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SPECTRAL ORDER PRESERVING MATRICES AND MUIRHEAD'S THEOREM

BY

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ABSTRACT. In this paper, a characterization is given for matrices which preserve the Hardy-Littlewood-Pólya spectral order relation $\prec$ for n-vectors in R^n. With this characterization, a new proof is given for the classical Muirhead theorem and some Muirhead-type inequalities are obtained. Moreover, sufficient conditions are also given for matrices which preserve the Hardy-Littlewood-Pólya weak spectral order relation $\prec\prec$.

Introduction. Spectral inequalities (i.e. expressions of the form $f \prec g$ or $f \prec\prec g$ where $\prec$ and $\prec\prec$ denote the Hardy-Littlewood-Pólya spectral order relations between measurable functions [2, §1]) are of considerable importance in various branches of analysis (see [1] through [7]) and also of great interest in themselves (see [5] for example). In this paper, we make a further investigation into spectral inequalities by introducing a new concept called spectral order preserving matrices. We characterize this concept and show how this characterization can be used to give a new proof of the classical Muirhead theorem. With this, we also obtain some Muirhead-type inequalities.

1. **Preliminaries.** In the sequel, we employ the following notation. If $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in R^n$ is any n-tuple of real numbers, we denote by $\mathbf{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ the n-tuple in R^n whose components are those of $\mathbf{x}$ arranged in decreasing order of magnitude, i.e., $x_1^* \geqslant x_2^* \geqslant \cdots \geqslant x_n^*$ and there exists a permutation π of the intergers $1, 2, \cdots, n$ such that $x^*_{\pi(i)} = x_i$, $1 \leqslant i \leqslant n$. If $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in R^n$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in R^n$, then we say that the *weak spectral inequality* $\mathbf{a} \prec\prec \mathbf{b}$ holds whenever

$$\sum_{i=1}^{k} a_i^* \leqslant \sum_{i=1}^{k} b_i^* \tag{1.1}$$

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for $1 \leq k \leq n$, and that the *strong spectral inequality* $\mathbf{a} \prec \mathbf{b}$ holds if $\mathbf{a} \prec\prec \mathbf{b}$ and if, in addition, there is equality in (1.1) for $k = n$. Moreover, we say that $\mathbf{a}$ and $\mathbf{b}$ are *spectrally equivalent* or $\mathbf{a}$ is a *rearrangement* of $\mathbf{b}$ (written $\mathbf{a} \sim \mathbf{b}$) whenever the components of $\mathbf{a}$ form a permutation of those of $\mathbf{b}$. Thus, $\mathbf{a} \sim \mathbf{b}$ if and only if $\mathbf{a}^* = \mathbf{b}^*$ or, equivalently, both $\mathbf{a} \prec \mathbf{b}$ and $\mathbf{b} \prec \mathbf{a}$ hold.

As in [2] and [5], the spectral inequality $\mathbf{a} \prec \mathbf{b}$ (respectively $\mathbf{a} \prec\prec \mathbf{b}$) is said to be *strictly strong* (respectively *strictly weak*) if $\mathbf{a} \not\sim \mathbf{b}$ (respectively if the inequality (1.1) is strict for $k = n$).

For any vectors $\mathbf{a}, \mathbf{b} \in R^n$, the strong spectral inequality

$$\mathbf{a} + \mathbf{b} \prec \mathbf{a}^* + \mathbf{b}^* \tag{1.2}$$

is easily seen to hold, by virtue of the fact that there exists a permutation π of the integers $1, 2, \cdots, n$ such that

$$a_{\pi(1)} + b_{\pi(1)} \geq a_{\pi(2)} + b_{\pi(2)} \geq \cdots \geq a_{\pi(n)} + b_{\pi(n)}$$

and that

$$\sum_{i=1}^{k} (a_{\pi(i)} + b_{\pi(i)}) \leq \sum_{i=1}^{k} (a_i^* + b_i^*)$$

for $1 \leq k \leq n$.

If $\mathbf{a} \prec \mathbf{c}$ and $\mathbf{b} \prec \mathbf{c}$, then it follows from (1.2) that

$$r\mathbf{a} + (1 - r)\mathbf{b} \prec \mathbf{c} \tag{1.3}$$

whenever $0 \leq r \leq 1$.

Let $\mathbf{a}_i, \mathbf{b}_i \in R^{n_i}, i = 1, 2, \cdots, m$, be m pairs of vectors such that $\mathbf{a}_i \prec \mathbf{b}_i$, $1 \leq i \leq m$. If $\mathbf{a}, \mathbf{b} \in R^{n_1+n_2+\dots+n_m}$ are vectors whose components are those of $\mathbf{a}_i, \mathbf{b}_i$, $1 \leq i \leq m$, respectively, then it is easily seen that $\mathbf{a} \prec \mathbf{b}$, i.e., $(a_{i1}, \cdots, a_{in_i}) \prec (b_{i1}, \cdots, b_{in_i})$, $1 \leq i \leq m$, implies

$$\begin{aligned}(a_{11}, \cdots, a_{1n_1}, \cdots, a_{m1}, \cdots, a_{mn_m})\\ \prec (b_{11}, \cdots, b_{1n_1}, \cdots, b_{m1}, \cdots, b_{mn_m}).\end{aligned} \tag{1.4}$$

Furthermore, if $\mathbf{a} = (a_1, \cdots, a_n)$ and $\mathbf{b} = (b_1, \cdots, b_n)$ are such that $(a_1, \cdots, a_k) \sim (b_1, \cdots, b_k)$ for some $1 \leq k < n$, then it is clear that $\mathbf{a} \sim \mathbf{b}$ if and only if

$$(a_{k+1}, \cdots, a_n) \sim (b_{k+1}, \cdots, b_n). \tag{1.5}$$

2. **Spectral order preserving matrices.** If P is an $n \times n$ permutation matrix, i.e., a matrix whose rows (or columns) form a rearrangement of those of an identity matrix, it is clear that $P\mathbf{a} \sim \mathbf{a}$ for all (column) vectors $\mathbf{a} \in R^n$. Observe that P also preserves spectral orders and spectral equivalency, i.e., $P\mathbf{a} \prec P\mathbf{b}, P\mathbf{a} \prec\prec P\mathbf{b}$ or $P\mathbf{a} \sim P\mathbf{b}$ whenever $\mathbf{a} \prec \mathbf{b}, \mathbf{a} \prec\prec \mathbf{b}$ or $\mathbf{a} \sim \mathbf{b}$ respectively. It is therefore natural to extend this notion to general $m \times n$ matrices by calling a matrix *strong spectral order preserving, weak spectral order preserving* or *spectral equivalency preserving* if it preserves the corresponding notions of spectral orders between vectors in R^n. Note that nontrivial spectral order preserving matrices exist, e.g., the matrix $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ preserves all spectral orders, i.e., $\prec$, $\prec\prec$ and $\sim$. If confusion does not arise, we shall simply call a strong spectral order preserving matrix spectral order preserving. It is clear that the product of two spectral order preserving matrices is again spectral order preserving.

PROPOSITION 2.1. *A positive* (*strong*) *spectral order preserving matrix is weak spectral order preserving.*

PROOF. Let A be a positive spectral order preserving $m \times n$ matrix. Let $\mathbf{a}, \mathbf{b} \in R^n$ be such that $\mathbf{a} \prec\prec \mathbf{b}$. Then, by Theorem 1.1 in [6], there exists $\mathbf{c} \in R^n$ such that $\mathbf{c} \geqslant 0$ and $\mathbf{a} + \mathbf{c} \prec \mathbf{b}$. Since A is positive spectral order preserving, we have $A\mathbf{a} \leqslant A(\mathbf{a} + \mathbf{c}) \prec A\mathbf{b}$ which implies that $A\mathbf{a} \prec\prec A\mathbf{b}$.

The following theorem shows that there are actually only two types of spectral order preserving matrices, rather than three.

THEOREM 2.2. *An $m \times n$ matrix is* (*strong*) *spectral order preserving if and only if it preserves spectral equivalency.*

PROOF. Clearly, the condition is necessary. To prove that it is also sufficient, let A be any $m \times n$ matrix which is spectral equivalency preserving, i.e., $A\mathbf{a} \sim A\mathbf{b}$ whenever $\mathbf{a} \sim \mathbf{b}$ where $\mathbf{a}, \mathbf{b} \in R^n$. Let $\mathbf{c}, \mathbf{d} \in R^n$ be such that $\mathbf{c} \prec \mathbf{d}$. We may assume $\mathbf{c} \not\sim \mathbf{d}$, otherwise, there is nothing to prove. Then, by Lemma 2 on p. 47 in [8], $\mathbf{c}$ can be derived from $\mathbf{d}$ by a finite number of transformations of the form $rI + (1 - r)P$, where $0 < r < 1$, I is the identity matrix and P is a permutation matrix. Hence we need only prove that $A\mathbf{c} \prec A\mathbf{d}$ whenever $\mathbf{c} = (rI + (1 - r)P)\mathbf{d}$ for some $0 < r < 1$ and for some permutation matrix P. But

$$A\mathbf{c} = rA\mathbf{d} + (1 - r)AP\mathbf{d} \prec A\mathbf{d}$$

by (1.3), since $P\mathbf{d} \sim \mathbf{d}$ implies $AP\mathbf{d} \sim A\mathbf{d}$.

Alternatively, by Rado's theorem [9, Theorem 1a, p. 321], there exist $\mathbf{d}_i \in R^n, i = 1, 2, \cdots, n$ such that $0 \leqslant r_i \leqslant 1, \mathbf{d}_i \sim \mathbf{d}, i = 1, 2, \cdots, n$ and

$\Sigma_{i=1}^n r_i = 1$ and $\mathbf{c} = \Sigma_{i=1}^n r_i \mathbf{d}_i$. Thus

$$A\mathbf{c} = \sum_{i=1}^{n} r_i A\mathbf{d}_i \prec A\mathbf{d}$$

by (1.3), since $A\mathbf{d}_i \sim A\mathbf{d}$, $i = 1, 2, \cdots, n$.

There is a general method by which we can generate all (strong) spectral order preserving matrices. Before coming to it, we give the following definition made specially for this purpose.

DEFINITION 2.3. Let $C = \{\mathbf{x}_i = (x_{i1}, \cdots, x_{in})\colon i = 1, 2, \cdots, m\}$ be any finite collection of not necessarily distinct vectors in R^n. Then a full permutation matrix generated by C is a matrix whose rows are all the distinct rearrangements of $\mathbf{x}$, $\mathbf{x} \in C$, i.e., whose rows consist of all the different permutations of $(x_{i1}, x_{i2}, \cdots, x_{in})$, $i = 1, 2, \cdots, m$, and possibly with each distinct rearrangement of $\mathbf{x}$ repeating itself as many times as $\mathbf{x}$ appearing in C.

LEMMA 2.4. *An $m \times n$ matrix A is spectral order preserving if, given any $n \times n$ permutation matrix P, there exists an $m \times m$ permutation matrix Q such that $QA = AP$, i.e. if any rearrangement of the columns is equivalent to some corresponding rearrangement of the rows.*

PROOF. Let $\mathbf{a} \sim \mathbf{b}$, where $\mathbf{a}, \mathbf{b} \in R^n$, then $\mathbf{a} = P\mathbf{b}$ for some $n \times n$ permutation matrix. Since there exists an $m \times m$ permutation matrix Q such that $QA = AP$, we have $A\mathbf{a} = AP\mathbf{b} = QA\mathbf{b} \sim A\mathbf{b}$.

LEMMA 2.5. *A full permutation matrix is (strong) spectral order preserving.*

PROOF. We need only prove the lemma for the case of a full permutation matrix generated by a single vector (in R^n), the general case then follows easily, by virtue of (1.4).

Let A be any full permutation matrix generated by a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. It is not hard to see that two distinct permutations of $\mathbf{x}$ remain distinct under the same transposition, i.e. only the elements in two fixed positions are interchanged. Regarding the transposition between any two columns of A as a mapping from the set of rows of A into itself, we see that this mapping is 1-1 and hence onto. Since any permutation is a product of transpositions, any rearrangement of columns of A consequently establishes a one-to-one and onto correspondence between the set of rows of A and itself. Thus a rearrangement of the columns of A corresponds to a rearrangement of its rows, and so, by Lemma 2.4, A is spectral equivalency preserving. Hence A is spectral order preserving, by Theorem 2.2.

LEMMA 2.6. *Let* $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ *and* $\mathbf{c} = (c_1, c_2, \cdots, c_n)$ *be vectors in* R^n *such that* $\mathbf{a} \sim \mathbf{b}$ *and* $\Sigma_{i=1}^n c_i^2 \leqslant \Sigma_{i=1}^n a_i^2$, *then*

$$\sum_{i=1}^n a_i c_i < \sum_{i=1}^n a_i^2$$

unless $\mathbf{a} = \mathbf{c}$ *and, in particular,*

$$\sum_{i=1}^n a_i b_i < \sum_{i=1}^n a_i^2$$

unless $\mathbf{a} = \mathbf{b}$.

PROOF. This is a direct consequence of Cauchy-Schwarz inequality [8, Theorem 7, p. 16].

THEOREM 2.7. *A matrix is (strong) spectral order preserving if and only if it is a full permutation matrix.*

PROOF. We need only prove the necessity of the condition since its sufficiency has been established in Lemma 2.5.

Let A be a spectral order preserving matrix. Suppose by contradiction that A is not a full permutation matrix. Let $\mathbf{a} = (a_1, \cdots, a_n)$ be a row vector of A whose components have the largest sum of squares, i.e. $\Sigma_{i=1}^n a_i^2$ is greatest. Let C be the collection of row vectors of A. Since A is not a full permutation matrix, C does not contain all rearrangements of its elements. In view of (1.5), Theorem 2.2 and Lemma 2.5, there is no loss in generality in assuming that C does not contain a rearrangement $\mathbf{b}$ of $\mathbf{a}$, i.e., $\mathbf{b} \notin C$ but $\mathbf{b} \sim \mathbf{a}$ and $\mathbf{b} \neq \mathbf{a}$. Let "superscript t" denote transpose (of a matrix); it then follows from Lemma 2.6 that $A\mathbf{a}^t$ contains a component (viz., $\Sigma_{i=1}^n a_i^2$) which is strictly greater than any of the components of $A\mathbf{b}^t$, and so $A\mathbf{a}^t \not\sim A\mathbf{b}^t$, showing that A is not spectral order preserving, by Theorem 2.2, since $\mathbf{a}^t \sim \mathbf{b}^t$.

COROLLARY 2.8. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are (column) vectors in* R^n, *then* $\mathbf{a} \prec \mathbf{b}$ *if and only if* $A\mathbf{a} \prec A\mathbf{b}$ *for all full permutation matrices* A.

PROOF. The condition is clearly sufficient, for the identity matrix is a full permutation matrix.

That the condition is necessary follows immediately from Theorem 2.7.

The following theorem is very useful in enabling us to discuss the case of equalities in inequalities arising in connection with spectral order preserving matrices.

THEOREM 2.9. *Let* A *be any* $m \times n$ *spectral order preserving matrix. If* $\mathbf{a}, \mathbf{b} \in R^n$ *are such that* $\mathbf{a} \prec \mathbf{b}$, *then* $A\mathbf{a} \sim A\mathbf{b}$ *if and only if either* $\mathbf{a} \sim \mathbf{b}$ *or all entries in each row of* A *are equal.*

PROOF. Clearly, the condition is sufficient, by Theorem 2.2. To prove that the condition is necessary, we first consider the case that $A\mathbf{a} = A\mathbf{b}$. Since A is spectral order preserving, A is a full permutation matrix generated by some collection C of vectors in R^n, by Theorem 2.7. If not all row vectors in A are constant, then there is a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in C$ with $x_i \neq x_j$, for some i, j such that $1 \leqslant i < j \leqslant n$. Since $A\mathbf{a} = A\mathbf{b}$, we have $x_1a_1 + \cdots + x_ia_i + \cdots + x_ja_j + \cdots + x_na_n = x_1b_1 + \cdots + x_ib_i + \cdots + + x_jb_j + \cdots + x_nb_n$ which also holds if we interchange x_i and x_j. Thus, it follows that

$$(x_i - x_j)((a_i - b_i) - (a_j - b_j)) = 0$$

or $a_i - b_i = a_j - b_j$. Similarly, it can be proved that $a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = c$, say, since there are rows of A which are permutations of $(x_1, \cdots, x_n)$. But $\mathbf{a} \prec \mathbf{b}$ and so $c = 0$ which implies that $\mathbf{a} = \mathbf{b}$.

Next, suppose $A\mathbf{a} \sim A\mathbf{b}$. If $\mathbf{a} \prec \mathbf{b}$ but $\mathbf{a} \not\sim \mathbf{b}$, then, by Lemma 2 on p. 47 of [8], there exists a finite sequence $\mathbf{c}_i$, $i = 0, 1, \cdots, n$, of vectors in R^n such that $\mathbf{a} = \mathbf{c}_0 \prec \mathbf{c}_1 \prec \cdots \prec \mathbf{c}_n = \mathbf{b}$ where $\mathbf{c}_{i-1} = r_i\mathbf{c}_i + (1 - r_i)\mathbf{c}_i'$, $0 < r_i < 1$, $\mathbf{c}_i' \sim \mathbf{c}_i$, $i = 1, 2, \cdots, n$. Since

$$A\mathbf{a} = A\mathbf{c}_0 \prec A\mathbf{c}_1 \prec \cdots \prec A\mathbf{c}_n = A\mathbf{b}$$

and $A\mathbf{a} \sim A\mathbf{b}$ imply $A\mathbf{a} = A\mathbf{c}_0 \sim A\mathbf{c}_1 \sim \cdots \sim A\mathbf{c}_n = A\mathbf{b}$, we need only consider the case that $\mathbf{a} = r\mathbf{b} + (1 - r)\mathbf{b}'$ where $0 < r < 1$ and $\mathbf{b}' \sim \mathbf{b}$. Now, $A\mathbf{a} = rA\mathbf{b} + (1 - r)A\mathbf{b}'$ and $A\mathbf{a} \sim A\mathbf{b} \sim A\mathbf{b}'$, so $A\mathbf{b} \sim rA\mathbf{b} + (1 - r)A\mathbf{b}' \sim A\mathbf{b}'$. By Proposition (2.15) on p. 22 in [7], we conclude that $A\mathbf{b} = A\mathbf{b}'$. But $A\mathbf{a} = rA\mathbf{b} + (1 - r)A\mathbf{b}'$ and so $A\mathbf{a} = A\mathbf{b}$ which, by the preceding paragraph, implies $\mathbf{a} = \mathbf{b}$, a contradiction. Hence $\mathbf{a} \sim \mathbf{b}$.

REMARK. The first part of the proof given for Theorem 2.9 clearly indicates that a spectral order preserving matrix with at least two entries unequal is 1-1 on (column) vectors whose sums of components are equal.

In view of Theorem 2.9, it is natural to give the following definition.

DEFINITION 2.10. A spectral order preserving matrix is said to be strict if at least two entries in one of its rows are unequal.

We can now restate Theorem 2.9 using the above definition.

THEOREM 2.11. *Let A be a strict spectral order preserving $m \times n$ matrix. If the spectral inequality $\mathbf{a} \prec \mathbf{b}$ is strictly strong, where $\mathbf{a}, \mathbf{b} \in R^n$, then the spectral inequality $A\mathbf{a} \prec A\mathbf{b}$ is also strictly strong.*

3. **Muirhead-type inequalities.** We can now show how to obtain the classical Muirhead's theorem from Theorem 2.7.

THEOREM 3.1 (MUIRHEAD). *If* $\mathbf{a}, \mathbf{b}, \mathbf{x} \in R^n$ *are such that* $\mathbf{a} \prec \mathbf{b}$ *and* $x_i > 0, i = 1, 2, \cdots, n,$ *then*

$$\sum! x_1^{a_1} \cdots x_n^{a_n} \leqslant \sum! x_1^{b_1} \cdots x_n^{b_n}$$

where $\Sigma!$ *denotes summation over all the distinct permutations of* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. *There is equality if and only if either* $\mathbf{a} \sim \mathbf{b}$ *or* $x_1 = x_2 = \cdots = x_n$.

PROOF. Let A be a full permutation matrix generated by the vector $\log \mathbf{x} = (\log x_1, \log x_2, \cdots, \log x_n)$. Then A is spectral order preserving, by Theorem 2.7, and so $A\mathbf{a} \prec A\mathbf{b}$. Now, by Corollary 2.6 in [2], we have $e^{A\cdot\mathbf{a}} \prec\prec e^{A\mathbf{b}}$ which clearly implies that

$$\sum! \exp(a_1 \log x_1 + \cdots + a_n \log x_n) \leqslant \sum! \exp(b_1 \log x_1 + \cdots + b_n \log b_n),$$

i.e., $\Sigma! x_1^{a_1} \cdots x_n^{a_n} \leqslant \Sigma! x_1^{b_1} \cdots x_n^{b_n}$.

Since exponential is a strictly convex function, it follows from [2, Theorem 2.5] that there is equality if and only if $A\mathbf{a} \sim A\mathbf{b}$ which is the case if and only if either $\mathbf{a} \sim \mathbf{b}$ or $\log x_1 = \log x_2 = \cdots = \log x_n$, i.e., $x_1 = x_2 = \cdots = x_n$, by Theorem 2.10.

REMARKS. 1. The present version of Muirhead's theorem is clearly equivalent to the one given in [8, Theorem 45, p. 45] where $\Sigma!$ denotes summation over all $n!$ permutations of $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, since each distinct permutation of $\mathbf{x}$ repeats the same number of times.

2. Theorem 3.1 remains valid if $\mathbf{a} \prec\prec \mathbf{b}$ and $\mathbf{x} \geqslant 1$, i.e., $x_i \geqslant 1$, $1 \leqslant i \leqslant n$. This can be easily verified using Proposition 2.1 and [2, Theorem 2.3].

When Theorem 2.7 is used in conjunction with Theorem 2.9 and [2, Theorems 2.3 and 2.5], many new rearrangement inequalities can be generated. We do not go into the details here and we content ourselves with the following Muirhead-type inequality.

THEOREM 3.2. *If* $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ *and* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ *are n-tuples in* R^n *such that* $\mathbf{a} \prec \mathbf{b}$ *and* $x_i > 0, i = 1, 2, \cdots, n,$ *then*

$$\prod!(1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) \leqslant \prod!(1 + x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n})$$

where $\Pi!$ *denotes multiplication over all the distinct permutations of* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. *There is equality if and only if either* $\mathbf{a} \sim \mathbf{b}$ *or* $x_1 = x_2 = \cdots = x_n$.

In particular,

$$(1+\sqrt[n]{x_1x_2\cdots x_n})^n<(1+x_1)(1+x_2)\cdots(1+x_n)$$

unless $x_1=x_2=\cdots=x_n$.

PROOF. The first part follows in exactly the same way that Theorem 3.1 was proved, except that we use the strictly convex function $t\mapsto\log(1+e^t)$, $t\in R$, instead of the exponential function. The last part is easily obtained on substituting $(1/n, 1/n, \cdots, 1/n), (1, 0, 0, \cdots, 0)$ for $\mathbf{a}, \mathbf{b}$ respectively.

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