

THE SPACE OF CONJUGACY CLASSES OF A TOPOLOGICAL GROUP

BY

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ABSTRACT. The space $G^\#$ of conjugacy classes of a topological group G is the orbit space of the action of G on itself by inner automorphisms. For a class of connected and locally connected groups which includes all analytic $[Z]$ -groups, the universal covering space of $G^\#$ may be obtained as the space of conjugacy classes of a group which is locally isomorphic with G , and the Poincaré group of $G^\#$ is found to be isomorphic with that of G/G' , the commutator quotient group. In particular, it is shown that the space $G^\#$ of a compact analytic group G is simply connected if and only if G is semisimple. The proof of this fact has not appeared in the literature, even though more specialized methods are available for this case.

I. Definitions and elementary properties. Two elements x, y of a topological group G are called *conjugate*, and we write $x \approx y$, if there is an element $t \in G$ such that $y = txt^{-1}$. The equivalence class of a point x under this relation is called the *conjugacy class* of x , denoted I_x . A subset of G which is a union of conjugacy classes is invariant under inner automorphisms and will be said to be *invariant*.

If G acts on itself by inner automorphisms, the inner automorphisms determined by the center $Z(G)$ of G are trivial and $G/Z(G)$ acts effectively on G . The orbit space under the action of G or $G/Z(G)$ is called the *space of conjugacy classes* of G , denoted $G^\#$. If G is the direct product of groups G_i , then $G^\#$ is homeomorphic with the Cartesian product of the spaces $G_i^\#$ (see [5, p. 130]).

The space $G^\#$ of a compact analytic group G is homeomorphic with the orbit space T/W of the action of the Weyl group W on a maximal toroid T of G [1, p. 95]. If G is semisimple, $G^\#$ may be obtained by identifying certain boundary points of a compact convex polyhedron in the Lie algebra of T (see [2, Example 6]). Some elementary proofs and [11, p. 231] give the following:

LEMMA 1. *If G is a compact analytic group, then $G^\#$ is compact, Hausdorff, second countable, and locally arcwise simply connected.*

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The natural map $p: G \rightarrow G^\#$ may not be closed if G is not discrete, and $G/Z(G)$ is not compact. From [4, p. 303], we have the following:

PROPOSITION 2. *If G is a connected, locally compact group, then the following are equivalent:*

- (i) *the natural map $p: G \rightarrow G^\#$ is closed;*
- (ii) *each neighborhood of e contains an invariant neighborhood of e (the [SIN] property);*
- (iii) *G is the direct product of a compact group and a vector group.*

EXAMPLE 1. Let H be the subgroup of $SL(3, R)$ consisting of matrices of the form

$$M(r, s, t) = \begin{pmatrix} 1 & r & . \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily checked that

$$M(a, b, c)M(r, s, t)(M(a, b, c))^{-1} = M(r, s, t + as - br)$$

and that

$$M(a, b, c)M(r, s, t)(M(a, b, c))^{-1}(M(r, s, t))^{-1} = M(0, 0, as - br).$$

An element of the form $M(0, 0, t)$ is central, and $D = \{M(0, 0, n): n \in Z\}$ is a discrete central subgroup. The conjugacy class of a noncentral element $M(r, s, t)$ is $\{M(r, s, w): w \in R\}$. In particular, $\{M(1/n, 0, w): w \in R\}$ is a conjugacy class for each $n \in Z^+$. Hence, H is not an [SIN] group. The quotient group H/D has compact conjugacy classes and is not an [SIN] group.

EXAMPLE 2. In the group H of Example 1, consider the subgroup $G = \{M(m, n, t); m, n \in Z; t \in R\}$. The conjugacy class of a noncentral element $M(m, n, t)$ is $\{M(m, n, t + kd): k \in Z; d \text{ the greatest common divisor of } m \text{ and } n\}$. The space $G^\#$ is normal, because each component of $G^\#$ is homeomorphic with R/dZ for some $d \in Z$. The component of e is exactly the center, so that G is an [SIN] group which is not the direct product of a vector group and a compact group.

For connectedness, we have

PROPOSITION 3. *Suppose that p is a closed map or that G is locally connected or that each conjugacy class is connected. Then $G^\#$ is connected if and only if G is connected.*

PROOF (of the nontrivial implication). Let C be the (invariant) component subgroup of G . If $C \neq G$, there is an open and closed set E which does not meet C . If p is closed, then $p(E)$ is an open and closed set which does not meet $p(C)$.

In the other two cases, consider the space $(G/C)^\#$ and the diagram:

$$\begin{array}{ccccc} G & \longrightarrow & G/C & \longrightarrow & (G/C)^\# \\ & \searrow & & \nearrow & \\ & & G^\# & & \end{array}$$

If G is locally connected, then G/C and $(G/C)^\#$ are discrete. If each conjugacy class is connected, then $G/C = (G/C)^\#$ is totally disconnected [8, p. 60]. But in either case, $(G/C)^\#$ is connected, hence, trivial. Thus, $G = C$.

Clearly, if $x, y \in G$ and $z \in Z(G)$, then $x \approx y$ if and only if $zx \approx zy$. This suggests that we define an action of $Z(G)$ on $G^\#$ by •

$$(*) \quad zI_x = I_{zx}.$$

This action of $Z(G)$ on $G^\#$ constitutes a transformation group, in the sense of [11], except that $G^\#$ may not be a Hausdorff space.

LEMMA 4. If D is a closed subgroup of $Z(G)$, then the orbit space $G^\#|D$ is homeomorphic with $(G/D)^\#$.

PROOF. Consider the diagram:

$$\begin{array}{ccccc} & & G/D & \longrightarrow & (G/D)^\# \\ & \nearrow & & & \downarrow \\ G & & & & \\ & \searrow & & & \\ & & G^\# & \longrightarrow & G^\#|D \end{array}$$

II. Stability subgroups for the action of $Z(G)$ on $G^\#$. The stability subgroups for the action $(*)$ are conveniently described in terms of the sets $I_x I_x^{-1}$. For each $x \in G$, the set $I_x I_x^{-1}$ is invariant and inversion-invariant and $e \in I_x I_x^{-1} \subset (G, G)$, the algebraic commutator subgroup. The following theorem, which was proved by Gotô in [6], will be used to show that the main result of this paper (Theorem 16) holds for analytic $[Z]$ -groups:

THEOREM 5 (GOTÔ). If G is a compact semisimple analytic group, then there is an element $x \in G$ such that $I_x I_x^{-1} = G$.

In a more general situation, we have the following relationship between the algebraic and topological structure of the conjugacy classes:

PROPOSITION 6. Suppose that the set $I_x I_x^{-1}$ is locally compact in its relative topology. Then $\bar{I}_x \subset I_x I_x^{-1} I_x \subset x(G, G)$. Moreover, if the set $I_x I_x^{-1}$ is

closed under the group operation, then it is a closed invariant subgroup of G contained in (G, G) .

PROOF. The second part follows from [8, p. 35], for then $I_x I_x^{-1}$ is a locally compact subgroup.

For the first part, let U, V be neighborhoods of e with $\bar{U} \cap I_x I_x^{-1}$ compact and $V^2 \subset U$. Let $w \in \bar{I}_x$, $z \in I_x^{-1} \cap w^{-1}V$, and let $\{w_i\}$ be a net in I_x converging to w . Then, eventually,

$$w_i z \in V w w^{-1} V \cap I_x I_x^{-1} = V^2 \cap I_x I_x^{-1} \subset U \cap I_x I_x^{-1}.$$

Thus, wz is in the closed set $\bar{U} \cap I_x I_x^{-1}$ and $w = wzz^{-1} \in I_x I_x^{-1} I_x$.

EXAMPLE 3. If G is the affine group, $\{(\begin{smallmatrix} r & s \\ 0 & 1 \end{smallmatrix}) : r \in R^+, s \in R\}$, and $x = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$, then $e \in \bar{I}_x$, and $I_x I_x^{-1} = I_x \cup I_x^{-1} \cup \{e\} = (G, G)$ (see [8, p. 350]).

We now identify the stability subgroups for the action $(*)$.

LEMMA 7. If D is a closed subgroup of $Z(G)$, then the set $D_x = D \cap I_x I_x^{-1} = D \cap x I_x^{-1}$ is the stability subgroup in D of $I_x \in G^\#$.

PROOF. An element $d \in D$ is in the stability subgroup if and only if it translates some (and hence, every) conjugate of x to another conjugate of x . Then, for some $s, t \in G$, $d = sxs^{-1}tx^{-1}t^{-1} = s^{-1}ds = xs^{-1}tx^{-1}t^{-1}s$.

These stability subgroups are related to the zeros of characters of finite-dimensional irreducible representations:

COROLLARY 8. Let π be a finite-dimensional irreducible representation of G and let $x \in G$. If $\text{trace}(\pi(x)) \neq 0$, then $Z(G) \cap I_x I_x^{-1} \subset \text{kernel}(\pi)$. If moreover, π is faithful, then the stability subgroup of $I_x \in G^\#$ under $(*)$ is trivial.

PROOF. Let $z \in Z(G) \cap I_x I_x^{-1}$, then Schur's lemma shows that $\text{trace}(\pi(x)) = \text{trace}(\pi(zx)) = \text{trace}(\pi(x))\text{trace}(\pi(z))/\text{trace}(\pi(e))$.

COROLLARY 9. If G is a compact semisimple analytic group and $x \in G$ is a regular point, that is, a point whose centralizer has minimum dimension, then D_x is isomorphic with a subgroup of the Weyl group W of G .

PROOF. There is a maximal toroid T which contains x (and $Z(G)$) (see [1]) and for each $d \in D_x$ there is exactly one $nT \in W$ such that $dx = nxn^{-1}$. This correspondence effects an isomorphism between D_x and an Abelian subgroup of W .

EXAMPLE 4. If $G = SU(2)$, there is only one conjugacy class with a non-trivial stability subgroup, that of $(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix})$. The Weyl group is of order two.

III. The Poincaré group of $G^\#$. The relationship between the structures $G^\#$ and $G^a = G/G'$, where G' is the closed commutator subgroup of G , is a consequence of the fact that the natural map from G to G^a factors through $G^\#$ (see [8, p. 358]).

LEMMA 10. *The map $q: G^\# \rightarrow G^a$ defined by $q(I_x) = xG'$ is continuous, open and surjective.*

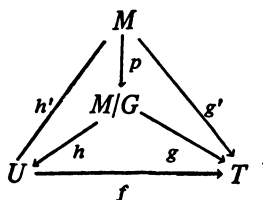
A connected and locally connected space S will be said to be *simply connected* if, for each covering space (U, f) of a space T and continuous map $g: S \rightarrow T$, there is a unique continuous map $h: S \rightarrow U$ such that $f \circ h = g$ and $h(s) = u$, where s and u are prescribed points such that $f(u) = g(s)$.

This is the definition used by Hochschild [10], and is equivalent to that used by Chevalley [3], except that we do not require the Hausdorff property. The stability of this lifting property under two types of maps which appear leads to sufficient conditions for the spaces $G^\#$ and G^a to be simply connected. For analytic $[Z]$ -groups, we show that these spaces are *arcwise* simply connected if they are simply connected.

A space is said to be locally simply connected if each point has a simply connected neighborhood. A connected space has a simply connected covering space if and only if it is locally simply connected (see [10] and [3]).

LEMMA 11. *Let G be a group which acts on a simply connected space M with a fixed point m . Then the orbit space M/G is simply connected.*

PROOF. Consider the commutative diagram



where f is a covering, g is continuous, $g' = g \circ p$ and h' is a specified lift of g' . To show that there is a map h as indicated, we show that h' is constant on G -orbits. Since h' is the unique map taking m to $h'(m)$ and satisfying $f \circ h' = g'$, precomposition of h' with an action of G does not alter h' , that is h' is constant on G -orbits.

We can now give some sufficient conditions for the spaces $G^\#$ and G^a to be simply connected:

PROPOSITION 12. *If G is locally connected and $G^\#$ is simply connected, then $G^a = G/G'$ is simply connected.*

PROOF. By Proposition 3, G is connected. Hence, the G' -cosets are connected [9, p. 142]. Use [10, p. 56], and Lemma 10.

PROPOSITION 13. *If G is simply connected, then $G^\#$ is simply connected.*

PROOF. The stability subgroup of e for the action of G on itself by inner automorphisms is G . Use Lemma 11.

PROPOSITION 14. *Let D be a discrete subgroup of $Z(G)$ and let D_x be the stability subgroup of $I_x \in G^\#$ under $(*)$. If $G^\#$ is simply connected, then $(G/D_x)^\#$ is simply connected.*

PROOF. The subgroup D_x is closed (we have not assumed any separation properties for $G^\#$). Use Lemmas 11 and 4.

PROPOSITION 15. *Let D be a discrete subgroup of $Z(G)$ which is generated by the stability subgroups D_x under $(*)$. If $G^\#$ is simply connected, then $(G/D)^\#$ is simply connected.*

PROOF. Partially order by inclusion the collection of subgroups D^* of D such that the orbit space $G^\#/D^*$ is simply connected, and use Zorn's lemma. The uniqueness of lifts in the definition of "simply connected" implies that the union of the elements of a chain is an upper bound for the chain, and D is the only possible maximal element because of Proposition 14.

EXAMPLE 5. In Example 1, the group H is the universal covering group of H/D and $(H/D)^\#$ is simply connected.

We are now ready to prove the main result.

THEOREM 16. *Let G be a connected and locally simply connected group with universal covering group $\tilde{G}$. If D is a discrete subgroup of $Z(\tilde{G})$ such that $G \cong \tilde{G}/D$ and $D \cap (\tilde{G})'$ is generated by the stability groups D_x under $(*)$, and $D(\tilde{G})'/(\tilde{G})'$ is discrete in $(\tilde{G})^a = \tilde{G}/(\tilde{G})'$, then the Poincaré groups of $G^\#$ and G^a are isomorphic with $D/(D \cap (\tilde{G})')$.*

PROOF. Let $D_1 = D \cap (\tilde{G})'$ and let $f_1: G \rightarrow G/D_1$ and $f_2: G/D_1 \rightarrow G/D$ be the natural covering maps. Since $D(\tilde{G})'/(\tilde{G})'$ is closed in $(\tilde{G})^a$, $f_1((\tilde{G})') = (\tilde{G}/D_1)'$ and $f_2(f_1(D(\tilde{G})')) = (\tilde{G}/D)'$. Thus, we have the diagram

$$\begin{array}{ccccc}
 \tilde{G} & \longrightarrow & (\tilde{G})^\# & \longrightarrow & (\tilde{G})^a \\
 f_1 \downarrow & & \downarrow & & \downarrow f_1^a \\
 \tilde{G}/D_1 & \longrightarrow & (G/D_1)^\# & \longrightarrow & (\tilde{G}/D_1)^a \\
 f_2 \downarrow & & \downarrow f_2^\# & & \downarrow f_2^a \\
 \tilde{G}/D & \longrightarrow & (\tilde{G}/D)^\# & \longrightarrow & (\tilde{G}/D)^a
 \end{array}$$

where f_1^a is the topological isomorphism induced by $f_1, f_2^\#$ and f_2^a are induced by f_2 and other maps are as in Lemmas 4 and 10.

Propositions 13, 15, and 12 show that the spaces $(\tilde{G}/D_1)^\#$ and $(\tilde{G}/D_1)^a$ are simply connected. It remains to show that D/D_1 is a properly discontinuous group of homeomorphisms of these spaces [12, p. 87]. In $(\tilde{G}/D_1)^a$, $D/D_1 \cong (D(\tilde{G})'/D_1)/((\tilde{G})'/D_1)$, a discrete subgroup. The action $(*)$ of D/D_1 on $(\tilde{G}/D_1)^\#$ is also properly discontinuous, because the elements of a D/D_1 orbit are conjugacy classes lying in distinct $(\tilde{G}/D_1)'$ -cosets (in $\tilde{G}$, we have $(dI_x)I_x^{-1} \subset (\tilde{G})'$ only if $d \in I_x I_x^{-1}(\tilde{G})' = (\tilde{G})'$).

COROLLARY 17. *If G is an analytic $[Z]$ -group, the spaces $G^\#$ and G^a are locally arcwise simply connected and have isomorphic fundamental groups.*

PROOF. First of all, the group G is the direct product of a vector group and a compact group (Proposition 2), so we may assume that G is compact. Then $\tilde{G}$ is the direct product of a vector group and a simply connected compact semisimple analytic group H (see [10] and [13]). If D is a discrete subgroup of $Z(\tilde{G})$ such that $\tilde{G}/D = G$, then $DH/H = D(\tilde{G})'/(\tilde{G})'$ is discrete [10, p. 6] and $D \cap H = D \cap (\tilde{G})' = D_x$ for some $x \in H$ (Theorem 5). The result follows from [12, p. 88], because $H^\#$ is arcwise simply connected (see Proposition 13, Lemma 1, and [10]).

COROLLARY 18. *A compact analytic group is semisimple if and only if $G^\#$ is simply connected.*

PROOF. A compact analytic group G is semisimple if and only if the toroid G^a is trivial; use Corollary 17.

EXAMPLE 6. One maximal toroid of $G = SO(4)$ consists of matrices

$$M(\theta, \varphi) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \varphi & -\sin \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix}$$

and the nontrivial elements of the Weyl group are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$C = AB = BA = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

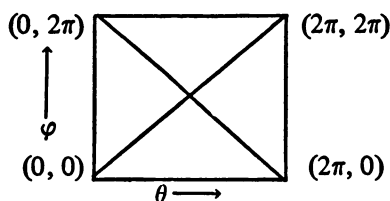
One checks easily that

$$A(M(\theta, \varphi))A^{-1} = M(2\pi - \theta, 2\pi - \varphi),$$

$$B(M(\theta, \varphi))B^{-1} = M(\varphi, \theta),$$

$$C(M(\theta, \varphi))C^{-1} = M(2\pi - \varphi, 2\pi - \theta),$$

so that each conjugacy class is represented by a matrix $M(\theta, \varphi)$ with $0 \leq \theta \leq \pi$ and $\theta \leq \varphi \leq 2\pi - \theta$. The space $G^\#$ may be realized as the small triangle on the left in the square



where pairs $M(0, \varphi)$, $M(0, 2\pi - \varphi)$ on the left-hand boundary must be identified. The space $G^\#$ is simply connected, as indicated by Corollary 18.

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