

LEMMA 3.1. *Let a be an element in A and $0 < p \leq \infty$. Then*

- (i) $\|a\| = |a|_\infty \leq |a|_p$. Thus $A_\infty = A$.
- (ii) $|a|_p = |[a]|_p$. Hence $a \in A_p$ if and only if $[a] \in A_p$.
- (iii) If $p \leq q$, then $|a|_p \geq |a|_q$ and so $A_p \subset A_q$.
- (iv) If e is a hermitian minimal idempotent in A , then $|e|_p = 1$ and so $e \in A_p$.
- (v) $|a|_p = |a^*|_p$. Hence $a \in A_p$ if and only if $a^* \in A_p$.

PROOF. Let $a^*a = \sum_n r_n e_n$ be a spectral representation of a^*a and $[a] = \sum_n k_n e_n$ with $k_n = \sqrt{r_n}$.

- (i) This follows from $\|a\|^2 = \|a^*a\|$ and [7, p. 112, Corollary (3.1.7)].
- (ii) This follows from $[a] = [[a]] = \sum_n k_n e_n$.
- (iii) and (iv). This is clear.
- (v) We can assume that $a \neq 0$. Put $f_n = k_n^{-2} a e_n a^*$. Then $aa^* = \sum_n k_n^2 f_n$ is a spectral representation of aa^* (see §2). Therefore it follows that $|a^*|_p = |a|_p$. This completes the proof of the lemma.

Let a be a positive element in A and B_0 the intersection of all maximal commutative $*$ -subalgebras of A containing a . If $\{f_\beta\}$ is the maximal orthogonal family of hermitian minimal idempotents in B_0 , then $a = \sum_\beta a f_\beta = \sum_\beta \lambda_\beta f_\beta$, where λ_β are nonnegative constants.

DEFINITION. For $0 < p < \infty$, we define $a^p = \sum_\beta \lambda_\beta^p f_\beta$.

REMARK. Let $a = \sum_\alpha k_\alpha e_\alpha = \sum_n k_n e_n$ be a spectral representation of a . If a^p exists, then by the proof of Lemma 2.3 $a^p = \sum_\alpha k_\alpha^p e_\alpha = \sum_n k_n^p e_n$ is a spectral representation of a^p .

LEMMA 3.2. *Let a be a positive element in A and $0 < p, q < \infty$. If a^q exists, then $|a^q|_{p/q} = |a|_p^q$.*

PROOF. This is clear.

LEMMA 3.3. Let $a \in A$ and $0 < p < \infty$. Then the following statements are equivalent:

- (i) $a \in A_p$.
- (ii) $[a]^p \in A_1$.
- (iii) $[a]^{p/2} \in A_2$.

If any of these conditions holds, then $|a|_p^p = \sum_{\beta} \|f_{\beta} [a]^p f_{\beta}\|$, where $\{f_{\beta}\}$ is a maximal orthogonal family of hermitian minimal idempotents in A .

PROOF. Let $[a] = \sum_{\alpha} k_{\alpha} e_{\alpha} = \sum_n k_n e_n$ be a spectral representation of $[a]$.

- (i) $\Leftrightarrow$ (ii) This follows from the equality $|a|_p^p = \sum_n k_n^p = |[a]^p|_1$.
- (ii) $\Leftrightarrow$ (iii) This follows from the equality $\|[a]^p\|_1 = \sum_n k_n^p = \|[a]^{p/2}\|_2^2$.

If any of these conditions holds, then by Lemma 2.2, we have

$$\begin{aligned} |a|_p^p &= \sum_{\alpha} k_{\alpha}^p = \sum_{\alpha} \|[a]^{p/2} e_{\alpha}\|^2 \\ &= \sum_{\beta} \|[a]^{p/2} f_{\beta}\|^2 = \sum_{\beta} \|f_{\beta} [a]^p f_{\beta}\|. \end{aligned}$$

This completes the proof.

LEMMA 3.4. Let a be a positive element in A and f a hermitian minimal idempotent in A . Then

- (i) $\|fa^p f\| \geq \|faf\|^p$ ($1 \leq p < \infty$).
- (ii) $\|fa^p f\| \leq \|faf\|^p$ ($0 < p \leq 1$).

PROOF. Let $a = \sum_{\alpha} k_{\alpha} e_{\alpha}$ be a spectral representation of a .

(i) Clearly we can assume that $1 < p < \infty$. Then by Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} \|faf\| &= \|a^{1/2} f\|^2 = \sum_{\alpha} \|e_{\alpha} a^{1/2} f\|^2 = \sum_{\alpha} k_{\alpha} \|e_{\alpha} f\|^2 \\ &\leq \left(\sum_{\alpha} k_{\alpha}^p \|e_{\alpha} f\|^2 \right)^{1/p} \left(\sum_{\alpha} \|e_{\alpha} f\|^2 \right)^{(p-1)/p} \\ &= \left(\sum_{\alpha} \|e_{\alpha} a^{p/2} f\|^2 \right)^{1/p} (\|f\|^2)^{(p-1)/p} = \|fa^p f\|^{1/p}. \end{aligned}$$

- (ii) Replacing a by a^p and p by $1/p$ in (i), we get (ii).

LEMMA 3.5. Let $a \in A_p$ and $\{f_{\beta}\}$ be a maximal orthogonal family of hermitian minimal idempotents in A . Then

- (i) $|a|_p^p \leq \sum_{\beta} \|af_{\beta}\|^p$ ($1 \leq p \leq 2$).
- (ii) $|a|_p^p \geq \sum_{\beta} \|af_{\beta}\|^p$ ($2 \leq p < \infty$).

If $[a] = \sum_n k_n e_n$ is a spectral representation of $[a]$, then $|a|_p^p = \sum_n \|ae_n\|^p$ ($0 < p < \infty$).

PROOF. (i) If $1 \leq p \leq 2$, then by Lemma 3.4(ii), we have $\|f_\beta[a]^p f_\beta\| \leq \|f_\beta[a]^2 f_\beta\|^{p/2} = \|af_\beta\|^p$. Therefore (i) follows now from Lemma 3.3.

(ii) This can be proved similarly.

If $[a] = \sum_n k_n e_n$, then $\|ae_n\| = \|e_n a^* a e_n\|^{1/2} = k_n$. Therefore $|a|_p^p = \sum_n \|ae_n\|^p$ ($0 < p < \infty$). This completes the proof.

LEMMA 3.6. Suppose $a, b \in A$ and $1 \leq p \leq \infty$, then the following statements hold:

- (i) If $a \in A_p$ and S is a right centralizer on A , then $Sa \in A_p$ and $|Sa|_p \leq \|S\| |a|_p$.
- (ii) If $a \in A_p$ and $b \in A$, then $|ab|_p \leq \|b\| |a|_p$ and $|ba|_p \leq \|b\| |a|_p$. Hence ab and ba are in A_p .
- (iii) If a, b are in A_p , then $|ab|_p \leq |a|_p |b|_p$.

PROOF. Clearly we can assume that $1 \leq p < \infty$.

(i) Suppose $1 \leq p \leq 2$. Let $[a] = \sum_\alpha k_\alpha e_\alpha$ be a spectral representation of $[a]$. Then by Lemma 3.5, we have

$$|Sa|_p^p \leq \sum_\alpha \|(Sa)e_\alpha\|^p \leq \|S\|^p \sum_\alpha \|ae_\alpha\|^p = \|S\|^p |a|_p^p.$$

If $2 \leq p < \infty$, let $[Sa] = \sum_\alpha k_\alpha e_\alpha$ be a spectral representation of $[SA]$. Then by a similar argument, we have $|Sa|_p \leq \|S\| |a|_p$. This proves (i).

(ii) This follows easily from (i) and Lemma 3.1(v).

(iii) This follows from (ii) and Lemma 3.1(i).

LEMMA 3.7. Let $a \in A_p$ and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A . Then

$$(3.2) \quad \sum_\beta \|f_\beta a f_\beta\|^p \leq |a|_p^p \quad (1 \leq p < \infty).$$

PROOF. Let W be the partial isometry associated with a and $b = W[a]^{1/2}$. Then $a = W[a] = b[a]^{1/2}$. It follows from Cauchy's inequality that

$$(3.3) \quad \sum_\beta \|f_\beta a f_\beta\|^p \leq \left(\sum_\beta \|f_\beta b\|^{2p} \right)^{1/2} \left(\sum_\beta \|[a]^{1/2} f_\beta\|^{2p} \right)^{1/2}.$$

By Lemma 3.3 and Lemma 3.4, we have

$$(3.4) \quad \sum_\beta \|[a]^{1/2} f_\beta\|^{2p} = \sum_\beta \|f_\beta[a]f_\beta\|^p \leq \sum_\beta \|f_\beta[a]^p f_\beta\| = |a|_p^p.$$

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have

$$(3.5) \quad \begin{aligned} \sum_\beta \|f_\beta b\|^{2p} &\leq \sum_\beta \|f_\beta(bb^*)^p f_\beta\| = |b^*|_{2p}^{2p} = |b|_{2p}^{2p} \\ &\leq \|[a]^{1/2}\|_{2p}^{2p} = |a|_p^p. \end{aligned}$$

Substituting (3.4) and (3.5) into (3.3), we get (3.2). This completes the proof.

In order that $|\cdot|_p$ be a norm on A_p ($1 \leq p \leq \infty$), it is sufficient now to show the triangle inequality.

LEMMA 3.8. *Let $a, b \in A_p$, then $|a + b|_p \leq |a|_p + |b|_p$ ($1 \leq p \leq \infty$). Hence $a + b \in A_p$.*

PROOF. We can assume $1 \leq p < \infty$. Write $[a + b] = \sum_{\alpha} k_{\alpha} e_{\alpha}$ and $[a + b] = W^*(a + b)$ (see (2.8)). Then by Lemma 3.5, Lemma 3.7 and Minkowski's inequality, we have

$$\begin{aligned} |a + b|_p &= \left(\sum_{\alpha} \|e_{\alpha}[a + b]e_{\alpha}\|^p \right)^{1/p} \\ &\leq \left(\sum_{\alpha} \|e_{\alpha}W^*ae_{\alpha}\|^p \right)^{1/p} + \left(\sum_{\alpha} \|e_{\alpha}W^*be_{\alpha}\|^p \right)^{1/p} \\ &\leq |W^*a|_p + |W^*b|_p \leq |a|_p + |b|_p. \end{aligned}$$

This completes the proof.

Now we have the main result of this section.

THEOREM 3.9. *For $1 \leq p \leq \infty$, A_p is a dual A^* -algebra which is a dense two-sided ideal of A .*

PROOF. By a similar argument in the proof of [4, p. 265, Corollary 3.2], we can show that A_p is complete. (We use maximal orthogonal families of hermitian minimal idempotents instead of orthonormal bases.) Hence A_p is an A^* -algebra which is a two-sided ideal of A . It follows from Lemma 3.1(iv) that A_p contains the socle S of A . Since S is dense in A , so is A_p . We claim that, for each a in A_p , a belongs to the closure of aA_p in A_p . In fact, let $[a] = \sum_{i=1}^{\infty} k_i e_i$ be a spectral representation of $[a]$ and W the partial isometry associated with a . Put $f_n = \sum_{i=1}^n e_i$ ($n = 1, 2, \dots$). Then

$$|a - af_n|_p \leq |[a] - [a]f_n|_p = \left| \sum_{i=n+1}^{\infty} k_i e_i \right|_p = \left(\sum_{i=n+1}^{\infty} k_i^p \right)^{1/p}.$$

Since $a \in A_p$, it follows that $|a - af_n|_p \rightarrow 0$ as $n \rightarrow \infty$. Hence by [5, p. 29, Lemma 8 (3)], A_p is a dual algebra. This completes the proof.

We shall need the following result.

COROLLARY 3.10. *Let $\{e_{\gamma}\}$ be any orthogonal family of hermitian minimal idempotents of A and $x \in A_p$ ($1 \leq p \leq \infty$), then $\sum_{\gamma} e_{\gamma}x$ is summable in $|\cdot|_p$ and especially when $\{e_{\gamma}\}$ is a maximal family $x = \sum_{\gamma} e_{\gamma}x$ in A_p .*

PROOF. This follows from Theorem 3.9 and Theorem 5.2 in [12].

Finally we remark that many statements and proofs in this section are similar to those given in [4] and [11].

4. The algebras A_1 and A_2 . We have a characterization of a proper H^* -algebra.

THEOREM 4.1. *The algebra A_2 is a proper H^* -algebra. Conversely, every proper H^* -algebra is of the form A_2 for some dual B^* -algebra A .*

PROOF. Let $a, b \in A_2$ and $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A . Then $f_\beta b^* a f_\beta = \lambda_\beta f_\beta$ for some constant λ_β . We claim that $\sum_\beta \lambda_\beta$ is absolutely summable and independent of the choice of $\{f_\beta\}$. In fact, let $x, y \in A f_\beta$. Then $y^* x = \langle x, y \rangle_\beta f_\beta$ for some constant $\langle x, y \rangle_\beta$. It follows from [7, p. 261, Theorem (4.10.3)] and [7, p. 263, Theorem (4.10.6)] that $\langle x, y \rangle_\beta$ defines a complete inner product on $A f_\beta$ such that $\langle x, x \rangle_\beta = \|x\|^2$. Now by Lemma 2.2 and the proof of [10, p. 30, Lemma 4], we can show that $\sum_\beta \lambda_\beta$ is absolutely summable and independent of $\{f_\beta\}$. Define

$$(4.1) \quad (a, b) = \sum_\beta \lambda_\beta \quad (a, b \in A_2).$$

Then by the proof of [10, p. 31, Lemma 5], $(,)$ is an inner product on A_2 such that $(xa, b) = (a, x^*b)$ and $(ax, b) = (a, bx^*)$ for all x in A . Also $|a|_2^2 = (a, a)$. Therefore A_2 is a proper H^* -algebra.

Conversely, let B be a proper H^* -algebra. Then B is a dense two-sided ideal of some dual B^* -algebra A . We can show that $B = A_2$ and this completes the proof.

LEMMA 4.2. *Let $1/p + 1/q = 1$, where $1 \leq p, q \leq \infty$. If $a \in A_p$ and $b \in A_q$, then $ab \in A_1$ and $|ab|_1 \leq |a|_p |b|_q$.*

PROOF. Suppose first that $2 \leq p < \infty$, $1 < q \leq 2$. Let $[b] = \sum_\alpha k_\alpha e_\alpha$ be a spectral representation of $[b]$. Also write $[ab] = W^*ab$. Then by Lemma 3.3, Lemma 3.5 and Hölder's inequality, we have

$$(4.2) \quad \begin{aligned} |ab|_1 &= \sum_\alpha \|e_\alpha [ab] e_\alpha\| = \sum_\alpha \|e_\alpha W^* a b e_\alpha\| \\ &\leq |W^* a|_p |b|_q \leq |a|_p |b|_q. \end{aligned}$$

By a similar argument, we can show that (4.2) holds for $1 < p \leq 2$, $2 \leq q < \infty$. We now identify A_1 .

THEOREM 4.3. $A_1 = \{xy : x, y \in A_2\}$.

PROOF. If $a \in A_1$, then by Lemma 3.3, $[a]^{1/2} \in A_2$. Let W be the

partial isometry associated with a . Then $a = W[a] = (W[a]^{1/2})([a]^{1/2}) \in \{xy : x, y \in A_2\}$. The converse follows from Lemma 4.2 and this completes the proof.

Let $a \in A_1$. Then by Theorem 4.3, $a = c^*b$ for some b, c in A_2 .

Define

$$(4.3) \quad \text{tr } a = (b, c) \quad (a \in A_1),$$

where (b, c) is given by (4.1).

LEMMA 4.4. Let $a \in A_1$, $\{f_\beta\}$ a maximal orthogonal family of hermitian minimal idempotents in A and $\lambda_\beta f_\beta = f_\beta a f_\beta$. Then $\text{tr } a$ is well defined, $\text{tr } a = \sum_\beta \lambda_\beta = \sum_\beta (a f_\beta, f_\beta)$ and $|\text{tr } a| \leq |a|_1$.

PROOF. By the proof of Theorem 4.1, $\sum_\beta \lambda_\beta$ is absolutely summable and independent of $\{f_\beta\}$. It is clear that $\text{tr } a = \sum_\beta \lambda_\beta = \sum_\beta (a f_\beta, f_\beta)$. Therefore $\text{tr } a$ is well defined. By Lemma 3.7, $|\text{tr } a| \leq \sum_\beta \|f_\beta a f_\beta\| \leq |a|_1$.

5. The uniform convexity of A_p ($1 < p < \infty$). For each a in A , we define a linear operator L_a on A_2 by

$$(5.1) \quad L_a(x) = ax \quad (x \in A_2).$$

Since $|ax|_2 \leq \|a\| |x|_2$, it follows that L_a is bounded on A_2 . Let $(,)$ be the given inner product on A_2 .

LEMMA 5.1. Let a be a positive element in A . Then L_a is positive and $L_a r = (L_a)^r$ ($0 < r < \infty$).

PROOF. This is clear.

We now establish [4, p. 260, Lemma 2.6] for A_p .

LEMMA 5.2. Let a and b be two positive elements in A and $0 < r < \infty$. If $(a+b)^r, a^r$ and b^r are in A_1 , then

$$(i) \quad \text{tr } (a+b)^r \leq \text{tr } a^r + \text{tr } b^r \quad (0 < r \leq 1).$$

$$(ii) \quad \text{tr } (a+b)^r \geq \text{tr } a^r + \text{tr } b^r \quad (1 \leq r < \infty).$$

PROOF. We assume first that $0 < r \leq 1$. Let $S = L_a$, $T = L_b$ and $U = L_{a+b}$. Then by the proof of [4, p. 260, Lemma 2.6], there exist operators C and D on A_2 such that

$$\|C\| \leq 1, \quad \|D\| \leq 1, \quad CU^{1/2} = S^{1/2}, \quad DU^{1/2} = T^{1/2},$$

$$U^r = U^{r/2} C^* C U^{r/2} + U^{r/2} D^* D U^{r/2}.$$

Let $\{f_\beta\}$ be a maximal orthogonal family of hermitian minimal idempotents in A . Then by Lemma 5.1, we have

$$\begin{aligned}
 \text{tr } (a+b)^r &= \sum_{\beta} ((a+b)^r f_{\beta}, f_{\beta}) = \sum_{\beta} (U^r f_{\beta}, f_{\beta}) \\
 (5.2) \quad &= \sum_{\beta} (CU^{r/2} f_{\beta}, CU^{r/2} f_{\beta}) + \sum_{\beta} (DU^{r/2} f_{\beta}, DU^{r/2} f_{\beta}).
 \end{aligned}$$

Since $C(a+b)^{r/2} \in A_2$ and $CU^{r/2} f_{\beta} = C(a+b)^{r/2} f_{\beta}$, it follows from (5.2) that

$$\begin{aligned}
 \text{tr } (a+b)^r &= |C(a+b)^{r/2}|_2^2 + |D(a+b)^{r/2}|_2^2 \\
 (5.3) \quad &= |(C(a+b)^{r/2})^*|_2^2 + |(D(a+b)^{r/2})^*|_2^2.
 \end{aligned}$$

Let $a = \sum_{\alpha} k_{\alpha} e_{\alpha}$ be a spectral representation of a . Since $(C(a+b)^{r/2})^* e_{\alpha} = (CU^{r/2})^* e_{\alpha} = (a+b)^{r/2} C^* e_{\alpha}$, it follows from [4, p. 252, Lemma 2.1] that

$$\begin{aligned}
 ((C(a+b)^{r/2})^* e_{\alpha}, (C(a+b)^{r/2})^* e_{\alpha}) &= ((a+b)^r C^* e_{\alpha}, C^* e_{\alpha}) \\
 &\leq ((a+b) C^* e_{\alpha}, C^* e_{\alpha})^r = (ae_{\alpha}, e_{\alpha})^r = k_{\alpha}^r = (a^r e_{\alpha}, e_{\alpha}).
 \end{aligned}$$

Therefore $|C(a+b)^{r/2}|_2^2 \leq \text{tr } a^r$. Similarly $|D(a+b)^{r/2}|_2^2 \leq \text{tr } b^r$. Hence by (5.3), we have $\text{tr } (a+b)^r \leq \text{tr } a^r + \text{tr } b^r$. The case $1 \leq r < \infty$ can be proved in a similar way and the proof is complete.

By using maximal orthogonal families of hermitian minimal idempotents and a similar argument in the proof of [4, p. 259, Lemma 2.5], we have:

LEMMA 5.3. *Let a be a positive element in A and b a hermitian element in A such that $a+b$ and $a-b$ are positive. Suppose $(a+b)^r$, $(a-b)^r$ and a^r are in A_1 . Then*

- (i) $\text{tr } (a+b)^r + \text{tr } (a-b)^r \leq \text{tr } a^r$ ($0 < r \leq 1$).
- (ii) $\text{tr } (a+b)^r + \text{tr } (a-b)^r \geq \text{tr } a^r$ ($1 \leq r < \infty$).

Now we have the following result.

THEOREM 5.4. *Let a and b be two elements in A_p and $1/p + 1/q = 1$. Then*

- (i) $2^{p-1}(|a|_p^p + |b|_p^p) \leq |a+b|_p^p + |a-b|_p^p \leq 2(|a|_p^p + |b|_p^p)$ ($0 < p \leq 2$)
- (ii) $|a+b|_p^q + |a-b|_p^q \leq 2(|a|_p^p + |b|_p^p)^{q/p}$ ($1 < p \leq 2$)
- (iii) $2(|a|_p^p + |b|_p^p) \leq |a+b|_p^p + |a-b|_p^p \leq 2^{p-1}(|a|_p^p + |b|_p^p)$ ($2 \leq p < \infty$)
- (iv) $2(|a|_p^p + |b|_p^p)^{q/p} \leq |a+b|_p^q + |a-b|_p^q$ ($2 \leq p < \infty$).

PROOF. This can be proved by using Lemma 5.2, Lemma 5.3 and the proof of [4, p. 261, Theorem 2.7]. We omit the details.

As observed in [4], we have:

COROLLARY 5.5. *For $1 < p < \infty$, A_p is uniformly convex and reflexive.*

6. The conjugate space of A_p . In this section, we always assume that $1 \leq p < \infty$ and $1/p + 1/q = 1$. Let A_p^* be the conjugate space of A_p . We shall show that $A_p = A_q^*$ in a natural way.

For each a in A_p ($1 \leq p < \infty$), we define

$$(6.1) \quad F_a(x) = \operatorname{tr} ax \quad (x \in A_q).$$

THEOREM 6.1. *For each a in A_p ($1 \leq p < \infty$), $F_a \in A_q^*$ and $\|F_a\| = |a|_p$.*

PROOF. By Lemma 4.2, F_a is well defined. It is clear that $F_a \in A_q^*$ and $\|F_a\| \leq |a|_p$. By a similar argument in the proof of [11, p. 786, Proposition 3.26], we can show that $\|F_a\| \geq |a|_p$. This completes the proof.

We now establish a converse of Theorem 6.1.

THEOREM 6.2. *For $1 \leq p < \infty$, every continuous linear functional F on A_q is of the form F_a for some a in A_p , where F_a is defined in (6.1).*

PROOF. We assume first that $p = 1$ and $F \in A_\infty^* = A^*$. Then it is clear that $F \in A_2^*$. Since A_2 is a Hilbert space, by the Riesz representation theorem, there exists some a in A_2 such that $F(x) = (x, a^*) = \operatorname{tr} ax$ for all x in A_2 . By the proof of Theorem 6.1, we can show that $a \in A_1$ and so $F = F_a$.

Now we consider the case $1 < p < \infty$ and assume $F \in A_q^*$. Then $F \in A_1^*$. Hence by the proof of [8, p. 103, Theorem 2], there exists a right centralizer S on A_2 such that

$$(6.2) \quad F(y) = \operatorname{tr} Sy \quad (y \in A_1).$$

Let $\{e_\alpha\}$ be a maximal orthogonal family of hermitian minimal idempotents in A and $\{E_\gamma\}$ the direct set of all finite sums $e_{\alpha_1} + e_{\alpha_2} + \cdots + e_{\alpha_n}$. Define F_γ on A_q by

$$(6.3) \quad F_\gamma(x) = F(E_\gamma x) \quad (x \in A_q).$$

Since $S(E_\gamma x) = (SE_\gamma)(E_\gamma x) = ((SE_\gamma)E_\gamma)x = (SE_\gamma)x$ and $E_\gamma x \in A_1$, by (6.2) and (6.3) we have

$$(6.4) \quad F_\gamma(x) = \operatorname{tr} S(E_\gamma x) = \operatorname{tr} (SE_\gamma)x \quad (x \in A_q).$$

Since $SE_\gamma = (SE_\gamma)E_\gamma \in A_p$, by (6.4) and Theorem 6.1, $|SE_\gamma|_p = \|F_\gamma\| \leq \|F\|$. Therefore $\{SE_\gamma\}$ is a bounded set in A_p . Since A_p is reflexive (Corollary 5.5), we can assume that $SE_\gamma \rightarrow a$ weakly for some a in A_p . Hence $ae_\alpha = Se_\alpha$ for all α . Therefore by (6.2), $F(e_\alpha x) = \operatorname{tr} ae_\alpha x$. For each x in A_q , by

Corollary 3.10, $x = \sum_{\alpha} e_{\alpha} x$ in $|\cdot|_q$. Hence it follows that $F(x) = \text{tr } ax$ ($x \in A_q$). This completes the proof.

REMARK. Some arguments in the above proof are similar to those in the proof of [3, p. 130, Theorem III. 12.2].

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