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k-REGULAR ELEMENTS IN SEMISIMPLE ALGEBRAIC GROUPS

BY

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ABSTRACT. In this paper, Steinberg's concept of a regular element in a semisimple algebraic group defined over an algebraically closed field is generalized to the concept of a k-regular element in a semisimple algebraic group defined over an arbitrary field of characteristic zero. The existence of semisimple and unipotent k-regular elements in a semisimple algebraic group defined over a field of characteristic zero is proved. The structure of all k-regular unipotent elements is given. The number of minimal parabolic subgroups containing a k-regular element is given. The number of conjugacy classes of R-regular unipotent elements is given, where R is the real field. The number of conjugacy classes of Q_p-regular unipotent elements is shown to be finite, where Q_p is the field of p-adic numbers.

1. Introduction. In [11], R. Steinberg defined a regular element in a semisimple algebraic group defined over an algebraically closed field to be one whose centralizer had minimal dimension. He showed that there exists one conjugacy class of unipotent regular elements and that a unipotent element was regular if and only if it was contained in a unique Borel subgroup. He also showed that a unipotent element u was regular if and only if

$$u = x_{\alpha_1} \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m}$$

where $x_{\alpha_i} \neq e$ for all simple roots α_i.

In this paper, we extend Steinberg's results to groups defined over a field k of characteristic zero which is not necessarily algebraically closed. We define a k-regular element to be an element in the split radical of a minimal parabolic subgroup whose centralizer has the same dimension as the centralizer of a maximal split torus. If k is algebraically closed, then a k-regular element is regular in Steinberg's sense.

We will show the existence of both semisimple and unipotent k-regular elements. We shall also see that a unipotent element u is k-regular if

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and only if $u = x_{\alpha_1} \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m}$ where $x_{\alpha_i} \neq e$ for all restricted simple roots α_i. The number of minimal parabolic subgroups containing a k-regular element x will be shown to be $|W_k(G)|/|W_k(C^0(x_s))|$ where $W_k(G)$ is the restricted Weyl group of G. We will show that there is one G_R-conjugacy class of R-regular unipotent elements if G is adjoint or if G is not split and is not one of three possible exceptions. A method for calculating the number of G_R-conjugacy classes of R-regular unipotent elements in a split group is given. For the p-adics, the number of conjugacy classes of Q_p-regular unipotent elements is shown to be finite.

In [6], Kostant and Rallis examined an equivalent concept to k-regularity in the $\mathfrak{p}$-space of a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of a complex semisimple Lie algebra.

In [8], L. P. Rothschild discusses the G-conjugacy classes of R-regular nilpotent elements in a real Lie algebra which are the Lie algebra analogues to R-regular unipotent elements.

2. **Notation.** We will be concerned with a semisimple algebraic group G defined over a field k of characteristic zero. The group of k-rational points of G will be denoted G_k. Associated with such an algebraic group is a Lie algebra $\mathfrak{g}$ whose k-rational points will be denoted $\mathfrak{g}_k$. In general, Gothic letters will be used to denote Lie algebras of algebraic subgroups.

Throughout the paper we will use P to denote a minimal parabolic subgroup of G. S will represent a maximal k-split torus contained in P and U the maximal unipotent subgroup contained in P. The action of S on U gives rise to a restricted root system Δ_k on S. We will call the set of roots α such that the root group $U_{(\alpha)}$ is contained in P, the positive roots and denote the set Δ_k^+. The set Π_k such that every root in Δ_k^+ can be written uniquely as a positive integral combination of roots in Π_k is called the set of simple roots.

The paper makes strong use of Morosov's lemma, which can be found in [4] or [5]. The results on reductive algebraic groups used in this paper can be found in [2], [3], or [13].

3. **Simply written elements.**

DEFINITION 1. A *Lie triple* $[X, H, Y]$ is a subset of three elements $\{X, H, Y\}$ of a Lie algebra such that $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$.

PROPOSITION 1. *Let X be a nilpotent element in $\mathfrak{g}_k$, the k-rational points of semisimple Lie algebra $\mathfrak{g}$. Then there exists a Lie triple, $[X, H, Y]$ such that $H \in \mathfrak{s}_k$, the k-rational points of the Lie algebra $\mathfrak{s}$ of a maximal k-split torus of G.*

PROOF. Since the characteristic of $\mathfrak{g}_k$ is zero, X can be embedded in a Lie triple $[X, H', Y']$ in $\mathfrak{g}_k$. The Lie algebra $\mathfrak{g}'_k$ spanned over k by $\{X, H', Y'\}$ is semisimple and thus forms the k-rational points of an algebraic Lie algebra $\mathfrak{g}'$ by Corollary 7.9, p. 195 of [2]. Let G' be the group whose Lie algebra is $\mathfrak{g}'$. The Lie algebra $\mathfrak{n}'$ spanned by X is the Lie algebra of a unipotent group U' in G'. The normalizer of $\mathfrak{n}'$ in G' is a Borel subgroup of G' and contains a maximal k-split torus S' of G'. Let S be a maximal k-split torus of G containing S'. Let $\mathfrak{s}$ be the Lie algebra of S and $\mathfrak{s}'$ be the Lie algebra of S'. Since ad $\mathfrak{s}'$ normalizes $\mathfrak{n}'$, this defines a root α on $\mathfrak{s}'$. Let $\mathfrak{n}''$ be the root space in $\mathfrak{g}'$ for the root $-\alpha$. The elements of $\mathfrak{n}''$ are nilpotent elements of $\mathfrak{g}'$. The spaces $\mathfrak{n}', \mathfrak{s}'$ and $\mathfrak{n}''$ span $\mathfrak{g}'$. Let $H \in \mathfrak{s}'_k$ be such that $[H, X] = 2X$. Thus $[H, N] = -2N$ for any $N \in \mathfrak{n}''_k$.

$$[[N, X], H] = -[[X, H], N] - [[H, N], X] = 2[X, N] - 2[X, N] = 0.$$

Thus $[N, X] \in \mathfrak{s}'_k$. Pick $Y \in \mathfrak{n}''_k$ such that $[X, Y] = H$. Thus $[X, H, Y]$ is a Lie triple and $H \in \mathfrak{s}'_k \subset \mathfrak{s}_k$ where $\mathfrak{s}$ is the Lie algebra of a maximal k-split torus.

For the remainder of this section, P will be a fixed minimal parabolic subgroup and S a fixed maximal k-split torus contained in P. U will be the maximal unipotent subgroup of P. Then $U = U_{(\alpha_1)}U_{(\alpha_2)} \cdots U_{(\alpha_r)}$ where $U_{(\alpha_i)}$ is the α_i-root group and α_i is a root with respect to S.

DEFINITION 2. A nilpotent element $N \in \mathfrak{n}_k$, the k-rational points of the Lie algebra $\mathfrak{n}$ of the group U, is *simply written* if $N = \Sigma_{\alpha \in \Pi_k} X_\alpha$ where $X_\alpha \in \mathfrak{n}_\alpha$, α-root space and $X_\alpha \neq 0$ for all $\alpha \in \Pi_k$.

PROPOSITION 2. *A simply written nilpotent element $N \in \mathfrak{n}_k$ is contained as the first member of a Lie triple $[N, H, Y]$ in $\mathfrak{g}_k$ such that $H \in \mathfrak{s}_k$, the k-rational points of the Lie algebra $\mathfrak{s}$ of the maximal k-split torus S.*

PROOF. By Proposition 1, N can be embedded in a Lie triple $[N, H', Y']$ where H' is an element of the k-rational points of the Lie algebra $\mathfrak{s}''$ of a maximal k-split torus S''. Let $\mathfrak{n}'$ be the $\bar{k}$-space spanned by N. Let $R(\mathfrak{n}')$ be the normalizer of $\mathfrak{n}'$ in G. Let $x \in R(\mathfrak{n}')_k$. Then, by the Bruhat decomposition, $x = u_1 n_w u_2$ where $u_1, u_2 \in U_k$ and $n_w \in N(S)_k$. Since $x \in R(\mathfrak{n}')_k$, $u_1 n_w u_2 N u_2^{-1} n_w^{-1} u_1^{-1} = tN$ where $t \in k$. Let $u_2 N u_2^{-1} = Y = \Sigma Y_\alpha + \Sigma Y_\beta$ with $Y_\alpha \neq 0$ for all $\alpha \in \Pi_k$. But

$$n_w Y n_w^{-1} = \sum n_w Y_\alpha n_w^{-1} + \sum n_w Y_\beta n_w^{-1} = \sum Y'_{w^{-1}(\alpha)} + \sum Y'_{w^{-1}(\beta)}$$

which is not in $\mathfrak{p}$, the Lie algebra of the minimal parabolic subgroup P, unless $w = e$ and $n_w \in P$. Thus $R(\mathfrak{n}')_k \subset P$ and $R(\mathfrak{n}') \subset P$. Since H' is in the Lie algebra of $R(\mathfrak{n}')$, $H' \in \mathfrak{p}$.

H' is therefore contained in the Lie algebra of a maximal k-split torus contained in the minimal parabolic subgroup P. The maximal k-split tori of P are conjugate by elements in U_k. Therefore, there is an element $u \in U_k$ such that $uH'u^{-1} \in \mathfrak{s}_k$, the k-rational points of the Lie algebra $\mathfrak{s}$ of the maximal k-split torus S. Now, $uNu^{-1} = \Sigma_{\alpha\in\Pi_k} X_\alpha + \Sigma X_\beta$, β ranging over higher roots. Let $H = uH'u^{-1}$. Then,

$$\begin{aligned}[H, uNu^{-1}] = 2uNu^{-1} &= [H, \textstyle\sum X_\alpha + \sum X_\beta] \\ &= \textstyle\sum \alpha(H)X_\alpha + \sum \beta(H)X_\beta = \sum 2X_\alpha + \sum 2X_\beta.\end{aligned}$$

But, then $\alpha(H) = 2$ for all $\alpha \in \Pi_k$. Thus $\beta(H) > 2$ for all higher roots. But $\beta(H) = 2$ unless $X_\beta = 0$ for all β, and $uNu^{-1} = N$. Therefore, the Lie triple $[N, H, uY'u^{-1}]$ is the Lie triple which we want.

COROLLARY 1. *Let $N \in \mathfrak{n}_k$ be simply written. In the Lie triple $[N, H, Y]$ such that $H \in \mathfrak{n}_k$, H and Y are unique.*

PROOF. If $N = \Sigma X_\alpha$, $\alpha \in \Pi_k$. Then $\alpha(H) = 2$ for all $\alpha \in \Pi_k$. Since the simple roots form a basis for $\mathfrak{s}_k^*$, H is uniquely determined. Y is unique from p. 984 in [5].

4. k-split elements.

DEFINITION 3. The *split radical* $SR(P)$ of a minimal parabolic subgroup P is the group generated by a maximal split torus S and the unipotent radical U of P.

DEFINITION 4. An element $g \in G_k$ is a *k-split element* if it is contained in the split radical of a minimal parabolic subgroup.

LEMMA 1. *Every k-rational element of $SR(P)$ is conjugate by an element in U_k to an element of the form of su where $s \in S_k$ and $u \in U_k$ and s commutes. with u.*

PROOF. Order the positive roots Δ_k^+ starting with roots of height one and then height two and so on. Let $x \in SR(P)_k$. Then $x = s_1 u_1$ where $s_1 \in S_k$ and $u_1 = \Pi x_{\alpha_{i_i}} \in U_k$ where $x_{\alpha_i} \in U_{\alpha_{i_k}}$, the k-rational points of the α_i-root group, and $\alpha_i < \alpha_j$ if $i < j$.

If s_1 commutes with x_{α_1}, proceed to x_{α_2}. If s_1 does not commute with x_{α_1}, then $\alpha_1(s_1) \neq 1$. If $x_{\alpha_1} = \exp(X_{\alpha_1})$, construct

$$u_{\alpha_1} = \exp\left(\frac{\alpha_1(s_1)}{1 - \alpha_1(s_1)} X_{\alpha_1}\right).$$

Then

$$s_1^{-1}u_{\alpha_1}s_1u_{\alpha_1}^{-1} = s_1^{-1}\exp\left(\frac{\alpha_1(s_1)}{1-\alpha_1(s_1)}X_{\alpha_1}\right)s_1\exp\left(\frac{\alpha_1(s_1)}{\alpha_1(s_1)-1}X_{\alpha_1}\right)$$

$$= \exp\left(\frac{-1}{\alpha_1(s_1)-1}X_{\alpha_1}\right)\exp\left(\frac{\alpha_1(s_1)}{\alpha_1(s_1)-1}X_{\alpha_1}\right) = \exp(X_{\alpha_1}) = x_{\alpha_1}.$$

Thus $u_{\alpha_1}^{-1}s_1x_{\alpha_1}u_{\alpha_1} = s_1$ and $u_{\alpha_1}^{-1}s_1u_1u_{\alpha_1} = s_1x'_{\alpha_2}\cdots x'_{\alpha_n}$.

Proceeding inductively, s_1u_1 is conjugate by an element in U_k after r steps to $s_1\Pi x'_{\beta_i}\Pi x'_{\gamma_j}$ where x'_{β_i} commutes with s_1 and $\gamma_j > \alpha_i$ for all α_i, $1 \leqslant i \leqslant r$.

PROPOSITION 3. *Every semisimple k-split element is contained in a maximal k-split torus.*

PROOF. Let x be a semisimple element of $SR(P)_k$. By Lemma 1, x is conjugate by u_1 in U_k to an element of the form su where $s \in S_k$ and $u \in U_k$ and $su = us$. But by the uniqueness of the Jordan decomposition, $u = e$ and $u_1xu_1^{-1} = s_1$ since $u_1xu_1^{-1}$ is semisimple. Thus x is an element of the maximal k-split torus $u_1^{-1}Su_1$.

5. k-regular elements.

DEFINITION 5. A k-split element x is *k-regular* if $\dim C(x) = \dim C(S)$ where $C(x)$ is the centralizer of x in G and $C(S)$ is the centralizer of S in G.

PROPOSITION 4. *If x is a k-regular semisimple element and y is a k-split semisimple element then* $\dim C(x) \geqslant \dim C(y)$.

PROOF. By Proposition 3, every semisimple k-split element is contained in a maximal k-split torus. Thus y is an element of a k-split torus S_1. But, $\dim C(y) \leqslant \dim C(S_1) = \dim C(S) = \dim C(x)$.

PROPOSITION 5. *There exists a k-regular semisimple element.*

PROOF. Since $C^0(x)$ is reductive, where $C^0(x)$ is the connected component of the identity and x is an element of S_k, $C^0(x)$ is generated by $C(s)$ and the root groups $U_{(\alpha)}$ where $\alpha(x) = 1$. Since S is k-split, $S_k \cong \mathrm{Hom}(X^*(S), k^*)$. Since k is infinite, there exists $s \in S_k$ such that $\alpha(s) \neq 1$ for all $\alpha \in X^*(S)$. Therefore, $\alpha(s) \neq 1$ for all $\alpha \in \Delta_k$. Therefore, $C^0(s) = C(S)$ and s is k-regular.

DEFINITION 6. A unipotent element $u \in U_k$ is *simply written* with respect to a maximal k-split torus S, if $u = \exp(N)$ and N is a simply written nilpotent element.

PROPOSITION 6. *There exists a k-regular unipotent element.*

PROOF. Assume that G is not anisotropic. Let x be a simply written unipotent element of U_k with respect to the maximal k-split torus S. Then $x =$

$\exp(X)$ where $X = \Sigma X_\alpha$, $\alpha \in \Pi_k$ and $X_\alpha \neq 0$. By Proposition 2, X is contained in a Lie triple $[X, H, Y]$ with $H \in \mathfrak{s}_k$, the k-rational points of the Lie algebra $\mathfrak{s}$ of the group S. Let $\mathfrak{a}$ be the k-Lie algebra spanned by $\{X, H, Y\}$. $\mathfrak{g}_k$ is a completely reducible $\mathfrak{a}$-module. Therefore, we can write $\mathfrak{g}_k$ as the sum of irreducible $\mathfrak{a}$-modules $\mathfrak{g}_k^{(i)}$. Each $\mathfrak{g}_k^{(i)}$ is the direct sum of eigenspaces $(\mathfrak{g}^{(i)}_{-m_i+2p})_k$ of H such that if $N \in (\mathfrak{g}^{(i)}_{-m_i+2p})_k$, then $[H, N] = (-m_i + 2p)N$.

dim $C(X) = n$, the number of irreducible components. $C(H)$ has a one-dimensional intersection with $\mathfrak{g}^{(i)}$ if m_i is even. If m_i is odd, $C(H) \cap \mathfrak{g}^{(i)} = \{0\}$.

Now, $2X = [H, X] = [H, \Sigma X_\alpha] = \Sigma\alpha(H)X_\alpha$. Thus $\alpha(H) = 2$ for all $\alpha \in \Pi_k$ and $\beta(H)$ is even for all $\beta \in \Delta_k^+$. Thus m_i cannot be odd for any i. Therefore, dim $C(H) =$ dim $C(X) = n$.

But, $\beta(H) \neq 0$ for any $\beta \in \Delta_k^+$. Thus dim $C(H) =$ dim $C(\mathfrak{s}) =$ dim $C(S)$. Since $C(X)$ is the Lie algebra of $C(x)$, dim $C(x) =$ dim $C(X) =$ dim $C(H) =$ dim $C(S)$. Therefore, x is k-regular.

If G is anistropic, then the identity is a k-regular unipotent element.

Proposition 7. *Let x be a k-regular unipotent element and y a unipotent element in G_k. Then* dim $C(k) \leqslant$ dim $C(y)$.

Proof. Let $y = \exp Y$. By Proposition 1, Y is contained in a Lie triple $[Y, H, Z]$ such that $H \in \mathfrak{s}'_k$, the k-rational points of the Lie algebra of a maximal k-split torus S'. Let $\mathfrak{a}$ be the k-Lie algebra generated by $\{Y, H, Z\}$. As in Proposition 6, $\mathfrak{g}_k$ is the direct sum of irreducible $\mathfrak{a}$-modules $\mathfrak{g}_k^{(i)}$. But dim $C(H) \leqslant$ dim $C(Y)$ and dim $C(x) =$ dim $C(S) =$ dim $C(S') \leqslant$ dim $C(H)$. Thus dim $C(x) \leqslant$ dim $C(Y) =$ dim $C(y)$.

Proposition 8. *A k-split element x is k-regular if and only if x_u is k-regular in $C^0(x_s)$, the Zariski component of the identity in $C(x_s)$ where $x = x_s x_u$ is the Jordan decomposition of x.*

Proof. We know that $C(x) \subseteq C(x_s)$, since anything which commutes with x commutes with both its semisimple and unipotent parts. Thus, the Zariski-connected component of $C(x)$ containing the identity is equal to $C_H^0(x_u)$, the Zariski-connected component of the centralizer of x_u in $C^0(x_s)$, which we denote by H. Let x be k-regular. Then it is k-split and thus x_s and x_u are k-split. Therefore, x_s is contained in a maximal k-split torus S and $S \subseteq C^0(x_s) = H$. Since $x_s \in S$, then $C(S) \subseteq C^0(x_s) = H$ and dim $C_G(S) =$ dim $C_H(S)$. Since dim $C_G(x) =$ dim $C_H(x)$, x is k-regular in H. But, dim $C_H(x) =$ dim $C_H(x_u)$ and x_u is k-regular in $H = C^0(x_s)$.

Suppose x_u is k-regular in $C^0(x_s)$. Then dim $C_G(x) =$ dim $C_H(x) =$

$\dim C_H(x_u) = \dim C_H(S) = \dim C_G(S)$ and x is k-regular in G.

COROLLARY 2. *Every semisimple k-split element is the semisimple part of a k-regular element.*

PROOF. Let x_s be a semisimple k-split element. By Proposition 6, there exists a k-regular unipotent element x_u in $C^0(x_s)$. The element $x_s x_u$ is k-regular in G and $x = x_s x_u$ is the Jordan decomposition.

6. **k-regular unipotent elements.**

PROPOSITION 10. *A unipotent element $u \in U_k$ is k-regular if $u = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m}$ where $x_{\alpha_i} \in U_{\alpha_i}$, the α_i root group, $x_{\beta_i} \in U_{\beta_i}$ the β_i root group, $\alpha_i \in \Pi_k$ and $x_{\alpha_i} \neq e$ for all $\alpha_i \in \Pi_k$ and $x_{\alpha_i} \notin U_{2\alpha_i}$ if $2\alpha_i$ is a root, β_i range over positive roots of higher order.*

PROOF. It suffices to show that u is G_k-conjugate to a simply written unipotent element x.

Let $u = \exp N$ where $N = \Sigma X_\alpha + \Sigma X_\beta$ where $\alpha \in \Pi_k$ and ranges over positive roots of height greater than one. We will show that N is G_k-conjugate to $X = \Sigma X_\alpha$. Note that $X_\alpha \neq 0$ for all $\alpha \in \Pi_k$. The conjugation will be carried out sequentially. X will be conjugated by an element in U_k so that it will have the same components of N in the root spaces of height one and two and then that element will be conjugated so that it agrees with N in all root spaces of height one, two and three and so on.

Let $\mathfrak{n}_k$ be the k-rational points of the Lie algebra $\mathfrak{n}$ of U. Let $\mathfrak{n}_k^+$ be the k-rational points of the Lie algebra $\mathfrak{n}^+$ spanned by the root spaces $\mathfrak{n}_{\beta_i}$ where β_i has height greater than one. We shall first prove that $[X, \mathfrak{n}_k] = \mathfrak{n}_k^+$.

Since X is simply written it can embedded in a Lie triple $[X, H, Y]$ such that $H \in \mathfrak{s}_k$, the Lie algebra of S. Let $\mathfrak{a}$ be the k-Lie algebra spanned by $[X, H, Y]$. Then $\mathfrak{g}_k$ is the direct sum of irreducible $\mathfrak{a}$-modules $\mathfrak{g}_k^{(i)}$. Each $\mathfrak{g}_k^{(i)}$ is the direct sum of eigenspaces $(\mathfrak{g}^{(i)}_{-m_i+2p})_k$ where $[H, Z] = (-m_i + 2p)Z$ if $Z \in (\mathfrak{g}^{(i)}_{-m_i+2p})_k$. Let $\mathfrak{g}_{R_k} = \Sigma_{i=1}^n (\mathfrak{g}_R^{(i)})_k$. Note that, for $R > 0$, $[X, \mathfrak{g}_{R_k}] = (\mathfrak{g}_{R+2})_k$. Let $\mathfrak{g}_k^+ = \Sigma_{R>0}(\mathfrak{g}_R)_k$ and $\mathfrak{g}_k^{++} = \Sigma_{R>2}(\mathfrak{g}_R)_k$. Therefore, $[X, \mathfrak{g}_k^+] = \mathfrak{g}_k^{++}$. Since X is simply written, $X = \Sigma X_\alpha$, $\alpha \in \Pi_k$. Then $[H, X] = [H, \Sigma X_\alpha] = \Sigma\alpha(H)X_\alpha = 2\Sigma X_\alpha$. Therefore, $\alpha(H) = 2$ for all simple roots α, and $\beta(H)$ is positive and even for all positive roots $\beta \in \Delta_k^+$. Thus $\mathfrak{n}_k \subset \mathfrak{g}_k^+$ and $\mathfrak{n}_k^+ \subset \mathfrak{g}_k^{++}$ but $\dim \mathfrak{n}_k \geqslant \dim \mathfrak{g}_k^+$ and $\dim \mathfrak{n}_k^+ \geqslant \dim \mathfrak{g}_k^{++}$. Therefore, $\mathfrak{n}_k = \mathfrak{g}_k^+$ and $\mathfrak{n}_k^+ = \mathfrak{g}_k^{++}$. Thus $[X, \mathfrak{n}_k] = \mathfrak{n}_k^+$.

Let N_i be the sum of all X_{β_j} in the expansion of U into components such that the height of β_j is i. By above, there exists $Y_2 \in \mathfrak{n}_k$ such that $[X, Y_2] = N_2$ and

$$X_2 = \exp(Y_2)X\exp(-Y_2) = X + [X, Y_2] + \text{higher terms}$$
$$= X + N_2 + X_3^{(2)} + X_4^{(2)} + \cdots$$

where $X_i^{(2)}$ is the component of $\exp(Y_2)X\exp(-Y_2)$ in the root spaces of roots of height i.

There exists a $Y_3 \in \mathfrak{n}_k$ such that $[X, Y_3] = N_3 - X_3^{(2)}$. Then

$$\exp(Y_3)X_2\exp(-Y_3) = X_2 + [X_2, Y_3] + \text{higher terms}$$
$$= X + N_2 + X_3^{(2)} + N_3 - X_3^{(2)} + \text{higher terms}$$
$$= X + N_2 + N_3 + \text{higher terms.}$$

Proceeding sequentially we can conjugate X by a suitable element in U_k and obtain N. Thus $u = \exp(N)$ is G_k-conjugate to $x = \exp(X)$ and u is k-regular.

PROPOSITION 11. *Let u be a k-regular element in U_k. The G_k-conjugacy class of u is dense in U.*

PROOF. Let $\phi: P \longrightarrow U$ be defined by $\phi(p) = pup^{-1}$. Since ϕ is a regular, $\phi(p)$ contains an open subset of $\overline{\phi(P)}$. However,

$$\dim C(S) = \dim C_G(u) \geqslant \dim C_P(u)$$
$$= \dim(\phi^{-1}(u)) \geqslant \dim P - \dim \overline{\phi(P)}.$$

But $\dim P = \dim C(S) + \dim U$. Thus $\dim \overline{\phi(P)} = \dim U$ and $\overline{\phi(P)} = U$. Since P_k is dense in P, $\phi(P_k)$ is dense in $\overline{\phi(P)}$. Thus $\phi(P_k)$ is dense in U.

PROPOSITION 12. *Let $u \in U_k$ such that $u = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n}x_{\beta_1}\cdots x_{\beta_m}$, $\alpha_i \in \Pi_k$ and $x_{\alpha_j} = e$ for some $\alpha_j \in \Pi_k$ and β_i range over higher roots. Then u is not k-regular.*

PROOF. Suppose u is k-regular. The P_k-conjugacy class of u is then dense in U and contains an element x such that $x = x_{\alpha_1}x_{\alpha_1}\cdots x_{\alpha_n}x_{\beta_1}\cdots x_{\beta_m}$ and $x_{\alpha_i} \neq e$ for all $\alpha_i \in \Pi_k$. However, no P_k conjugate of u can have a nonidentity α_j component. Thus u is not k-regular.

COROLLARY 3. *A unipotent element $u \in U_k$ is k-regular if and only if $u = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n}x_{\beta_1}\cdots x_{\beta_m}$ where $x_{\alpha_i} \neq e$ for all $\alpha_i \in \Pi_k$ and β_i range over higher roots.*

7. k-regular elements and minimal parabolic subgroups.

LEMMA 3. *Let x be a k-regular element which is a member of two minimal parabolic subgroups P_1 and P_2. Let x be an element of the split radical*

$SR(P_1)$ of P_1 and x_s be an element of S, a maximal k-split torus contained in P_1. Then there exists an element n of $N(S)_k$, the k-rational points of the normalizer of S, such that $nP_2n^{-1} = P_1$.

PROOF. Since P_1 and P_2 are minimal parabolic subgroups, there exists $g \in G_k$ such that $gP_2g^{-1} = P_1$. By the Bruhat decomposition, $g = u_1n_wu_2$ where $u_1, u_2 \in RU(P_1)$ and $n_w \in N(S)_k$. Since u_1 is an element of P_1, $n_wu_2P_2u_2^{-1}n_w^{-1} = P_1$.

Let $u = x_{\alpha_1}x_{\alpha_2} \cdots x_{\alpha_n}x_{\beta_1} \cdots x_{\beta_m}$ where $w^{-1}(\alpha_i)$ is positive and $w^{-1}(\beta_i)$ is negative. Then

$$n_wu_2 = n_wx_{\alpha_1}x_{\alpha_2} \cdots x_{\alpha_n}x_{\beta_1} \cdots x_{\beta_m}$$

$$= x_{w^{-1}(\alpha_1)}x_{w^{-1}(\alpha_2)} \cdots x_{w^{-1}(\alpha_n)}n_wx_{\beta_1} \cdots x_{\beta_m} = u_3n_wu_4.$$

But then $n_wu_4P_2u_4^{-1}n_w^{-1} = P_1$.

Let $\Delta'_k = \{\alpha \in \Delta_k | \alpha(x_s) = 1\}$. Δ'_k is a root system for $C^0(x_s)$ and since x is k-regular, x_u is a k-regular unipotent element in $C^0(x_s)$. If Π'_k is a set of simple roots for Δ'_k such that $\Pi'_k \subset \Delta_k^+$, then $x_u = x_{\gamma_1}x_{\gamma_2} \cdots x_{\gamma_r}x_{\delta_1} \cdots x_{\delta_p}$ where $x_{\gamma_i} \neq e$ for all $\gamma_i \in \Pi'_k$ and δ_i ranges over the higher roots of $\Delta_k'^+ = \Delta_k^+ \cap \Delta'_k$.

Suppose $\beta_i(x_s) \neq 1$ for some root β_i where $x_{\beta_i} \neq e$ in the decomposition of u_4. Then pick the root β_j of lowest height such that $\beta_j(x_s) \neq 1$. By the above, we have

$$n_wx_{\beta_1}(t_1)x_{\beta_2}(t_2) \cdots x_{\beta_n}(t_n)x_sx_ux_{\beta_n}(-t_n) \cdots x_{\beta_1}(-t_1)n_w$$

$$= n_wx_sx_{\beta_1}(t_1/\beta_1(x_s)) \cdots x_{\beta_n}(t_n/\beta_n(x_s))x_ux_{\beta_n}(-t_n) \cdots x_{\beta_1}(-t_1).$$

Since β_j is the root of lowest height such that $\beta_j(x_s) \neq 1$, it is not a positive integral combination of the other β_i's, γ_i's and δ_i's. Therefore, the β_j component of $u_4x_sx_uu_4^{-1}$ is not the identity, since

$$x_{\beta_j}(t_j/\beta_j(x))x_{\beta_j}(-t_j) = x_{\beta_j}\left(t_j\left(\frac{1-\beta_j(x_s)}{\beta_j(x_s)}\right)\right) \neq e.$$

But if $u_4x_sx_uu_4^{-1}$ is written $x_sx_{\rho_1} \cdots x_{\rho_q}$ where $\rho_i \in \Delta_k^+$, $\rho_1 = \beta_j$, then

$$n_wu_4x_sx_uu_4^{-1}n_w^{-1} = x'_sx'_{w^{-1}(\rho_1)} \cdots x'_{w^{-1}(\rho_q)} \quad \text{where } n_wx_sn_w^{-1} = x'_s.$$

But $n_wu_4x_sx_uu_4^{-1}n_w^{-1}$ is an element of P_1, while $w^{-1}(\rho_1)$ is negative and

$x'_{w^{-1}(\rho_1)}$ is not in P_1. Therefore, by contradiction, $\beta_i(x_s) = 1$ for all roots β_i such that $x_{\beta_i} \neq e$ in the expansion of u_4.

Assume that there is a β_j such that $x_{\beta_j} \neq e$ in the expansion of u_4. Then β_j is a positive integral combination of the roots in Π'_k, since $\beta_j(x_s) = 1$. But $w^{-1}(\beta_j)$ is negative, thus $w^{-1}(\gamma_i)$ is negative for some $\gamma_i \in \Pi'_k$. Since x_u is k-regular in $C^0(x_s)$, $x_{\gamma_i} \neq e$. Since γ_i is simple in Δ'^+_k, the γ_i component of $u_4 x_u u_4^{-1}$ is not the identity. Thus

$$n_w u_4 x_s x_u u_4^{-1} n_w^{-1} = x'_s n_w x'_{\gamma_1} \cdots x'_{\gamma_r} x'_{\delta_1} \cdots x'_{\delta_p} n_w$$
$$= x'_s x''_{w^{-1}(\gamma_1)} \cdots x''_{w^{-1}(\gamma_r)} x''_{w^{-1}(\delta_1)} \cdots x''_{w^{-1}(\delta_p)},$$

and $x''_{w^{-1}(\gamma_j)}$ is not an element of P_1. This leads to a contradiction since $n_w u_4 x u_4^{-1} n_w^{-1}$ is an element of P_1. Therefore $x_{\beta_j} = e$ for all β_j and $n_w P_2 n_w^{-1} = P_1$.

PROPOSITION 13. *A k-regular unipotent element is contained in only one minimal parabolic subgroup.*

PROOF. Let u be a unipotent k-regular element such that u is an element of two minimal parabolic subgroups P_1 and P_2. Then $u = eu$ is the Jordan decomposition of u and e is an element of every maximal k-split torus of P_1 and u is an element of $SR(P_1)$. By Lemma 3, there exists $n_w \in N(S)_k$ such that $n_w P_2 n_w^{-1} = P_1$. Let $u = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m}$ where $x_{\alpha_i} \neq e$ for all $\alpha_i \in \Pi_k$ and β_i range over higher roots. If $w \neq e$, then $w^{-1}(\alpha_i)$ is negative for some $\alpha_i \in \Pi_k$ and $x'_{w^{-1}(\alpha_i)}$ is not an element of P_1. But

$$n_w u n_w^{-1} = x'_{w^{-1}(\alpha_1)} x'_{w^{-1}(\alpha_2)} \cdots x'_{w^{-1}(\alpha_n)} x'_{w^{-1}(\beta_1)} \cdots x'_{w^{-1}(\beta_m)}$$

is an element of P_1. This is a contradiction. Hence, $w = e$ and n_w is an element of P_1. Therefore $P_1 = P_2$.

COROLLARY 5. *The centralizer of a k-regular unipotent element is contained in P, the unique minimal parabolic subgroup containing u.*

PROOF. Let g be an element of G_k such that $gug^{-1} = u$. Then gPg^{-1} is a minimal parabolic subgroup containing u. Thus $gPg^{-1} = P$ and g is an element of P since P is its own normalizer. Since $C(u)_k$ is dense in $C(u)$, $C(u)$ is contained in P.

PROPOSITION 14. *A k-regular semisimple element is contained in $|W_k|$ minimal parabolic subgroups where W_k is the restricted Weyl group.*

PROOF. Let s be a k-regular semisimple element contained in a maximal k-split torus $S \subseteq P_1$ and let s be an element of P_2. Then by Lemma 3, there exists $n_w \in N(S)_k$ such that $n_w P_2 n_w^{-1} = P_1$. Since the elements in $N(S)_k$ normalize S, s is an element of $nP_1 n^{-1}$ for all $n \in N(S)_k$. Since $N(S)_k \cap P_1 = C(S)_k$ and P_1 is its own normalizer, the number of minimal parabolic subgroups of the form $n_w P_1 n_w^{-1}$ is $|N(S)_k / C(S)_k| = |W_k|$.

LEMMA 4. *Let s be a semisimple k-split element. A minimal parabolic subgroup P_1 of $C^0(s)$ is equal to the intersection of $C^0(s)$ and a minimal parabolic subgroup P_2 of G such that s is an element of $SR(P_2)$.*

PROOF. Let g be an element of $C^0(s)$ and let s be an element of the maximal k-split torus S where S is contained in the minimal parabolic subgroup P. Now by the Bruhat decomposition g can be written uniquely as $u_1 n_w u_2$ where u_1 and u_2 are elements of $U_k \subseteq P$, n_w is an element of $N(S)_k$ and $u_2 = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n}$ with $\alpha_i \in \Delta_k^+$ and $w^{-1}(\alpha_i)$ is negative. $sgs^{-1} = su_1 n_w u_2 s^{-1} = su_1 s^{-1} sn_w s^{-1} su_2 s^{-1} = g$. But $su_1 s^{-1}$ is an element of U_k, $sn_w s^{-1}$ is an element of $N(S)_k$ and $su_2 s^{-1} = x'_{\alpha_1} x'_{\alpha_2} \cdots x'_{\alpha_n}$ where $w^{-1}(\alpha_i)$ is negative. Hence by the uniqueness of the Bruhat decomposition, $su_1 s^{-1} = u_1$, $sn_w s^{-1} = n_w$, $su_2 s^{-1} = u_2$. Therefore, $C^0(s)$ is generated by $C(S)$ and the root groups $U_{(\alpha)}$ such that $\alpha(s) = 1$.

$P \cap C^0(s) = P_1$ is a minimal parabolic subgroup of $C^0(s)$, since $P \cap C^0(s)$ is generated by $C(S)$ and the root groups $U_{(\alpha)}$ such that $\alpha(s) = 1$ and $\alpha \in \Delta_k^+$. Given a root $\alpha \in \Delta'_k = \{\alpha \in \Delta_k | \alpha(s) = 1\}$ then either $U_{(\alpha)}$ is contained in P_1 or $U_{(-\alpha)}$ is contained in P_1.

Let P'_1 be another minimal parabolic subgroup of $C^0(s)$. Then $P'_1 = gP_1 g^{-1} = g(P \cap C^0(s))g^{-1} = gPg^{-1} \cap C^0(s)$ where g is an element of $C^0(s)_k$. But gPg^{-1} is a minimal parabolic subgroup of G and gSg^{-1} is a maximal k-split torus containing s.

PROPOSITION 15. *Let $x = x_s x_u$ be a k-regular element. Then x is contained in $|W_k(G)|/|W_k(C^0(x_s))|$ minimal parabolic subgroups of G.*

PROOF. Let x be contained in $SR(P_1)$, let x_s be contained in the maximal k-split torus S in P_1 and let x be contained in another minimal parabolic subgroup P_2. By Lemma 3, there exists $n_w \in N(S)_k$ such that $n_w P_2 n_w^{-1} = P_1$. Thus any minimal parabolic subgroup containing x also contains S. Hence, if P_2 is a minimal parabolic subgroup containing x then $P_2 \cap C^0(x_s)$ is a minimal parabolic subgroup of $C^0(x_s)$. By Proposition 8, x is k-regular in G if and only if x_u is k-regular in $C^0(x_s)$. Therefore, by Proposition 13, x is contained in only one minimal parabolic subgroup of $C^0(x_s)$. There are $|W_k(G)|$

minimal parabolic subgroups of G containing S and $|W_k(C^0(x_s))|$ minimal parabolic subgroups of $C^0(x_s)$ containing S. Since $C(S)$ is contained in $C^0(x_s)$ we can think of $W_k(C^0(x_s))$ as a subgroup of $W_k(G)$. Given a coset of $W_k(G)/W_k(C^0(x_s))$, there is a representative n_w in $N(S)_k$ such that $n_w P_1 n_w^{-1}$ contains x. Two such representatives come from the same coset in $N(S)_k/C(S)_k$. Therefore, the number of minimal parabolic subgroups containing x is $|W_k(G)|/|W_k(C^0(x_s))|$.

8. **The conjugacy classes of R-regular unipotent elements.** In the following section, the field k will be restricted to the field of real numbers R.

PROPOSITION 16. *The P_R-conjugacy class of an R-regular unipotent element is open in U_R, in the real topology, where U is the unipotent radical of the minimal parabolic subgroup P.*

PROOF. Let $u \in U_R$ be an R-regular unipotent element. Let $u = \exp(N)$. Then we see that $\dim C(u) = \dim C(N) = \dim C(S_R) = \dim C(\mathfrak{s}_R)$ where $\mathfrak{s}$ is the Lie algebra of the maximal R-split torus S. Let $\mathfrak{p}$ be the Lie algebra of P and let $\mathfrak{n}$ be the Lie algebra of U. Since $\mathfrak{n}_R$ is normal in $\mathfrak{p}_R$, $\operatorname{ad}(N)(\mathfrak{p}_R) \subseteq \mathfrak{n}_R$. But $\mathfrak{p}_R = C(\mathfrak{s}_R) + \mathfrak{n}_R$. Thus $\operatorname{ad}(N)(\mathfrak{p}_R) = \mathfrak{n}_R$. Therefore, P_R operating by conjugacy on u is open in U_R.

COROLLARY 6. *The P_R-conjugacy class of a unipotent R-regular element $u \in U_R$ is a union of connected components of the set of R-regular elements in U_R.*

DEFINITION 7. Let $L_i = \{u \in U_R | u = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m}\}$ where $\alpha_j \in \Pi_R$, $\beta_j \in \Delta_R^+$, $\beta_j \notin \Pi_R$ and $x_{\alpha_i} = e$.

Note 1. The union of the L_i is the full set of the non-R-regular elements in U. Thus the set of R-regular elements R_u is equal to $U_R - \bigcup_{i=1}^n L_i$. Each L_i partitions $U_R - L_i$ into two components if and only if $\dim U_{(\alpha_i)} = 1$. We shall call one such component M_i and the other N_i. The set of $\alpha_i \in \Pi_R$ such that $\dim U_{(\alpha_i)} = 1$ will be denoted M.

DEFINITION 8. Let $\dim U_{(\alpha_i)} = 1$ and A be a connected component of R_u. $\rho_i(A) = 1$ if A is contained in M_i and $\rho_i(A) = -1$ if A is contained in N_i.

PROPOSITION 17. *Two R-regular unipotent elements of U_R are G_R-conjugate if and only if they are P_R-conjugate.*

PROOF. Let u_1 and u_2 be two R-regular unipotent elements in U_R, and let $gu_1g^{-1} = u_2$ for some $g \in G_R$. Both u_1 and u_2 are contained in a unique minimal parabolic subgroup P. But gPg^{-1} is a minimal parabolic subgroup containing u_2. Thus $gPg^{-1} = P$. But since P is its own normalizer $g \in P_R$.

PROPOSITION 18. *An R-regular unipotent element in a connected component A of R_u is G_R-conjugate to an element in another connected component B of R_u if and only if there exists an element $s \in S_R$ such that $\alpha_i(s) \in R^+$, the positive real numbers, if $\rho_i(A) = \rho_i(B)$ and $\alpha_i(s) \in R^-$, the real numbers, if $\rho_i(A) \neq \rho_i(B)$ for all $\alpha_i \in M$.*

PROOF. By Proposition 17, it is sufficient to restrict our discussion to P_R-conjugacy. Since conjugacy by elements in U_R does not change the components of an element in the simple root spaces, it is sufficient to restrict our attention to $C(S)_R$-conjugacy.

Let $C(S)_R = \mathcal{D}(C(S))_R S_R$ where $\mathcal{D}(C(S))$ is the derived group of $C(S)$ and $\mathcal{D}(C(S))$ is an anisotropic semisimple algebraic group. Let T' be a maximal torus defined over R in $\mathcal{D}(C(S))$ and $T = T'S$ is a maximal torus defined over R in G. The restricted roots of S are the restrictions of roots on T to S. Since $\dim u_{(\alpha_i)} = 1$ for $\alpha_i \in M$, α_i is defined over R. Thus the restriction of $\alpha_i \in M$ to T' is the zero map. Let $\dim U_{(\alpha_i)} = 1$. Then $U_{(\alpha_i)}$ is a root group for T as well as a restricted root group of S. Thus for any root β such that $\beta(S) = 1$, $\alpha_i + n\beta$ is not a root and therefore $U_\beta \in C(S)$. The elements in U_β with $\beta(S) = 1$ commute with the elements in $U_{(\alpha_i)}$. Likewise, conjugation by an element $t \in T'$ leaves x_{α_i} unchanged since $\alpha_i(T') = 1$. Therefore, since T' and the U_β generate $\mathcal{D}(C(S))$, the elements of $\mathcal{D}(C(S))_R$ commute with $x_{\alpha_i} \in U_{(\alpha_i)}$ where $\dim U_{(\alpha_i)} = 1$. Thus an R-regular element of a connected component A of R_u is G_R-conjugate to an element of another connected component B only if it is S_R-conjugate to some element of B.

If an R-regular unipotent element x in a connected component A of R_u is S_R-conjugate to some element in another connected component B then by Corollary 6, x is G_R-conjugate to all elements in B.

If x is an element of M_i and s is an element of S_R such that $\alpha_i(s) \in R^-$, the negative real numbers, then $sxs^{-1} \in N_i$, since

$$x = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_i}(t) \cdots x_{\alpha_n} x_{\beta_1} \cdots x_{\beta_m}$$

and

$$sxs^{-1} = x'_{\alpha_1} x'_{\alpha_2} \cdots x_{\alpha_i}(\alpha_i(s)t) \cdots x'_{\alpha_n} x'_{\beta_1} \cdots x'_{\beta_m} \in N_i.$$

Likewise if x is an element of M_i and sxs^{-1} is an element of N_i, then $\alpha_i(s)$ is negative.

A connected component A of R_u equals $\bigcap_{i=1}^{p} M_{n_i} \cap \bigcap_{j=1}^{q} N_{n_j}$, since A lies either in M_i or N_i for each i. $\rho_i(A) = \rho_i(B)$ if either both A and B lie in M_i or both lie in N_i. Thus if x is an element of A and sxs^{-1} is an element of B for $s \in S_R$, then $\alpha_i(s)$ is positive if $\rho_i(A) = \rho_i(B)$ and negative if $\rho_i(A) \neq \rho_i(B)$.

THEOREM 1. *Let G be an adjoint semisimple algebraic group. The R-regular unipotent elements are G_R-conjugate.*

PROOF. Since two minimal parabolic subgroups are G_R-conjugate, it suffices to show that two R-regular unipotent elements in U_R are P_R-conjugate. By Proposition 18, it is sufficient to show that given any partition of the set M into two sets I and J there exists an element $s \in S_R$ such that $\alpha_i(s)$ is positive for $i \in I$ and $\alpha_j(s)$ is negative for $j \in J$.

Extend S to maximal torus T of G defined over R. The characters of S are restrictions of characters of T. The restricted roots of S are restrictions of roots of T. The restricted simple roots of S are restrictions of the simple roots of T. Since the group is adjoint, the simple roots of T generate the character group of T. Therefore, the restricted simple roots of S generate the character group of S.

Now $S_R = \mathrm{Hom}(X^*(S), R^*)$. Since the simple roots are free generators for $X^*(S)$, there exists $s_i \in S_R$, such that $\alpha_i(s_i)$ is negative and $\alpha_j(s_i) = 1$ for $j \neq i$.

Thus given a partition $M = I \cup J$ where $I \cap J = \varnothing$, let $s = s_{r_1} s_{r_2} \cdots s_{r_p}$ where $\alpha_{r_i} \in I$. Then

$$\alpha_i(s) = \alpha_i(s_{r_1} s_{r_2} \cdots s_{r_p}) = \alpha_i(s_{r_1})\alpha_i(s_{r_2}) \cdots \alpha_i(s_{r_p}) = \alpha_i(s_i)$$

is negative if $i \in I$. But $\alpha_j(s) = 1$ if $j \in J$.

Therefore, the R-regular unipotent elements form a single G_R-conjugacy class.

An equivalent theorem to Theorem 1 was proved in [8] by Rothschild from the Lie algebra point of view.

THEOREM 2. *If G is an R-split semisimple algebraic group, then the number of conjugacy classes of R-regular unipotent elements is given by 2^q where q is the number of cyclic components of even order in a direct sum decomposition of $Z(G)$, the center of G.*

PROOF. As in Theorem 1, we are concerned with finding the number of orbits of S_R acting on the set of connected components of R_u.

Let $\phi \in X^*(S)$. We define $\bar{\phi}\colon S_R \longrightarrow Z_2$, by

$$\overline{\phi}(s) = \begin{cases} 0 & \text{if } \phi(s) \text{ is positive,} \\ 1 & \text{if } \phi(s) \text{ is negative.} \end{cases}$$

Define $\overline{X^*(S)} = \{\overline{\phi} | \phi \in X^*(S)\}$ and $\overline{\Delta}_R = \{\overline{\alpha} | \alpha \in \Delta_R\}$.

Since S is a torus, $\overline{X^*(S)}$ can be viewed as Z_2-vector space of dimension $n = \dim S$ and $\overline{X^*(S)}$ = number of connected components of R_u. Likewise we can form $Z_2(\overline{\Delta}_R)$, the Z_2-vector space generated by $\overline{\Delta}_R$. Let $s \in S_R$. Then $\hat{s}(\overline{\phi}) = \overline{\phi}(s)$ maps $Z_2(\overline{\Delta}_R)$ into Z_2.

$$\hat{s}(\overline{\phi} + \overline{\psi}) = (\overline{\phi} + \overline{\psi})(s) = \overline{\phi}(s) + \overline{\psi}(s) = \hat{s}(\overline{\phi}) + \hat{s}(\overline{\psi}).$$

Thus each element of S_R gives a linear functional on $Z_2(\overline{\Delta}_R)$ into Z_2. Suppose s_1 and s_2 are elements such that $\hat{s}_1 = \hat{s}_2$. Then $\overline{\alpha}_i(s_1) = \overline{\alpha}_i(s_2)$ for all restricted roots. Therefore $s_1 A s_1^{-1} = s_2 A s_2^{-1}$ where A is a connected component of R_u.

The set $\hat{S} = \{\hat{s} | s \in S_R\}$ is a subspace of the dual of $Z_2(\overline{\Delta}_R)$, since $\hat{s}_1(\overline{\phi}) + \hat{s}_2(\overline{\phi}) = \overline{\phi}(s_1) + \overline{\phi}(s_2) = \overline{\phi}(s_1 s_2) = \widehat{s_1 s_2}(\overline{\phi})$. If $\hat{S}$ is not the space of linear functionals, there must exist $\overline{\phi}$ in $Z_2(\overline{\Delta}_R)$ such that $s(\overline{\phi}) = \overline{\phi}(s) = 0$ for all $s \in S_R$. But, then $\overline{\phi} = 0$. Thus $\hat{S}$ is the dual space of $Z_2(\overline{\Delta}_R)$. Since the simple roots generate the full set of roots, a linear functional on $Z_2(\overline{\Delta}_R)$ is determined by its behavior on $\overline{\Pi}$.

Therefore, given a connected component A of R_u there is a one to one correspondence between the connected components B obtainable from A by conjugation by elements in S_R and the dual space of $Z_2(\overline{\Delta}_R)$. But the order of $Z_2(\overline{\Delta}_R)$ equals the order of its dual.

Thus the number of conjugacy classes of R-regular unipotent elements is given by the order of $\overline{X^*(S)}/Z_2(\overline{\Delta}_R)$, which is equal to 2^q where q is the $\dim \overline{X^*(S)}/Z_2(\overline{\Delta}_R)$.

Now $Z_2(\overline{\Delta}_R) = Z(\Delta_R)/(Z(\Delta_R) \cap 2X^*(S))$ and $\overline{X^*(S)} = X^*(S)/2X^*(S)$. But by the second isomorphism theorem we get

$$Z_2(\overline{\Delta}_R) \cong \frac{(Z(\Delta_R) + 2X^*(S))}{2X^*(S)}.$$

Therefore,

$$\overline{X^*(S)}/Z_2(\overline{\Delta}_R) \cong \frac{X^*(S)/2X^*(S)}{(Z(\Delta_R) + 2X^*(S))/2X^*(S)} = X^*(S)/(Z(\Delta_R) + 2X^*(S)).$$

By the fundamental theorem of finitely generated abelian groups, there exists a basis $\{\phi_1, \phi_2, \cdots, \phi_n\}$ of $X^*(S)$ such that $\{m_1\phi_1, m_2\phi_2, \cdots, m_n\phi_n\}$ is a basis for $Z(\Delta_R)$. If m_i is odd, $\phi_i \in Z(\Delta_R) + 2X^*(S)$. If m_i is even, $\phi_i \notin Z(\Delta_R) + 2X^*(S)$, but $2\phi_i \notin Z(\Delta_R) + 2X^*(S)$. Hence $\{r_1\phi_1, r_2\phi_2, \cdots, r_n\phi_n\}$ is a basis for $Z(\Delta_R) + 2X^*(S)$ with $r_i = 1$ if m_i is odd, $r_i = 2$ if m_i is even.

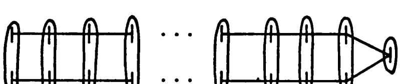
The center of G is isomorphic to $X^*(S)/Z(\Delta_R)$. The number of components of $\overline{X^*(S)}/Z_2(\overline{\Delta}_R)$ in the decomposition into a direct sum of cyclic groups is equal to the number of components q of $Z(G)$ with even order. Thus the number of G_R-conjugacy classes of an R-regular unipotent element is 2^q which is the order of $\overline{X^*(S)}/Z_2(\overline{\Delta}_R)$.

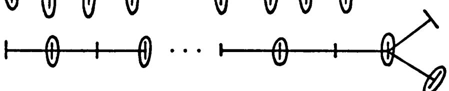
For R-split simply connected almost simple groups of the given type, we have the following number of conjugacy classes of R-regular unipotent elements.

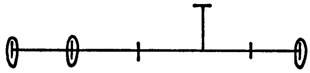
TABLE 1.

TYPE OF GROUP	$Z(G)$		NUMBER OF CONJUGACY CLASSES OF R-REGULAR UNIPOTENT ELEMENTS
A_n	Z_{n+1}		1 if n is even
			2 if n is odd
B_n	Z_2		2
C_n	Z_2		2
D_n	Z_4	if n is odd	2 if n is odd
	$Z_2 \times Z_2$	if n is even	4 if n is even
E_6	Z_3		1
E_7	Z_2		2
E_8	Z_1		1
F_4	Z_1		1
G_2	Z_1		1

THEOREM 3. *Let G be a nonsplit, semisimple group defined over R with irreducible restricted root system. Then there is only one conjugacy class of R-regular unipotent elements except possibly in the three cases whose restricted Dynkin diagrams are the following*:

(1) 

(2) 

(3) 

PROOF. If G is not absolutely almost simple, then $\dim U_{(\alpha_i)} > 1$ for all $\alpha_i \in \Pi_R$ and $M = \emptyset$. Therefore, we can restrict ourselves to the absolutely almost simple case.

The restrictions of roots and characters on a maximal torus T defined over R containing S are restricted roots and characters respectively of S. Therefore, $X^*(S)/Z(\Delta_R)$ is a homomorphic image of $X^*(T)/Z(\Delta)$ where Δ is an absolute root system for G. Therefore, if G is of type E_6, E_8, F_4 or G_2, $X^*(T)/Z(\Delta)$ is of odd order and hence $X^*(S)/Z(\Delta_R)$ is of odd order. Thus $\overline{\Pi}_R$ forms a basis for $\overline{X^*(S)}$. Therefore, for any subset $M \subseteq \Pi_R$, $\overline{M}$ can be extended to a basis of $\overline{X^*(S)}$.

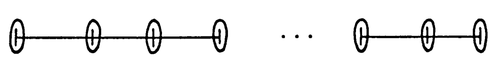
For the remainder of the proof, we will show that in all other forms M can be extended to a basis for $X^*(S)$, excluding the three possible exceptions.

Given a semisimple algebraic group G defined over R, we can, by a construction of Borel and Tits in [3], construct a maximal split group $\widetilde{G}$ such that $S \subseteq G$ and $\alpha \in \Pi_R$ is a simple root of $\widetilde{G}$ if $2\alpha \notin \Delta_R$ and if $2\alpha \in \Delta_R$ then 2α is a simple root of G. Let $\Delta_{\widetilde{G}}$ be the root system for $\widetilde{G}$. Let $V = X^*(S) \otimes_Z R$. We can induce onto V an inner product under the action of the Weyl group. To each element $\alpha \in \Delta_{\widetilde{G}}$, there exists a linear functional α^* on V defined by $\alpha^*(v) = -2(\alpha, v)/(\alpha, \alpha)$. Note that $\alpha^*(\beta)$ is an integer for all $\alpha, \beta \in \Delta_{\widetilde{G}}$. The set of weights of the root system $\Delta_{\widetilde{G}}$ denoted $\Lambda_{\widetilde{G}}$ is the set $\{v \in V | \alpha^*(v) \in Z, \text{ for all } \alpha \in \Delta_{\widetilde{G}}\}$. We can see that $\Lambda_{\widetilde{G}} \supseteq X^*(S) \supseteq \Delta_{\widetilde{G}}$. Therefore, it is sufficient to show that M can be extended to a basis for $\Lambda_{\widetilde{G}}$. Note that M cannot contain α if $2\alpha \in \Delta_R$, since $U_{(\alpha)}$ must be at least two dimensional. Therefore, $M \subseteq \Delta_{\widetilde{G}}$.

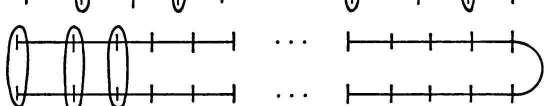
The remainder of the proof will consist of checking the possible real forms to verify that M can be extended to a basis for $\Lambda_{\widetilde{G}}$.

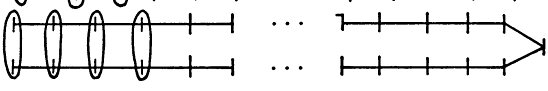
Let λ_α be the weight such that $\alpha^*(\lambda_\alpha) = 1$ but $\beta^*(\lambda_\alpha) = 0$ for $\alpha \neq \beta$ and $\alpha, \beta \in \Pi'_R$, the set of simple roots of $\widetilde{G}$. Therefore, by examining the Dynkin diagram of $\widetilde{G}$, we can write each root of Π'_R in terms of the λ_α's. It is clear that the λ_α's freely generate $\Lambda_{\widetilde{G}}$. Therefore, we can determine from the expression of the roots of M in terms of the λ_α's, whether M can be extended to a basis of $\Lambda_{\widetilde{G}}$. The restricted diagrams used in the following case by case study can be found in [12].

I. *Let G be of type A_n.* The restricted Dynkin diagrams are:

(a) 

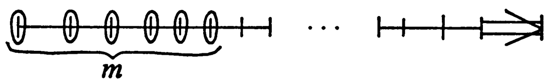
(b) 

(c) 

(d) 

In case (a), G is split. In case (b), $M=\varnothing$. In case (c), $M=\varnothing$. In case (d), $M=\varnothing$ except in the case which constitutes the first exception.

II. *Let* G *be of type* B_n. The restricted Dynkin diagrams are:

(a) 

$\tilde{G}$ is of type B_n with diagram:

$M=\{\alpha_2,\alpha_3,\cdots,\alpha_m\}$, $\alpha_m=-2\lambda_{\alpha_m}+\lambda_{\alpha_{m-1}}$, $\alpha_i=-2\lambda_{\alpha_i}+\lambda_{\alpha_{i-1}}+\lambda_{\alpha_{i+1}}$ for $2\leqslant i<m$. $M\cup\lambda_{\alpha_m}$ is a basis for $\Lambda_{\tilde{G}}$.

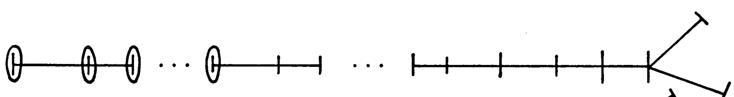
III. *Let* G *be of type* C_n. The restricted Dynkin diagrams are:

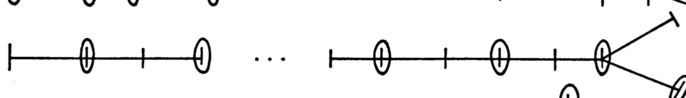
(a) 

(b) 

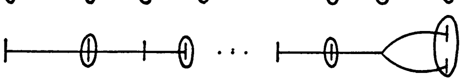
In case (a), G is split. In case (b), $M=\varnothing$.

IV. *Let* G *be of type* D_n. The restricted Dynkin diagrams are:

(a) 

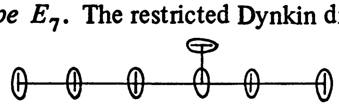
(b) 

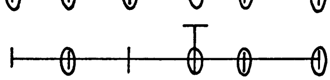
(c) 

(d) 

In case (a) either G is split or $\tilde{G}$ is of type B_n. In the latter case, the argument is the same as the above in type B_n. Case (b) is the second exception. In case (c), $\tilde{G}$ is of type B_n and the above argument again holds. In case (d), $M=\varnothing$.

V. *Let* G *be of type* E_7. The restricted Dynkin diagrams are:

(a) 

(b) 

(c) 

(d) 

In case (a), G is split. In case (b), $\widetilde{G}$ is of type F_4 and $\Lambda_{\widetilde{G}} = \Delta_{\widetilde{G}}$. Case (c) is the third exception. In case (d), $M = \emptyset$.

In each of the three exceptional cases G is of type C_n and M consists of the unique long simple root, α_m. In these cases, $\alpha_m = 2\lambda_{\alpha_{m-1}} - 2\lambda_{\alpha_m} \in 2\Lambda_{\widetilde{G}}$. There are, therefore, at most two G_R-conjugacy classes of R-regular unipotent elements.

9. Q_p-conjugacy classes of Q_p-regular unipotent elements. For the following results, k will be restricted to Q_p the field of p-adic numbers.

PROPOSITION 19. *The G_{Q_p}-conjugacy class of a Q_p-regular unipotent element is open in U_{Q_p}, in the p-adic topology.*

PROOF. Let u be a Q_p-regular unipotent element in U_{Q_p}. Then $u = \exp(N)$. From the argument of Proposition 16, we see that $\operatorname{ad} N(\mathfrak{p}_{Q_p}) = \mathfrak{p}_{Q_p}$. Thus if $\phi\colon P_{Q_p} \longrightarrow U_{Q_p}$ with $\phi(p) = pup^{-1}$ then the differential of ϕ is onto $\mathfrak{n}_{Q_p}$. But by LG 3.15 of [10], ϕ maps P_{Q_p} onto an open set in U_{Q_p}.

PROPOSITION 20. *Let x and y be two Q_p-regular unipotent elements in U_{Q_p} such that $x = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n}x_{\beta_1}\cdots x_{\beta_m}$ and $y = y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}y_{\beta_1}\cdots y_{\beta_m}$ where $x_{\alpha_i} \neq e \neq y_{\alpha_i}$ for all $\alpha_i \in \Pi_{Q_p}$, and β_i ranges over higher roots. x is G_{Q_p}-conjugate to y if and only if $x' = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n}$ is $C(S)_{Q_p}$ conjugate to $y' = y_{\alpha_1}y_{\alpha_2}\cdots y_{\alpha_n}$ where $C(S)$ is the centralizer of the maximal Q_p-split torus S.*

PROOF. By the proof of Proposition 10, x is U_{Q_p}-conjugate to x' and y is U_{Q_p}-conjugate to y'. Thus if x' is $C(S)_{Q_p}$-conjugate to y', then x is G_{Q_p}-conjugate to y.

If x is G_{Q_p}-conjugate to y then by the argument of Proposition 17, x is P_{Q_p}-conjugate to y where P is the unique minimal parabolic subgroup containing x and y. Thus x' is P_{Q_p}-conjugate to y'. But $P = C(S)U$ and the conjugation of x' with an element in U_{Q_p} cannot leave an element whose only components are in simple root spaces unless the element in U_{Q_p} commutes with x'. Since conjugation by $C(S)$ normalizes the restricted root groups, the only elements of P_{Q_p} which could conjugate x' into y' are of the form cu where $c \in C(S)_{Q_p}$, $u \in U_{Q_p}$ and u commutes with x'. But then $cxc^{-1} = y'$ and x' and y' are $C(S)_{Q_p}$-conjugate.

THEOREM 4. *There are only a finite number of G_{Q_p}-conjugacy classes of Q_p-regular unipotent elements.*

PROOF. From Proposition 20, it suffices to show that there are only a finite number of orbits of $C(S)_{Q_p}$ acting on $R = \{u|u = x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n},$

$\alpha_i \in \Pi_{Q_p}$ and $x_{\alpha_i} \neq e$ for each $\alpha_i \in \Pi_{Q_p}\}$. Topologically, $R = U'_{\alpha_1 Q_p} \times U'_{\alpha_2 Q_p} \times \cdots \times U'_{\alpha_n Q_p}$ where $U'_{\alpha_i Q_p} = U_{\alpha_i Q_p} - \{e\}$. Let $M_{\alpha_i Q_p}$ be the projective space produced from $U'_{\alpha_i Q_p}$ by identifying all points lying on a line through the origin. $C(S)_{Q_p}$ operates on $M_{\alpha_i Q_p}$ and hence on $M_{Q_p} = \Pi_{i=1}^n M_{\alpha_i Q_p}$. Since α_i is an integral combination of basis elements in $X^*(S)$, $Q_p^*/\alpha_i(S_{Q_p})$ is finite. Therefore, if $\phi: \Pi_{i=1}^n U'_{\alpha_i Q_p} \longrightarrow M_{Q_p}$ is the projection map, then an orbit in M_{Q_p} has only a finite number of orbits in its preimage in R. Since $M_{\alpha_i Q_p}$ is compact, M is compact. The orbits of $C(S)_{Q_p}$ are open in R, hence they are open in M. Therefore, there are only a finite number. Thus there are only a finite number of G_{Q_p}-conjugacy classes of Q_p-regular unipotent elements.

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