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DECOMPOSABLE BRAIDS AS SUBGROUPS OF BRAID GROUPS[1]

BY

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ABSTRACT. The group of all decomposable 3-braids is the commutator subgroup of the group I_3 of all 3-braids which leave strand positions invariant. The group of all 2-decomposable 4-braids is the commutator subgroup of I_4, and the group of all decomposable 4-braids is explicitly characterized as a subgroup of the second commutator subgroup of I_4.

Introduction. A braid on n strands is called *k-decomposable* iff whenever k arbitrary strands are removed, the remaining braid on $n-k$ strands is deformable into the identity braid. The set of all k-decomposable n-braids is denoted D_{kn}, and it shall be the task of this paper to determine D_{kn} as a subgroup of the braid group B_n in the cases where $n=3, k=1$, and $n=4, k=1, 2$. Based upon these cases, a reasonable conjecture is drawn as to the remainder of the D_{kn}.

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Decomposable 3-braids. We shall confine our attention to the subgroup of the braid group consisting of those braids which leave strand positions invariant. Denote this group by I_n.

Notation. For any elements u, v of a group:

$$(u, v) = u^{-1}v^{-1}uv; \qquad u^v = v^{-1}uv.$$

For any n elements $u_1, u_2, \cdots, u_n$,

$$(u_1, u_2, \cdots, u_n) = (\cdots((u_1, u_2), u_3), \cdots, u_n).$$

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For all normal subgroups $N_1, N_2, \cdots, N_n$ of a group: $(N_1, N_2, \cdots, N_n)$ shall denote the normal subgroup generated by the set of all $(u_1, u_2, \cdots, u_n)$, where $u_i \in N_1$. Let P_n denote the subgroup of B_n consisting of all "n-pure" braids (i.e. those braids in which strands numbered 1 through $n-1$ are uninvolved with each other and only strand n weaves its way among its lesser indexed straight companions).

LEMMA 1 (ARTIN). *P_n is normal in I_n.*

P_n is free on $n-1$ free generators, $t_1, t_2, \cdots, t_{n-1}$ defined by

$$t_i = \sigma_i\sigma_{i+1} \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}^{-1} \cdots \sigma_{i+1}^{-1}\sigma_i^{-1},$$

where the σ are the generators for B_n given in [1]. This is obvious since P_n is the fundamental group of the space between two planes from which $n-1$ straight line segments, parallel to each other, joining the two planes have been removed.

If $x_1, x_2, \cdots, x_n$ are the free generators of the free group on which B_n acts, then

$$t_i(x_j) = (x_i, x_n)x_j(x_n, x_i) \text{ for } i < j < n,$$

$$t_i(x_i) = x_n^{-1}x_ix_n = x_i(x_i, x_n),$$

$$t_i(x_n) = x_n^{-1}x_i^{-1}x_nx_ix_n = (x_n, x_i)x_n.$$

LEMMA 2. *$D_{k,n}$ is normal in B_n.*

PROOF. The conjugates of any k-decomposable n-braid are k-decomposable since if any k strands were removed the remnants of a conjugating braid and its inverse would be separated from each other by only a trivial braid, and would therefore be in a position to annihilate each other. Q.E.D.

LEMMA 3. *$D_{1,n}$ is a normal subgroup of P_n.*

PROOF. According to Artin [1] every braid may be written as a product, $\rho\pi$, where $\pi \in P_n$ and ρ is a braid in the subgroup I_{n-1} of I_n, generated by the first $n-2$ of the $n-1$ σ_i's generating I_n. ρ leaves the nth strand straight and uninvolved with any other of the first $n-1$ strands. The removal of the nth strand reduces π to a trivial braid on the first $n-1$ strands, and leaves ρ unchanged. Thus if $\pi\rho \in D_{1,n}$, then $\rho = 1$.

Since $D_{1,n}$ is normal in B_n, it is normal in every subgroup of B_n in which it is contained, in particular in P_n. Q.E.D.

LEMMA 4. *Let $\theta_i, i = 1, 2, \cdots, n-1$, be the normal closure of t_i in P_n. Then $D_{1,n}$ is the intersection of the groups θ_i.*

PROOF. $D_{1,n}$ is obviously a subset of the intersection of the θ_i. The removal of the ith strand of an element of P_n has the effect of mapping it into its coset with respect to the normal closure of t_i. We now note that

$$D_{1,n} \supseteq \prod_{\substack{\text{all permutations} \\ (i_1, \cdots, i_{n-1}) \text{ of } (1, \cdots, n-1)}} (\theta_{i_1}, \theta_{i_2}, \cdots, \theta_{i_{n-1}}). \quad \text{Q.E.D.}$$

We now consider the case where $n = 3$.

THEOREM 1. *$D_{1\ 3} = I_3'$ (the commutator subgroup of I_3). $D_{1\ 3}$ is generated by the conjugates of $(t_1, t_2) = (\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_2^2)$; i.e. by conjugates of the braid $(\sigma_1\sigma_2^{-1})^3$.*

PROOF. From the preceding lemmas, it follows that $D_{1\ 3}$ is generated by $\theta_1 \cap \theta_2$. Obviously this is (θ_1, θ_2), i.e. the commutator subgroup of P_3. Since I_3 is generated [1] by σ_1^2, σ_2^2, and $\sigma_1\sigma_2^2\sigma_1^{-1}$, and since

$$\begin{aligned}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_1^2\sigma_2^2 = \sigma_1\sigma_2^2\sigma_1\sigma_2^2 &= \sigma_1\sigma_2(\sigma_2\sigma_1\sigma_2)\sigma_2 \\ &= \sigma_1\sigma_2(\sigma_1\sigma_2\sigma_1)\sigma_2 = (\sigma_1\sigma_2)^3\end{aligned}$$

is in the center of B_3, it follows that I_3' is the normal closure of the commutator of any two of the elements $\sigma_1^2, \sigma_2^2, \sigma_1\sigma_2^2\sigma_1^{-1}$. We choose the last two which we had denoted t_1 and t_2 respectively. It must now be shown that (t_1, t_2) is a conjugate of $(\sigma_1\sigma_2^2\sigma_1^{-1}, \sigma_2^2)$.

Using the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, from B_3,

$$\begin{aligned}(t_1, t_2) &= \sigma_1\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-2}\sigma_1\sigma_2^2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_2^{-1}(\sigma_1\sigma_2)\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}(\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1})\sigma_2^{-1}(\sigma_2\sigma_1\sigma_2\sigma_1^{-1})\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_1\sigma_2^{-1}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2^2 \\ &= \sigma_2^{-1}((\sigma_2\sigma_1\sigma_2^{-1})\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}((\sigma_1^{-1}\sigma_2\sigma_1)\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}(\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2)\sigma_2 \\ &= \sigma_2^{-1}\sigma_1^{-1}(\sigma_1\sigma_2^{-1})^{-3}\sigma_1\sigma_2. \quad \text{Q.E.D.}\end{aligned}$$

Decomposable 4-braids.

THEOREM 2. *$D_{1\ 4}$ is the product $((\theta_1, \theta_2), \theta_3)((\theta_1, \theta_3), \theta_2) \doteq \theta^*$.*

PROOF. We shall prove $D_{1\ 4} = ((\theta_1, \theta_2), \theta_3)((\theta_1, \theta_3), \theta_2)((\theta_2, \theta_3), \theta_1)$. According to P. Hall [3] each of the three factors is contained in the product of the other two. In order to simplify notation, denote t_1, t_2, and t_3 by a, b, and c respectively. We wish to find the intersection of A, B, and C, the respective normal closures of a, b, and c.

The normal closure of a is freely generated by the elements a^α, where α runs through all freely reduced words in b and c. (This follows from elementary combinatorial arguments.) We now characterize which elements of A are also in B. If we map $b \longrightarrow 1$, this has the result that $\alpha \longrightarrow c^s$, where s is the exponent sum of the c's in α. We denote c^s by $\bar{\alpha}$, and observe that $a^\alpha a^{-\bar{\alpha}}$ is indeed in both A and B. We wish to show that $A \cap B = (A, B)$. For this purpose we write $\alpha^{-1} = b^{e_1}c^{f_1}b^{e_2}c^{f_2} \cdots b^{e_k}c^{f_k}$, where the e_i and f_i are nonzero integers with the possible exceptions of e_1 and f_k. Set $e = \Sigma_{i=1}^k e_i$ and $f = \Sigma_{i=1}^k f_i$, and rewrite α^{-1} as

$$\alpha^{-1} = b^{e_1}(c^{f_1}bc^{-f_1})^{e_2}(c^{f_1+f_2}bc^{-f_1-f_2})^{e_3} \cdots (c^{f_1+\cdots+f_{k-1}}bc^{-f_1 \cdots -f_{k-1}})^{e_k}c^f. \tag{1}$$

Now we use the identity

$$\begin{aligned} \Omega_k &= v_k^{-1}v_{k-1}^{-1} \cdots v_1^{-1}uv_1v_2 \cdots v_{k-1}v_ku^{-1} \\ &= v_k^{-1}v_{k-1}^{-1} \cdots v_1^{-1}uv_1v_2 \cdots v_{k-1}u^{-1}v_k(v_k, u^{-1}). \end{aligned} \tag{2}$$

Putting

$$u = c^fac^{-f}, \quad \text{and} \quad v_k = (c^{f_1+\cdots+f_{k-1}}bc^{-f_1-\cdots-f_{k-1}})^{e_k}, \tag{3}$$

we observe that $(v_k, u^{-1}) \in (A, B) = (B, A)$. Therefore $\Omega_k \in (A, B)$ if we can show that

$$\Omega_{k-1} = v_{k-1}^{-1} \cdots v_1^{-1}uv_1 \cdots v_{k-1}u^{-1} \in (A, B), \tag{4}$$

since

$$\Omega_k = v_k^{-1}\Omega_{k-1}v_k(v_k, u^{-1}). \tag{5}$$

However, this follows by induction since

$$\Omega_1 = v_1^{-1}uv_1u^{-1} \text{ is obviously in } (A, B) \text{ if } v_1 = b^e. \tag{6}$$

We now characterize which products of elements $a^\alpha a^{-\bar{\alpha}}$ are also in C. Mapping $c \longrightarrow 1$ has the effect of mapping $a^\alpha a^{-\bar{\alpha}} \longrightarrow b^eab^{-e}a^{-1}$. We must show that $a^\alpha a^{-\bar{\alpha}}ab^ea^{-1}b^{-e} \in \theta^*$.

We first show that for $v = c^\phi b^\epsilon c^{-\phi}$, and $u = c^\gamma ac^{-\gamma}$, that $(v, u^{-1}) \equiv (b, a^{-1}) \bmod \theta^*$.

$$(v, u^{-1})(a^{-1}, b^{\epsilon}) = c^{\phi}b^{-\epsilon}c^{-\phi}c^{\gamma}ac^{-\gamma}c^{\phi}b^{\epsilon}c^{-\phi}c^{\gamma}a^{-1}c^{-\gamma}(ab^{-\epsilon}a^{-1}b^{\epsilon})$$
$$= c^{\phi}(b^{-\epsilon}c^{\delta}ac^{-\delta}b^{\epsilon}c^{\delta}a^{-1}c^{\delta})c^{-\phi}(ab^{-\epsilon}a^{-1}b^{\epsilon}),$$

for $\delta = \gamma - \phi$. Thus

$$(v, u^{-1})(a^{-1}, b^{\epsilon}) = c^{\phi}(b^{\epsilon}, c^{\delta}a^{-1}c^{-\delta})c^{-\phi}(b^{\epsilon}, a^{-1})^{-1}.$$

Let $c^{\phi} = r$, $b^{-\epsilon} = s$, and $c^{\delta}ac^{-\delta} = t$. Then

$$(v, u^{-1})(a^{-1}, b^{\epsilon}) = rs^{-1}t^{-1}sts^{-1}asa^{-1}r^{-1}(r, as^{-1}a^{-1}s^{-1}).$$

Note that $(r, as^{-1}a^{-1}s^{-1})$ is an element of θ^*.

To show that $(s, t)(s, a^{-1}) \in \theta^*$, note that $t = c^{\delta}ac^{-\delta}$. By the Witt-Hall identities [3]

$$(s, t)(s, a^{-1}) = (s, ta^{-1})((s, t), a^{-1}).$$

$ta^{-1} \in (A, C), s \in B$ together imply that $(s, ta^{-1}) \in (B, (A, C)) \subseteq \theta^*$. $((s, t), a^{-1}) \in \theta^*$ also. Substitution into the argument in lines (1) through (6) yields the desired result. Q.E.D.

From this, we make the reasonable conjecture that D_{kn} is an $(n - k - 1)$-fold commutator subgroup of elements which are commutators from $n - k - 1$ distinct subgroups in $D_{k+1\ n}$, each isomorphic to $D_{k\ n-1}$. The proof of this is a problem in the commutator calculus which may, perhaps, be solved by mimicking algebraically the geometric constructions in [2].

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