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ON THE FRATTINI SUBGROUPS OF GENERALIZED FREE PRODUCTS AND THE EMBEDDING OF AMALGAMS

BY

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ABSTRACT. In this paper we shall prove a basic relation between the Frattini subgroup of the generalized free product of an amalgam $\mathfrak{A} = (A, B; H)$ and the embedding of $\mathfrak{A}$ into nonisomorphic groups, namely, if $\mathfrak{A}$ can be embedded into two nonisomorphic groups $G_1 = \langle A, B\rangle$ and $G_2 = \langle A, B\rangle$ then the Frattini subgroup of $G = (A * B)_H$ is contained in H. We apply this result to various cases. In particular, we show that if A, B are locally solvable and H is infinite cyclic then $\Phi(G)$ is contained in H.

1. Introduction. In [8] Higman and Neumann asked of the Frattini subgroup $\Phi(G)$ of the generalized free product $G = (A * B)_H$ of two groups A, B amalgamating the subgroup H: (i) Can $\Phi(G) = G$? (ii) If $\Phi(G) \subsetneqq G$, is $\Phi(G) \subseteq H$? Some progress towards answering these questions was made in [4], [12], [13], [14]. Here we first generalize a result in [13] to give the complete solution in the case of A, B being finitely generated nilpotent groups, thus generalizing Theorem 2 of [14] and also Theorem 3.6 of [13]. This is Theorem 3.1.

In §4 a method which can be used to reduce certain general problems to one of studying generalized free products of subamalgams is introduced. It is then used to show as a special case that in any generalized free product of locally solvable groups amalgamating an infinite cyclic subgroup the Frattini subgroup is contained in the amalgamated subgroup. §5 contains a further application of this reduction procedure.

The notation and terminology will be the same as in [13] and are essentially standard. The definitions and results we need concerning permutational products of groups are given in §4. Throughout E is the identity subgroup of any group appearing.

Most of the results presented here have already been announced in [2].

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2. The basic theorem. We first prove the following far reaching result which is of interest in its own right. The proof is essentially an extension of that of Theorem 2.1 in [13] and is, in fact, motivated by that theorem.

THEOREM 2.1. *Let* $G = (A * B)_H$. *If there exists a nontrivial normal subgroup* N *of* G *such that* $N \cap H = E$ *then* $\Phi(G) \subseteq H$.

PROOF. First suppose $1 \neq d \in N \cap \Phi(G)$. Since $N \cap H = E$ we see that $d \notin H$. Then it is easy to find an element $c \in N \cap \Phi(G)$ (see [13, p. 64]), such that c is of the form:

$$c = b_1 a_1 \cdots b_k a_k$$

where $b_i \in B \backslash H$ and $a_i \in A \backslash H$. Clearly $c^2 \neq 1$ in G. We shall show that $c \notin \langle A^{c^2}, B\rangle \subseteq G$. From this it follows that $c \notin \Phi(G)$, whence $N \cap \Phi(G) = E$.

If $c \in \langle A^{c^2}, B\rangle$ then c can be expressed in the following form:

$$\begin{aligned} c &= b_0 c^{-2} a_1 c^2 b_1 \cdots c^{-2} a_n c^2 b_n \\ &= b_0 c^{-1} c^{-1} a_1 c \cdot c b_1 c^{-1} c^{-1} \cdots c^{-1} \cdot c^{-1} a_n c \cdot c b_n. \end{aligned} \tag{1}$$

Here b_0 and b_n may both be 1. We choose the above way of representing c so that n is as small as possible. If $c^{-1} a_1 c \in H$ then because of the choice of c we must have $a_1 \in H$. But then $[a_1, c] \in H \cap N = E$. Thus a_1 and c commute. This means that $c^{-2} a_1 c^2 = a_1 \in H$. It follows that the first form in (1) may be rewritten as:

$$c = b_0^* c^{-2} a_2 c^2 b_2 \cdots c^{-2} a_n c^2 b_n$$

where $b_0^* = b_0 a_1 b_1 \in B$. This contradicts the minimal choice of n. Thus $c^{-1} a_1 c \notin H$, so that after performing as much cancellation and amalgamation as possible $c^{-1} a_1 c$ is seen to be an element of G of odd length beginning and ending with an element from $A \backslash H$.

Similar remarks apply to the terms $c^{-1} a_i c$ $(i = 2, 3, \cdots, n)$ and $c b_i c^{-1}$ $(i = 1, 2, \cdots, n-1)$. Then with c written as in (1) its normal form length as determined by the second form for c in (1) is clearly at least (length b_0 + length $c^{-1} + 2n - 1$ + length c + length b_n). This is clearly impossible for $n \geqslant 1$. On the other hand, $c = b_0$ is also impossible. Hence $c \notin \langle A^{c^2}, B\rangle$. It follows that $N \cap \Phi(G) = E$.

Now suppose $\Phi(G) \not\subseteq H$. Then, as above, there exists an element $x \in \Phi(G)$ of the form $b_1 a_1 \cdots b_r a_r$. Similarly there exists an element $y \in N$ of the form $b'_1 a'_1 \cdots b'_s a'_s$. Here $a_i, a'_i \in A \backslash H$ and $b_i, b'_i \notin B \backslash H$. But

$$[x, y] \in N \cap \Phi(G) = E.$$

This is clearly impossible if $(b'_1)^{-1}b_1 \notin H$. This can easily be arranged by careful choice of x, y except perhaps in the case of $[B : H] = 2$. If $[A : H] \neq 2$, an obvious modification of the above choice of x, y yields noncommuting x', y' whose commutator is in $N \cap \Phi(G) = E$, a contradiction. Hence $\Phi(G) \subseteq H$. If $[A : H] = [B : H] = 2$ we see that $H \lhd G$ so that $\Phi(G) \subseteq H$ follows by mapping G onto the ordinary free product $A/H * B/H$ which has, as in [8], trivial Frattini subgroup. This completes the proof.

An immediate consequence of Theorem 2.1 is the following corollary:

COROLLARY 2.2. *Let $G = (A * B)_H$. If there exists a group $P = \langle A, B\rangle$ embedding $\mathfrak{A}$ such that P is not isomorphic to G then $\Phi(G) \subseteq H$.*

The significance of Theorem 2.1 is now clear, since in general a given amalgam $\mathfrak{A} = (A, B; H)$ has more than one nonisomorphic embedding.

Applying Corollary 2.2 to Corollary 6.5 and Theorems 6.6, 8.4 of [11] we immediately have the following results:

THEOREM 2.3. *Let $G = (A * B)_H$. If A, B have finite exponent and H is central in one of them then $\Phi(G)$ is contained in H.*

THEOREM 2.4. *Let $G = (A * B)_H$. If A is periodic and H is central in A then $\Phi(G)$ is contained in H.*

THEOREM 2.5. *Let $G = (A * B)_H$. If A, B are locally finite and H is central and of countable index in one of them then $\Phi(G)$ is contained in H.*

3. **Finitely generated nilpotent groups.** In [13] the following result was proved (Theorem 3.5(i)):

THEOREM T1. *Let $G = (A * B)_H$ be residually finite. If A, B each satisfies a nontrivial identical relation not satisfied by the infinite dihedral group then $\Phi(G)$ is contained in H.*

Applying this result it was shown in [13] that if $G = (A * B)_H$ where A, B are finitely generated torsion-free nilpotent groups and H is closed in A, B then $\Phi(G)$ is contained in H. We can now generalize this result to the following:

THEOREM 3.1. *Let A, B be finitely generated nilpotent groups and let $G = (A * B)_H$. Then $\Phi(G)$ is contained in H.*

PROOF. Firstly suppose A, B are torsion-free. Then, by Theorem 4 of [3], G is a residually finite extension of a free group F (say), which is such that $F \cap H = E$. If $F \neq E$ then, by Theorem 2.1 (with $F = N$), we have $\Phi(G) \subseteq H$. If, however, $F = E$ then G is itself residually finite whence, by Theorem T1, we again have $\Phi(G) \subseteq H$.

Now let $\tau(A), \tau(B), \tau(H)$ be the torsion subgroups of A, B, H respectively. Then $\tau(A), \tau(B), \tau(H)$ are respectively the sets of elements of finite order in A, B, H so that

$$\tau(A) \cap H = \tau(H) = \tau(B) \cap H.$$

In proving Proposition 1(b) [5, p. 134] Dyer shows that the normal subgroup K of G generated by $\tau(A)$ and $\tau(B)$ is an extension of a free group S by a finite group R, say. If R has order n then K^n (the characteristic subgroup of K generated by the nth powers of all elements of K) is contained in S. Since S clearly misses H $(S \cap H \subseteq \tau(H))$, an application of Theorem 2.1 immediately shows that $\Phi(G) \subseteq H$ if $S \neq E$. If, however, $S = E$ then K is finite and $\tau(A) = \tau(H) = \tau(B)$. In this case $\tau(H) \lhd G$ and $G/\tau(H)$ is the generalized free product $(A/\tau(H) * B/\tau(H))_{H/\tau(H)}$. Clearly $A/\tau(H), B/\tau(H)$ are torsion free. Thus by the first part of the proof $\Phi(G/\tau(H)) \subseteq H/\tau(H)$, whence $\Phi(G) \subseteq H$ as required.

4. The reduction procedure. The reduction procedure mentioned in §1 involves the permutational products of groups introduced by Neumann [10]. Briefly: Let $(A, B; H)$ be an amalgam of groups with an amalgamated subgroup H. Choose (left) coset representatives $S = \{s_i\}$ for A modulo H and $T = \{t_j\}$ for B modulo H and consider the set W of all triples (s_i, t_j, h) where $s_i \in S$, $t_j \in T$ and $h \in H$. To each $a \in A$ define the permutation $\rho(a)$ on W by

$$(s_i, t_j, h)^{\rho(a)} = (s_i', t_j', h')$$

where $s_i'h' = s_i ha$ and $t_j' = t_j$. Then $\rho(A) = \{\rho(a); a \in A\}$ as shown in [10] is an isomorphic copy of A. Similarly one can define $\rho(B)$. It can be shown [10] that $\rho(H)$ is independent of whether one regards H as a subgroup of A or B. In fact $\rho(A) \cap \rho(B) = \rho(H)$ for any choice of transversals [10]. The subgroup of S_W, the unrestricted symmetric group on W, generated by $\rho(A)$ and $\rho(B)$ is called the permutational product on $(A, B; H)$ with transversals $\{s_i\}$, $\{t_j\}$. The isomorphism type of the various permutational products on $(A, B; H)$ depends heavily on the choice of the transversals (see, in particular, [1] and [10]).

Now let $\bar{A}, \bar{B}$ be subgroups contained in A and B respectively such that $\bar{A} \cap H = \bar{B} \cap H$. In general no real relationship exists between the various permutational products on $(A, B; H)$ and those obtainable on the subamalgam $(\bar{A}, \bar{B}; \bar{A} \cap H = \bar{B} \cap H)$ (see [11]). However if $H \subseteq \bar{A} \cap \bar{B}$ then, by choosing the transversals $\{s_i\}$, $\{t_j\}$ by first choosing coset representatives of $\bar{A}$ and $\bar{B}$ modulo H respectively and then representatives of A modulo $\bar{A}$ and B modulo $\bar{B}$, one sees easily that with regard to the transversals so chosen the permutational product on $(\bar{A}, \bar{B}; H)$ coincides with the subgroup of the permutational product on $(A, B; H)$ generated by $\rho(\bar{A})$ and $\rho(\bar{B})$.

A result similar to this holds for generalized free products of groups (see [9]). Our reduction procedure is then as follows. Let $(A, B; H)$ be an amalgam and $(\bar{A}, \bar{B}; H)$ a subamalgam. If we can find a permutational product P on $(\bar{A}, \bar{B}; H)$ which is not isomorphic to $(\bar{A} * \bar{B})_H$ then there exists a permutational product (choosing the transversals $\{s_i\}$, $\{t_j\}$ as described above) on $(A, B; H)$ which is not isomorphic to $(A * B)_H$ for otherwise the subgroup of $(A * B)_H$ generated by $\bar{A}, \bar{B}$ would be simultaneously isomorphic to P and isomorphic to $(\bar{A} * \bar{B})_H$. But then there clearly exists a homomorphism of $(A * B)_H$ to the constructed permutational product on $(A, B; H)$ with a nontrivial kernel. This kernel is the sort of normal subgroup required to apply Theorem 2.1.

As an easy application of these remarks and to indicate more clearly our point of view we prove the following theorem:

THEOREM 4.1. *Let $(A, B; H)$ be an amalgam where A, B each generate varieties other than the variety $\mathbf{0}$ of all groups. If H is properly contained in its centralizer $C_A(H)$ in A and if either $[C_A(H) : H]$ or $[B : H]$ is greater than 2, then $\Phi(A * B)_H$ is contained in H.*

PROOF. Let P be any permutational product on $(C_A(H), B; H)$. By Corollary 3 of [10], $[\rho(C_A(H)), \rho(B)]$ is isomorphic to a subgroup of the direct power of $[C_A(H) : H]$ copies of B. If A, B generated the varieties $\mathbf{V}_1, \mathbf{V}_2$ respectively then $[\rho(C_A(H)), \rho(B)] \in \mathbf{V}_2$ and so $P \in \mathbf{V}_2\mathbf{V}_1\mathbf{V}_2 \neq \mathbf{0}$. Thus $P \not\cong (C_A(H) * B)_H$ (see Lemma G1 below), and consequently there are permutational products on $(A, B; H)$ which are not isomorphic to $G = (A * B)_H$. Mapping G in a natural way onto such a permutational product on $(A, B; H)$ yields a kernel which is a nontrivial normal subgroup of G intersecting H trivially. Applying Theorem 2.1, the theorem follows immediately.

To prove the main results in this section we shall need the following results of Gregorac [6], [7].

LEMMA G1 [7]. *Let $G = (A * B)_H$, where $A \neq H \neq B$, satisfy a nontrivial identical relation. Then $[A : H] = [B : H] = 2$.*

Now let $(A, B; H)$ be an amalgam and U, V be normal subgroups of A, B respectively such that $U \cap H = V \cap H$. Then one can form the factor amalgam $(A/U, B/V; H/U \cap V)$. From now on we shall always choose transversals as follows:

First choose transversals $\{u_k\}$, $\{v_r\}$ for UH and VH modulo H respectively by taking the u_k from U and v_r from V. Now choose transversals $\{s_i\}$ and $\{t_j\}$ for A modulo UH and B modulo VH respectively. Then the $\{s_iu_k\}$ and the $\{t_jv_r\}$ are transversals for A and B modulo H respectively, whilst the $\{s_iU\}$ and the $\{t_jV\}$ are transversals for A/U and B/V

modulo $H/U \cap V$ respectively. Denoting by $\mathfrak{A}$ the amalgam $(A, B; H)$, by $\mathfrak{F}$ the factor amalgam $(A/U, B/V; H/(U \cap V))$ and by $P = P(\mathfrak{A}, S, T)$ the permutational product on $\mathfrak{A}$ using the transversals S, T, we have the following result of Gregorac [6]:

THEOREM G2 [6]. (i) *Let* $\mathfrak{A}, \mathfrak{F}$ *be as above. Let* $N = U = V \subseteq H$ *and let* J *be the semidirect product* $P''N$ *where* P'' *is the automorphism group generated on* N *by* A *and* B. *Then with* $S = \{s_i u_k\}$, $T = \{t_j v_r\}$, $S' = \{s_i N\}$, $T' = \{t_j N\}$ *and* $W' = S' \times T' \times H/N$, P *can be embedded in* $J \operatorname{Wr} (P', W')$ *where* $P' = P(\mathfrak{F}, S', T')$ *permutes the elements of the set* W' *in the obvious manner (see* [6, *p.* 114]) *and* $J \operatorname{Wr} (P', W')$ *is the unrestricted wreath product of* J *and* P'.

(ii) *Using the same notation: If* $U \cap H = V \cap H$ *is central in* A *and in* B *then* P *can be embedded in* $D \operatorname{Wr} (P', W')$ *where* D *is the generalized direct product of* U *and* V *with* $U \cap V$ *amalgamated.*

(i) and (ii) are Theorems 4.1 and 6.1 of [6]. We shall several times use (ii) taking $U = V \cap H = E$ and $VH = B$ so that $D = V$ and P' is then isomorphic to A.

We shall also need the following easy result concerning permutational products of groups.

LEMMA 4.2. *Let* $(A, B; H)$ *be an amalgam where* $A = H \times N$ *is abelian. Then there exists a permutational product which contains* N *in its centre. In particular such a permutational product cannot be isomorphic to* $(A * B)_H$.

PROOF. Choose as coset representatives for A modulo H the elements of N and coset representatives of B modulo H arbitrarily. Then a typical permuted triple is of the form $(n, \hat{b}, h)$ where $n \in N$, $h \in H$ and $\hat{b}$ a coset representative of B modulo H. It is clear that $\rho(N)$ is centralized by $\rho(H)$ in the given permutational product. Let $\hat{b}_0$ be any coset representative of B modulo H and $n_0 \in N$. Then,

$$
\begin{aligned}
(n, \hat{b}, h)^{\rho(n_0)^{-1}\rho(\hat{b}_0)^{-1}\rho(n_0)\rho(\hat{b}_0)} &= (nn_0^{-1}, \hat{b}, h)^{\rho(\hat{b}_0)^{-1}\rho(n_0)\rho(\hat{b}_0)} \\
&= (nn_0^{-1}, \hat{b}_1, h_1)^{\rho(n_0)\rho(\hat{b}_0)} \\
&= (n, \hat{b}_1, h_1)^{\rho(\hat{b}_0)} = (n, \hat{b}, h)
\end{aligned}
$$

where $b_1 h_1 = bhb_0^{-1}$ and hence $\hat{b}_1 h_1 \hat{b}_0 = \hat{b}h$. Thus $[\rho(n_0), \rho(\hat{b}_0)]$ permutes all triples identically. Hence $\rho(N)$ commutes with $\rho(B)$ elementwise. The lemma then follows immediately.

In the following we shall restrict ourselves to the case when the amalgamated subgroup H is infinite cyclic. The case of H being finite cyclic has been completely settled by Tang [12] in which it was shown that $\Phi(G)$ $(G = (A * B)_H$, and H finite cyclic) is precisely given by $\langle \Phi(A) \cap N, \Phi(B) \cap N \rangle$ where N is the maximal G-normal subgroup contained in H. A similar result for H infinite cyclic is not obtainable as shown by the example given on p. 572 of [12].

Our main theorem on embeddings is:

THEOREM 4.3. *Let A, B be solvable groups, $\mathfrak{A} = (A, B; H)$ where H is an infinite cycle and $[A : H]$, $[B : H]$ are not both equal to 2. Then there exists a permutational product on $\mathfrak{A}$ which is not isomorphic to the generalized free product $G = (A * B)_H$ on $\mathfrak{A}$.*

PROOF. Let X, Y denote the last nontrivial terms of the derived series of A, B and let U, V be respectively the terms of these derived series which precede X, Y so that $U' = X$, $V' = Y$. (Note that our theorem is certainly true when one of A, B is abelian (Theorem 4.1) so we may assume throughout that A, B are both solvable of length $\geqslant 2$.) Note also that then $U \not\subseteq H$, $V \not\subseteq H$ since U, V are not abelian. Let $H = \langle h \rangle$.

Case 1. $X \subseteq H$, $Y \subseteq H$ and $[UH : H] = [VH : H] = 2$. Then $[U : U \cap H] = [V : V \cap H] = 2$. Now U is not abelian and $U \cap H \lhd UH$ since $H \lhd UH$. Therefore there exists an element $a \in U \backslash H$ such that $a^2 \in U \cap H$ and $a^{-1}ua = u^{-1}$ where $U \cap H = \langle u \rangle$. Let $a^2 = u^i$ for some integer i. Then

$$a^2 = a^{-1}a^2a = a^{-1}u^ia = u^{-i} = a^{-2}.$$

Thus $a^4 = 1$. But $U \cap H$ is infinite cyclic. It follows that $a^2 = 1$. In the same way there exists an element $b \in V \backslash H$ such that $b^2 = 1$ and $b^{-1}vb = v^{-1}$ where $V \cap H = \langle v \rangle$. Consider the permutational product P on $(UH, VH; H)$ with $\{1, a\}$, $\{1, b\}$ as the chosen transversals. It is easy to check that, in P, $\rho(a)$ and $\rho(b)$ commute whereas a, b do not commute in $(UH * VH)_H$. Hence $P \not\cong (UH * VH)_H$ as required.

Case 2. $X \subseteq H$, $Y \subseteq H$ and at least one of $[UH : H]$ and $[VH : H]$ is greater than 2. Suppose $[UH : H] \geqslant 3$. Choose $u_1, u_2, u_3 \in U$ to be distinct coset representatives of H in UH. Let $\bar{A} = \langle h, u_1, u_2, u_3 \rangle$ and $\bar{U} = U \cap \bar{A}$. Then $[\bar{A} : \bar{U}] < \infty$ and $[\bar{A} : H] \geqslant 3$. Thus $\bar{U}$ is a finitely generated metabelian group containing X. Consequently setting $Z = X \cap Y$, we have $\bar{A}/Z$ is a finite extension of the finitely generated metabelian (whence residually finite) group $\bar{U}/Z$. Thus $\bar{A}/Z$ is residually finite. This means there exists a normal subgroup T of finite index in $\bar{A}$ such that T contains Z and $T/Z \cap H/Z = E$.

First assume $[VH : H] < \infty$. Consider the amalgam $(\bar{A}/Z, VH/Z; H/Z)$. By Theorem G2(ii) we see that, by choosing suitable coset representatives, there exist permutational products P on $(\bar{A}/Z, VH/Z; H/Z)$ and Q on $(\bar{A}/Z/T/Z, VH/Z; H/Z)$ such that P is embeddable in T/Z Wr Q. Now T/Z is a subgroup of a finite extension of a metabelian group and Q is finite. Hence P satisfies a nontrivial identical relation. An application of Theorem G2(i) shows that there exists a permutational product R on $(\bar{A}, VH; H)$ which is embeddable in C_2Z Wr P (again assuming that the coset representatives are chosen correctly and C_2 being cyclic of order 2). Thus R satisfies a nontrivial identical relation. Since $[\bar{A} : H] \geqslant 3$ we have $R \not\cong (\bar{A} * VH)_H$.

If $[VH : H] = \infty$ then certainly $[VH : H] \geqslant 3$. Proceeding as we did above for the group A we set $\bar{B} = \langle h, v_1, v_2, v_3\rangle$ where v_1, v_2, v_3 are three distinct coset representatives of H in VH. We then choose S normal and of finite index in $\bar{B}$ (note $\bar{B}$, like $\bar{A}$, is residually finite) and such that $S/Z \cap H/Z = E$. Now consider the amalgam $(\bar{A}/Z, \bar{B}/Z; H/Z)$ and its finite factor amalgam $(\bar{A}/Z/T/Z, \bar{B}/Z/S/Z; H/Z)$. By Theorem G2(ii) there exist permutational products P, Q on these amalgams such that P is embeddable in $(T/Z \times S/Z)$ Wr Q. We then complete the proof as above.

In the remaining cases we may assume either $X \not\subseteq H$ or $Y \not\subseteq H$. Suppose, without loss of generality, the former.

Case 3. $[XH : H] = 2$ and $Y \subseteq H$. Here $[X : X \cap H] = 2$ and so X has at most two generators. If X is cyclic with generator x then either X has infinite order whence $x^2 = h^s$ or else x has order 2. (For $x^2 \in H$; hence if x is of finite order $x^2 = 1$.) In the first case $h^{-1}xh = x^{\pm 1}$. This implies

$$h^{-1}x^r h = h^{-1}h^s h = h^s = x^r$$

and $h^{-1}xh = x$ follows. Thus XH is abelian. Since U is not abelian we see that $UH \supset XH$ and so $[UH : H] \geqslant 3$. Thus we can proceed as in Case 2 but replacing the subgroup X there by the subgroup $X \cap H$ here, noting that, in Case 2, only the fact that X was a normal cyclic subgroup of A was used. Here $X \cap H$ is characteristic in $X \lhd A$ and so $X \cap H \lhd A$ as required.

If X has order 2 then X is central in XH and $XH = X \times H$ follows. Lemma 4.2 is then applicable.

If X is 2-generator then $[XH : H] = 2$ implies X is the direct product of $X \cap H$ and a 2-cycle S, say. Thus S is central in XH, $XH = H \times S$ and we proceed as for the case where X is of order 2.

Case 4. $[XH : H] = 2$ and $Y \not\subseteq H$. In this case $[YH : H] \geqslant 2$. For the case $[YH : H] > 2$ see the cases below. If $[YH : H] = 2$ we see from Case 3 that XH, YH are both abelian. Thus all permutational products on $(XH, YH; H)$ are abelian [10, Theorem 1].

Case 5. $[XH : H] \geqslant 3$, $\tau(X) \neq E$. Here $\tau(X)$ is characteristic in $X \lhd A$. Hence $\tau(X) \lhd A$ and $\tau(X) \cap H = E$, clearly. Consider $\bar{B} = (\tau(X)H, B; H)$.

By G2(ii), there is a permutational product P on $\bar{B}$ which is embeddable in $\tau(X)$ Wr B, a solvable group. Thus, by G1, P is not isomorphic to $(\tau(X)H * B)_H$ except perhaps if $[\tau(X)H : H] = 2 = [B : H]$. But then $\tau(X)$ is central in $\tau(X)H$ (being normal and of order 2). Hence $\tau(X)H = \tau(X) \times H$ and Lemma 4.2 suffices.

Case 6. $[XH : H] \geqslant 3, \tau(X) = E$ and $Y \not\subseteq H$. The cases $X \cap H = E$ and $Y \cap H = E$ are soon cleared up by appealing to G2(ii) with the subamalgams $(XH, B; H)$, $(A, YH; H)$ respectively. So suppose $X \cap H \supset E$. Since $[XH : H] \geqslant 3$ we can find three distinct coset representatives x_1, x_2, x_3 of XH mod H, all from X. Consider the subamalgam $(\bar{A}, B; H)$ where $\bar{A} = \langle h, x_1, x_2, x_3\rangle$. Set $\bar{X} = X \cap \bar{A}$. Clearly $[\bar{A} : \bar{X}] < \infty$ and so $\bar{X}$ is finitely generated. Further $\bar{X} \cap H = X \cap H \supseteq X \cap Y \supset E$ and so we can find a basis $\bar{x}_1, \cdots, \bar{x}_m$ of X such that $\bar{x}_m^s = \hat{h}$, where $\hat{h}$ is a generator of $X \cap Y$. Then $\bar{X}^s$ is a normal subgroup of finite index in $\bar{A}$ and is such that $\bar{X}^s \cap H = \langle\hat{h}\rangle$. In a like manner we can, since $[YH : H] \geqslant 2$, find in $\cdot Y$ two distinct coset representatives y_1, y_2 of YH mod H. Then set $\bar{Y} = Y \cap \bar{B}$ where $\bar{B} = \langle h, y_1, y_2\rangle$. Then $\bar{Y}$ is finitely generated—say by $\bar{y}_1, \cdots, \bar{y}_n$ —and there exists t such that $\bar{y}_n^t = \hat{h}$. Thus $\bar{Y}^t \cap H = \langle\hat{h}\rangle$. Now $\hat{h}$ is central in $\bar{A}$ and in $\bar{B}$ and so by G2(ii) there is a permutational product P on $(\bar{A}, \bar{B}; H)$ contained inside D Wr (P', W') where D is the generalized direct product of $\bar{X}^s$ and $\bar{Y}^t$, and P' is a permutational product on $(\bar{A}/\bar{X}^s, \bar{B}/\bar{Y}^t; H/\langle\hat{h}\rangle)$ and consequently finite. Thus P satisfies a nontrivial identity and $P \not\approx (A * B)_H$ since, clearly, $[A : H] \geqslant 3$.

Case 7. Finally $[XH : H] \geqslant 3, \tau(X) = E, Y \subseteq H$. Here we choose $\bar{A}$ as in Case 6 but now we cannot choose $\bar{B}$ similarly. Note that $Y \subseteq H$ implies $[H : V \cap H] < \infty$ and so VH is metabelian by finite. We can then find a finitely generated metabelian by finite subgroup $\bar{B}$, say, between H and VH. Thus $\bar{B}$ and $\bar{B}/(X \cap Y)$ are residually finite. Hence we can find $S \subseteq \bar{B}$ such that S is normal and of finite index in $\bar{B}$ and such that $S \cap H = X \cap Y$ (as in Case 2 above). We then proceed as before using the amalgams $(\bar{A}/\langle\hat{h}\rangle, HS/\langle\hat{h}\rangle; H/\langle\hat{h}\rangle)$ and $(\bar{A}/\langle\hat{h}\rangle\big/\bar{X}^s/\langle\hat{h}\rangle, HS/\langle\hat{h}\rangle\big/S/\langle\hat{h}\rangle; H/\langle\hat{h}\rangle)$ where $\langle\hat{h}\rangle = X \cap Y$. If S can be chosen to be in H then $[\bar{B} : H]$ is finite and we replace HS in the above amalgams by $\bar{B}$ itself.

This completes the proof of Theorem 4.3.

The result promised in the Introduction now follows easily.

THEOREM 4.4. *Let* $G = (A * B)_H$. *If* A, B *are locally solvable and* H *is infinite cyclic then* $\Phi(G)$ *is contained in* H.

PROOF. If $H \lhd G$ we are done. If $H \ntriangleleft G$ then there exists an element $a \in A$ (or $b \in B$) such that $a^{-1}ha \notin H$ where $H = \langle h\rangle$. Let $b \in B\backslash H$. Let

$S_A = \langle a, h\rangle$ and $S_B = \langle b, h\rangle$. It is easy to see that $[S_A : H] > 2$. Now S_A and S_B are solvable. Hence, by Theorem 4.3, the theorem follows immediately.

Gregorac [6] has observed that certain amalgams are embeddable only in their generalized free products. Theorem 4.3 may be regarded as a step towards answering the general problem: "For which amalgams $\mathfrak{A}$ is there a permutational product not isomorphic to the generalized free product on $\mathfrak{A}$?" Related to this and the example given in [1] is the question: "If *one* of the permutational products on $\mathfrak{A}$ is isomorphic to the generalized free product on $\mathfrak{A}$, need they all be?"

5. **A further application.** We shall prove some results by direct application of both Whittemore's result [14, Proposition 2.1] and the reduction procedure. In particular, Theorem 5.1 can also be compared with Theorem 1 of [4].

THEOREM W1 [14]. *Let* $G = (A * B)_H$. *If there exists* $x \in G$ *such that* $H \cap H^x = E$ *then* $\Phi(G) = E$.

THEOREM 5.1. *Let* $G = (A * B)_H$ *where* H *is infinite cyclic. If* $[A, H]$ *and* $[B, H]$ *are finitely generated and satisfy the maximal condition on abelian subgroups then* $\Phi(G)$ *is contained in* H. *In particular if* A *and* B *satisfy the maximal condition on subgroups then* $\Phi(G)$ *is contained in* H.

PROOF. Let H^A and H^B be the normal closures of H in A, B respectively. Then $H^A = \langle a^{-1}ha; a \in A\rangle$ and $H^B = \langle b^{-1}hb; b \in B\rangle$ where $H = \langle h\rangle$. Now $H^A = H[H, A]$. Since $[H, A]$ is finitely generated, therefore, H^A is finitely generated. In particular, let H^A be generated by $a_1^{-1}ha_1, \cdots, a_u^{-1}ha_u$, where $a_j \in A$ and u is finite. If there exists a_j such that $\langle a_j^{-1}ha_j\rangle \cap H = E$ then, by Theorem W1, $\Phi(G) = E$. Thus we can assume that for all $j = 1, \cdots, u$, there exist integers r_j and s_j such that $a_j^{-1}h^{r_j}a_j = h^{s_j}$. Let $s = s_1s_2 \cdots s_u$ then h^s is central in H^A. In the same way there exists an integer t such that h^t is central in H^B. Now let C_A, C_B be the centres of H^A, H^B respectively. Clearly $C_A \cap C_B \neq E$.

Case 1. $C_A \subseteq H, C_B \subseteq H$. Let $Z = C_A \cap C_B$. Then Z is characteristic in C_A which is characteristic in $H^A \lhd A$. Then $Z \lhd A$. Similarly $Z \lhd B$. Hence $Z \lhd G$. Considering G/Z we have, by Theorem 1 [4], $\Phi(G/Z) \subseteq H/Z$, whence $\Phi(G) \subseteq H$.

Case 2. $C_A \nsubseteq H, C_B \nsubseteq H$. Consider the subamalgam $(HC_A, HC_B; H)$. Since HC_A and HC_B are abelian, therefore, the permutational products on $(HC_A, HC_B; H)$ are isomorphic to the generalized direct product on the subamalgam. Hence by the reduction process and Theorem 2.1 we have proved this case.

Case 3. $C_A \nsubseteq H$ (say), $C_B \subseteq H$. We first note that $[A, H]$ and $[B, H]$

satisfy the maximal condition on abelian subgroups. It is not difficult to see that HC_A, being abelian, satisfies the maximal condition on subgroups, whence HC_A is cyclic if it is locally cyclic.

Now suppose HC_A has a subgroup N (hence normal in HC_A since HC_A is abelian) such that $N \cap H = E$. Then $NH = N \times H$. Thus applying the reduction process and Lemma 4.2 we are done. This is the case in particular when HC_A has elements of finite order when we can take $N = \tau(HC_A)$ or when HC_A is not locally cyclic. For, in this latter case, there exists a two generator subgroup $\overline{H}$ such that $H \subsetneqq \overline{H} \subseteq HC_A$ with $\overline{H}$ not cyclic. Thus there exists a basis $\{\overline{h}_1, \overline{h}_2\}$ for $\overline{H}$ such that $H = \langle \overline{h}_2^r \rangle$ for some integer r. Writing $H_1 = \langle \overline{h}_1 \rangle$ and using $(H_1H, B; H)$ as the subamalgam we have reduced it to the above situation.

On the other hand if HC_A is locally infinite cyclic then, as observed earlier, HC_A must be cyclic. It follows that C_A is infinite cyclic, whence $Z = C_A \cap C_B$ would be a nontrivial normal subgroup in G. Considering G/Z and applying Theorem 1 [4] we immediately have that $\Phi(G)$ is contained in H.

This completes the proof.

It is to be noted that the maximal condition on abelian subgroups of $[A, H]$ and $[B, H]$ is needed mainly because we cannot prove Case 3 of the above theorem when HC_A is locally cyclic but not cyclic. If we can prove that $\Phi(G)$ is contained in H when HC_A is locally cyclic but not cyclic then Theorem 5.1 can be strengthened by dropping the maximal condition on subgroups.

6. **Final comments.** A proof of Theorem 3.1 can also be obtained by applying the reduction process. The proof is then much longer than that presented. However, using the reduction process we can generalize the theorem somewhat, namely, the condition that A, B be finitely generated can be replaced by $\tau(H)$ finite and $\tau(H) \subsetneqq \tau(A)$ or $\tau(B)$. We omit the details. Moreover the reduction process also enables one to prove the above results for amalgams containing more than two groups with a single amalgamated subgroup.

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