

INTERPOLATION POLYNOMIALS WHICH GIVE BEST ORDER OF APPROXIMATION AMONG CONTINUOUSLY DIFFERENTIABLE FUNCTIONS OF ARBITRARY FIXED ORDER ON $[-1, +1]$

BY

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Dedicated to Professor P. Turán

ABSTRACT. The object of this paper is to show that there exists a polynomial $P_n(x)$ of degree $\leq 2n - 1$ which interpolates a given function exactly at the zeros of n th Tchebycheff polynomial and for which $\|f - P_n\| \leq C_k w_k(1/n, f)$ where $w_k(1/n, f)$ is the modulus of continuity of f of k th order.

1. Introduction. The classical theorems of D. Jackson extend the Weierstrass approximation theorem by giving quantitative information on the degree of approximation to a continuous function in terms of its smoothness. Specifically, Jackson proved

THEOREM 1. *Let $f(x)$ be continuous for $|x| \leq 1$ and have modulus of continuity $w(t)$. Then there exists a polynomial $P(x)$ of degree n at most such that $|f(x) - P(x)| \leq Aw(1/n)$ for $|x| \leq 1$, where A is a positive numerical constant.*

In 1951 S. B. Stečkin [3] made an important generalization of the Jackson theorem.

THEOREM 2 (S. B. STEČKIN). *Let k be a positive integer; then there exists a positive constant C_k such that for every $f \in C[-1, +1]$ we can find an algebraic polynomial $P_n(x)$ of degree n so that*

$$(1.1) \quad \|f - P_n\| \leq C_k w_k(1/n, f),$$

where $w_k(1/n, f)$ is the modulus of continuity of $f(x)$ of k th order.

During personal conversation in Poznań, S. B. Stečkin raised the following

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question concerning Theorem 2: Does there exist an algebraic polynomial $P_n(x)$ of degree Cn ($C > 1$) interpolating at n points and which satisfies (1.1)? The case $k = 1$ was previously raised by P. L. Butzer [1] and solved consequently by G. Freud [2]. The object of this paper is to prove that an algebraic interpolatory polynomial satisfying (1.1) does exist. The approach we have adapted is to modify the classical Hermite-Fejér interpolation polynomials on the Tchebycheff nodes. Moreover the degree of the new interpolation process is still $2n - 1$. It may be interesting to point out that recently the author [5], [6] has solved the problem of obtaining a trigonometric polynomial which interpolates a given 2π periodic continuous function at $x_k = 2k\pi/n$, $k = 0, 1, \dots, n - 1$, and for which (1.1) is also true.

2. It is well known that the Hermite-Fejér interpolation polynomial of degree $\leq 2n - 1$ is defined by

$$(2.1) \quad H_n[f, x] = \sum_{k=1}^n f(x_{kn})(1 - xx_{kn}) \left(\frac{T_n(x)}{n(x - x_{kn})} \right)^2$$

where

$$(2.2) \quad x_{kn} = \cos((2k - 1)\pi/2n), \quad k = 1, 2, \dots, n,$$

are the zeros of Tchebycheff polynomial $T_n(x) = \cos(n \arccos x)$. Let us express $H_n[f, x]$ as a linear combination of $T_0(x)$, $T_1(x)$, $\dots$, $T_{2n-1}(x)$. For this purpose we define

$$(2.3) \quad C_0(f) = \frac{1}{n} \sum_{k=1}^n f(x_{kn}), \quad C_j(f) = \frac{2}{n} \sum_{k=1}^n f(x_{kn}) T_j(x_{kn})$$

for $j = 1, 2, \dots, 2n - 1$. A simple computation shows that

$$H_n[f, x] = \sum_{j=0}^{2n-1} C_j(f) \left(\frac{2n-j}{2n} \right) T_j(x).$$

This representation of $H_n[f, x]$ suggests the definition

$$(2.4) \quad R_n(f) = R_n[f, x] = \sum_{j=0}^{2n-1} C_j(f) \alpha_{j,M} T_j(x),$$

where M is an arbitrary fixed positive integer and

$$(2.5) \quad \alpha_{0,M} = 1, \quad \alpha_{j,M} + \alpha_{2n-j,M} = 1, \quad j = 1, 2, \dots, n, \quad \alpha_{j,M} = 0, \quad j > 2n.$$

A simple example of $\alpha_{j,M}$ satisfying (2.5) is given by

$$(2.6) \quad \begin{aligned} \alpha_{j,M} &= (2n-j)^M / ((2n-j)^M + j^M), & j = 0, 1, \dots, 2n-1, \\ &= 0, & j \geq 2n. \end{aligned}$$

For our purpose we make further restrictions on $\alpha_{j,M}$. We denote

$$(2.7) \quad \begin{aligned} \mu_{j,M} &= (1 - \alpha_{j,M})/j^M, & j = 1, 2, \dots, 2n, \\ &= 0, & j = 0. \end{aligned}$$

Let us suppose that

$$(2.8) \quad |\mu_{j+1,M} - \mu_{j,M}| = O(1/n^{M+1}), \quad j = 1, \dots, 2n-1,$$

and

$$(2.9) \quad |\mu_{j+1,M} - 2\mu_{j,M} + \mu_{j-1,M}| = O(1/n^{M+2}), \quad j = 1, \dots, 2n-1.$$

We also require that

$$(2.10) \quad 1 - \alpha_{1,M} = O(1/n^M),$$

$$(2.11) \quad |\alpha_{j+1,M} - 2\alpha_{j,M} + \alpha_{j-1,M}| = O(1/n^2), \quad j = 1, \dots, 2n-1.$$

We now state our main theorem.

THEOREM 3. *Let $f(x)$ be continuous for $|x| \leq 1$. Then $R_n(f)$ as defined by (2.4) and (2.5) satisfy*

$$(2.12) \quad R_n[f, x_{in}] = f(x_{in}), \quad i = 1, 2, \dots, n,$$

and

$$(2.13) \quad R_n[1, x] = 1.$$

Moreover under the assumptions (2.7)–(2.11) we have ($f \neq$ polynomial of degree $\leq m-1$)

$$(2.14) \quad \|R_n(f) - f\| \leq C_M w_{M-1}(1/n, f).$$

It is easy to verify that the choice of $\alpha_{j,M}$ given by (2.6) satisfies all the requirements needed in Theorem 3.

PROOF OF THEOREM 3. First we will prove that $R_n(f)$ as defined by (2.4) and (2.5) is an interpolation polynomial in x of degree $\leq 2n-1$ satisfying (2.12). For this purpose we express

$$(3.1) \quad R_n[f, x] = \sum_{j=0}^{n-1} C_j(f) \alpha_{j,M} T_j(x) + \sum_{j=n+1}^{2n-1} C_j(f) \alpha_{j,M} T_j(x).$$

In view of the fact that

$$(3.2) \quad T_{2n-j}(x_{in}) = -T_j(x_{in}), \quad C_{2n-j}(f) = -C_j(f), \quad j = 1, \dots, n,$$

we obtain, on using (2.5) and (3.1),

$$R_n[f, x_{in}] = C_0(f) + \sum_{j=1}^{n-1} C_j(f) T_j(x_{in}).$$

From the definition of $C_j(f)$ as given in (2.3) it follows that

$$C_0(f) + \sum_{j=1}^{n-1} C_j(f) T_j(x_{in}) = f(x_{in}), \quad i = 1, \dots, n.$$

Therefore we obtain $R_n[f, x_{in}] = f(x_{in})$, $i = 1, \dots, n$. This proves (2.12).

(2.13) is an immediate consequence of (2.3). For if $f(x) \equiv 1$ then $C_0(f) = 1$, $C_j(f) = 0$, $j = 1, \dots, 2n-1$.

Next we hope to prove the existence of a positive constant L independent of n and x such that

$$(3.3) \quad \|R_n[f]\| \leq L\|f\|.$$

To prove this we need some preliminary notation and estimates. We denote the Fejér kernel by

$$(3.4a) \quad \begin{aligned} t_j(\theta) &= 1 + \frac{2}{j} \sum_{i=1}^{j-1} (j-i) \cos i\theta, \quad j = 2, 3, \dots, \\ t_1(\theta) &\equiv 1. \end{aligned}$$

Associated with this kernel we introduce

$$(3.4) \quad \tau_{j,k}(\theta) = \frac{1}{2}(t_j(\theta + \theta_{kn}) + t_j(\theta - \theta_{kn})).$$

It is easy to verify that

$$(3.5) \quad (j+1)\tau_{j+1,k}(\theta) - 2j\tau_{j,k}(\theta) + (j-1)\tau_{j-1,k}(\theta) = 2 \cos j\theta \cos j\theta_{kn},$$

and

$$(3.6) \quad \sum_{k=1}^n |\tau_{j,k}(\theta)| = n, \quad j = 1, 2, \dots.$$

From (2.3) and (2.4) it follows that

$$(3.7) \quad R_n[f, x] = \sum_{k=1}^n f(x_{kn}) P_{kn}(x),$$

where

$$(3.8) \quad P_{kn}(x) = \frac{1}{n} \left[1 + 2 \sum_{j=1}^{2n-1} \alpha_{j,M} T_j(x_{kn}) T_j(x) \right].$$

Now we prove that

$$(3.9) \quad \sum_{k=1}^n |P_{kn}(x)| \leq L,$$

from which (3.3) follows on using (3.7). For this purpose we express (3.8) in terms of $\tau_{j,k}(\theta)$ as defined in (3.9). On using (3.5) we obtain

$$P_{kn}(x) = \frac{1}{n} \sum_{l=1}^{2n-1} (\alpha_{l+1,M} - 2\alpha_{l,M} + \alpha_{l-1,M}) \tau_{l,k}(\theta) + \tau_{2n,k}(\theta) \alpha_{2n-1,M}.$$

On using (2.10), (2.11) and (3.6), (3.9) follows immediately. This proves (3.9) and in turn (3.3). By proving (3.3) it follows easily that $R_n[f, x]$ converges uniformly to $f(x)$ on $[-1, +1]$ for every continuous function on $[-1, +1]$. Here our aim is to obtain error estimates (2.14). The proof of (2.14) is based on following [4]

THEOREM 4 [S. B. STEČKIN]. *Let P be a natural number and U_n ($n = 1, 2, \dots$) be a linear method of approximation of functions having the following properties:*

- (i) *for any function $\phi(\theta) \in C_{2\pi}$, $\|U_n(\phi)\| \leq L_0 \|\phi\|$;*
- (ii) *for any function $\phi(\theta) \in C_{2\pi}$ for which $\phi^{(P)}(\theta) \in C_{2\pi}$, $\|\phi - U_n(\phi)\| \leq L_P \|\phi^{(P)}\|/n^P$, $n = 1, 2, \dots$. Then for any function $\phi(\theta) \in C_{2\pi}$ we have $\|\phi - U_n(\phi)\| \leq B_P(L_0 + L_P)w_P(1/n, \phi)$.*

First let us choose U_n to be typical means of fourier series given by

$$(3.10) \quad X_{2n,M}(\phi, \theta) = \frac{1}{2}a_0 + \sum_{j=1}^{2n-1} (a_j \cos j\theta + b_j \sin j\theta)(1 - j^M/(2n)^M),$$

where a_j 's, b_j 's are fourier coefficients of $\phi(\theta)$. From a theorem of A. Zygmund [7] it follows that conditions (i) and (ii) are satisfied for $P = M - 1$ ($M > 1$) and, therefore, we conclude from Stečkin's theorem that

$$(3.11) \quad |X_{2n,M}(\phi, \theta) - \phi(\theta)| \leq C_M w_{M-1}(1/n, \phi).$$

Theorem 4 and (3.11) described above lead to a simple proof of (2.14). The representation of $R_n[f, x]$ as given by (3.7) suggests that we consider a trigonometric polynomial

$$(3.12) \quad A_n[\phi, \theta] = \sum_{k=1}^n \phi(\theta_{kn}) \left\{ \frac{1}{n} + \frac{2}{n} \sum_{j=1}^{2n-1} \alpha_{j,M} \cos j(\theta - \theta_{kn}) \right\}$$

where

$$(3.13) \quad \phi(\theta) = f(\cos \theta) \equiv f(x).$$

From (3.12) it follows that

$$(3.14) \quad A_n[1, \theta] \equiv 1,$$

$$(3.15) \quad A_n[\cos i\theta, \theta] - \cos i\theta = -\alpha_{2n-i}(\cos i\theta + \cos(2n-i)\theta),$$

for $i = 1, 2, \dots, 2n-1$. From (3.10); (3.12)–(3.15) it follows that

$$\begin{aligned} A_n[X_{2n,M}(\phi, \theta) - X_{2n,M}(\theta)] &= -(1 + \cos 2n\theta) \sum_{i=1}^{2n-1} a_i \cos i\theta \alpha_{2n-i}(1 - i^M/(2n)^M) \\ &\quad + \sin 2n\theta \sum_{i=1}^{2n-1} a_i \sin i\theta \alpha_{2n-i}(1 - i^M/(2n)^M). \end{aligned}$$

Since $\phi(\theta)$ is an even function of θ , its fourier coefficients b_i are all zero. Therefore we obtain

$$\begin{aligned} A_n[X_{2n,M}(t), \theta] - X_{2n,M}(\theta) \\ = -(1 + \cos 2n\theta) \sum_{i=1}^{2n-1} (a_i \cos i\theta - b_i \sin i\theta) \alpha_{2n-i} (1 - t^M / (2n)^M) \\ + \sin 2n\theta \sum_{i=1}^{2n-1} (b_i \cos i\theta - a_i \sin i\theta) \alpha_{2n-i} (1 - t^M / (2n)^M). \end{aligned}$$

On using an integral representation of fourier coefficients we obtain

$$\begin{aligned} (3.16) \quad X_{2n,M}(\theta) - A_n[X_{2n,M}(t), \theta] \\ = \frac{(1 + \cos 2n\theta)}{\pi} \sum_{i=1}^{2n-1} i \delta_{i,M} \int_0^{2\pi} \phi(u) \cos i(u - \theta) du \\ + \frac{\sin 2n\theta}{\pi} \sum_{i=1}^{2n-1} i \delta_{i,M} \int_0^{2\pi} \phi(u) \sin i(u - \theta) du, \end{aligned}$$

where

$$(3.17) \quad i \delta_{i,M} = \alpha_{2n-i,M} (1 - t^M / (2n)^M);$$

we put

$$(3.18) \quad F(\theta) = \frac{1}{\pi} \sum_{i=1}^{2n-1} \delta_{i,M} \int_0^{2\pi} \phi(u + \theta) \sin iu du,$$

and rewrite

$$(3.19) \quad X_{2n,M}(\theta) - A_n[X_{2n,M}(t), \theta] = (1 + \cos 2n\theta) F'(\theta) - \sin 2n\theta \tilde{F}'(\theta).$$

Now, we need to obtain estimates of $F'(\theta)$ and $\tilde{F}'(\theta)$. For this purpose we assume that $\phi(\theta)$ is $(M-1)$ times continuously differentiable of function of θ . Integrating by parts $(M-1)$ times we obtain after elementary calculation that

$$(3.20) \quad F(\theta) = \frac{(-1)^{M/2+1}}{\pi} \sum_{i=1}^{2n-1} \lambda_{i,M} \int_0^{2\pi} \phi^{(M-1)}(u + \theta) \cos iu du$$

for M even integer, where

$$(3.21) \quad i^{M-1} \lambda_{i,M} = \delta_{i,M}.$$

From (3.4a) we obtain

$$\begin{aligned} 2 \sum_{i=1}^{2n-1} \lambda_{i,M} \cos iu \\ = \sum_{i=1}^{2n-1} (\lambda_{i-1,M} - 2\lambda_{i,M} + \lambda_{i+1,M}) i t_i(u) + 2n \lambda_{2n-1,M} t_{2n}(u). \end{aligned}$$

Therefore on using (3.20) and (3.21) we obtain

$$(3.22) \quad F(\theta) = \frac{(-1)^{M/2+1}}{2\pi} \int_0^{2\pi} \phi^{(M-1)}(u + \theta) \times \left\{ \sum_{i=1}^{2n-1} (\lambda_{i,M} - 2\lambda_{i,M} + \lambda_{i+1,M}) it_i(u) + 2n\lambda_{2n-1} t_{2n}(u) \right\} du.$$

From (2.7), (3.17) and (3.21) it follows that

$$|\lambda_{i-1,M} - 2\lambda_{i,M} + \lambda_{i+1,M}| = O(1/n^{M+2}), \quad i = 1, \dots, 2n-1, \\ \lambda_{2n-1} = O(1/n^{M+1}).$$

From (3.4a) we obtain $\int_0^{2\pi} |t_i(u)| du = 2\pi$. On putting these estimates in (3.22) we obtain $|F(\theta)| \leq c_M \|\phi^{(M-1)}\|/n^M$. From (3.18) it follows that $F(\theta)$ is a trigonometric polynomial of order $\leq 2n$. On using a well-known theorem of S. N. Bernstein (see Zygmund [8, volume 1, p. 118]) we obtain

$$|F'(\theta)| \leq 2c_M \|\phi^{(M-1)}\|/n^{M-1}, \quad |\tilde{F}'(\theta)| \leq 2c_\phi \|\phi^{(M-1)}\|/n^{M-1}.$$

Therefore, under the assumption that $\phi^{(M-1)}(\theta) \in c_{2\pi}$ we obtain

$$(3.23) \quad |X_{2n,M}(\theta) - A_n[X_{2n}(t), \theta]| \leq 6c_M \|\phi^{(M-1)}\|/n^{M-1}.$$

Following a proof similar to that given for (3.3) it follows that for every $\phi \in c_{2\pi}$ we have

$$(3.24) \quad |A_n[\phi, \theta]| \leq B_M \|\phi\|.$$

Now we claim that for every $\phi^{(M-1)} \in c_{2\pi}$ we have

$$(3.25) \quad |\phi(\theta) - A_n[\phi, \theta]| \leq B_M \|\phi^{(M-1)}\|/n^{M-1}.$$

This follows from

$$\begin{aligned} \phi(\theta) - A_n[\phi, \theta] &= \phi(\theta) - X_{2n,M}(\theta) + X_{2n,M}(\theta) - A_n[X_{2n,M}(t), \theta] \\ &\quad + A_n[X_{2n,M}(t), \theta] - A_n[\phi, \theta], \end{aligned}$$

(3.23), (3.24) and (3.11). This proves (3.25).

(3.24) and (3.25) enable us to apply Stečkin Theorem 4 and we conclude that $|\phi(\theta) - A_n[\phi, \theta]| \leq B_M w_{M-1}(1/n, \phi)$.

But this inequality implies

$$|\phi(\theta) - \frac{1}{2}(A_n[\phi, \theta] + A_n[\phi, -\theta])| \leq B_M w_{M-1}(1/n, \phi).$$

Since $\phi(\theta) = f(\cos \theta) \equiv f(x)$ and $R_n[f, x] = \frac{1}{2}A_n[\phi, \theta] + A_n[\phi, -\theta]$, we obtain

$$(3.26) \quad |f(x) - R_n[f, x]| \leq B_M w_{M-1}(1/n, \phi).$$

It is well known that $w_{M-1}(1/n, \phi) \leq c_M w_{M-1}(1/n, f)$. We obtain from (3.26) that

$$|f(x) - R_n[f, x]| \leq e_M w_{M-1}(1/n, f).$$

This proves (2.14) and thus completes the proof of Theorem 3 for M even integer. For M -odd positive integer > 1 a similar proof can be given.

REFERENCES

1. P. L. Butzer, *On approximation theory* (Proc. Conf. Oberwolfach, 1963), Birkhäuser, Basel, 1964. MR 31 #2537.
2. G. Freud, *Über ein Jacksonsches Interpolationsverfahren*, On Approximation Theory (Proc. Conf. Oberwolfach, 1963), Birkhäuser, Basel, 1964, pp. 227–232. MR 32 #308.
3. S. B. Stečkin, *On the order of the best approximations of continuous functions*, Izv. Akad. Nauk SSSR Ser. Mat. 15 (1951), 219–242. (Russian) MR 13, 29.
4. ———, *The approximation of periodic functions by Fejér sums*, Trudy Mat. Inst. Steklov. 62 (1961), 48–60; English transl., Amer. Math. Soc. Transl. (2) 28 (1963), 269–282. MR 28 #5287a, b.
5. A. K. Varma, *The approximation of functions by certain trigonometric interpolation polynomials*, Proc. Internat. Sympos. Austin, Texas, edited by G. G. Lorentz, 1973, pp. 511–515.
6. ———, *Trigonometric interpolation polynomials which gives best order of approximation among continuously differentiable functions*, Proc. Internat. Sympos. Approximation Theory (Poznań, Poland, 1972) (to appear).
7. A. Zygmund, *The approximation of functions by typical means of their Fourier series*, Duke Math. J. 12 (1945), 695–704. MR 7, 435.
8. ———, *Trigonometric series*. Vols. 1, 2, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR 21, #6498.

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