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DUALITY THEORIES FOR METABELIAN LIE ALGEBRAS. II

BY

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ABSTRACT. In this paper I have replaced one of the axioms given in my *Duality theory for metabelian Lie algebras* (Trans. Amer. Math. Soc. 187 (1974), 89–102) concerning duality theories by a considerably more natural assumption which yields identical results—a uniqueness theorem.

1. **Introduction.** In [3] I defined and proved a uniqueness theorem for what I referred to as an algebraic duality theory for metabelian Lie algebras. In this paper I will investigate the duality theories which arise using axioms (I), (II), (III) and (V) of [3]. Axiom (V) was an easy consequence of the first four, and is really a more natural thing than (IV) to require of a duality theory. It will be shown that these weaker yet more natural axioms give rise to nearly identical results. To classify all metabelian Lie algebras N such that cod $N^2 = g$ one must consider the cases where $0 \leqslant \dim N^2 \leqslant \binom{g}{2}$. A duality theory nearly cuts this problem in half in the sense that classifying those N such that $\dim N^2 = p$, is the same as classifying their duals N_D which satisfy $\dim N_D^2 = \binom{g}{2} - p$. A duality theory shuffles among themselves those N for which $\dim N^2 = \binom{g}{2}/2$, and apparently contributes nothing to their classification. Using axioms (I), (II), (III), and (V) for the definition of an algebraic duality theory, we prove

MAIN THEOREM. *There is only one algebraic duality theory.*

The assumption that the ground field k is algebraically closed and of characteristic zero, together with the notation of §§1 and 2 of [3], remain in force throughout this article. [2], [3] and [5] provide a suitable background.

Suppose D_1, D_2 are duality theories satisfying axioms (I), (II), (III) and (V) of [3] and let V be a finite-dimensional k-vector space. Let $G_p(\bigwedge^2 V)$ represent the projective Grassmann variety of all p-dimensional subspaces of $\bigwedge^2 V$. Consider $D_1, D_2 : G_p(\bigwedge^2 V) \to G_{m-p}(\bigwedge^2(V^*))$ where $m = \dim \bigwedge^2 V$. By (II), $D_2^{-1} \circ D_1 = \rho$ is an automorphism of $G_p(\bigwedge^2 V)$. We can assume $p \leqslant \dim \bigwedge^2 V/2$ for otherwise we could consider $D_1 \circ D_2^{-1}$. Suppose $S_1, S_2 \in G_p(\bigwedge^2 V)$ and $V \oplus \bigwedge^2 V/S_1 \cong V \oplus \bigwedge^2 V/S_2$ (see §1 and Theorem 1 of [3]). Then by

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(V) for D_1 we have

$$V^* \oplus \bigwedge^2(V^*)/D_1(S_1) \cong V^* \oplus \bigwedge^2(V^*)/D_1(S_2).$$

By (V) and (III) for D_2, we see that

$$V \oplus \bigwedge^2 V/D_2^{-1}(D_1(S_1)) \cong V \oplus \bigwedge^2 V/D_2^{-1}(D_1(S_2)).$$

Thus, due to Theorem 1 of [3], the automorphism ρ of $G_p(\bigwedge^2 V)$ has the property that if $S_1, S_2 \in G_p(\bigwedge^2 V)$ and $S_1 \equiv S_2(GL(V))$, then $\rho(S_1) \equiv \rho(S_2)(GL(V))$. That is, ρ induces a bijection of the quotient set $G_p(\bigwedge^2 V)/GL(V)$. (Axiom (IV) of [3] had the effect of making ρ centralize the image of $GL(V)$ in Aut $G_p(\bigwedge^2 V)$. See §1 of [3].)

To establish a framework for the solution of this problem I wish to digress and consider the following situation. Let G be an algebraic group acting on a variety $\mathcal{V}$ by a morphism $G \times \mathcal{V} \to \mathcal{V}$. Then there is an induced map $G \to$ Aut $\mathcal{V}$—the group of automorphisms of $\mathcal{V}$. I wish to consider those automorphisms of $\mathcal{V}$ compatible with the G-orbit structure. The set $\mathcal{V}/G$ cannot always be given the structure of a variety in such a way that $\mathcal{V} \to \mathcal{V}/G$ is a morphism (in particular this would require orbits in $\mathcal{V}$ to be closed; see [1]). Therefore, the best one seems to be able to ask in general is: what is the subgroup $\mathcal{S}$ of Aut $\mathcal{V}$ consisting of those automorphisms which induce bijections on $\mathcal{V}/G$. $\mathcal{S}$ is called the orbit shuffling group. Certainly $\mathcal{S} \supseteq G$ (identifying G with its image in Aut $\mathcal{V}$), and sitting in between these two groups is the group $\mathcal{F}$ which consists of those automorphisms stabilizing each orbit. $\mathcal{F}$ is called the orbit fixing group. Thus $G \subseteq \mathcal{F} \subseteq \mathcal{S}$, and $\mathcal{F}$ is normal in $\mathcal{S}$ since it is the kernel of the homomorphism $\mathcal{S} \to \mathrm{Bij}(\mathcal{V}/G)$. In fact, $\mathcal{S}$ is the normalizer of $\mathcal{F}$ in Aut $\mathcal{V}$. For this, suppose $\tau \in$ Aut $\mathcal{V}$ normalizes $\mathcal{F}$, and let $v_1, v_2 \in \mathcal{V}$ satisfy $\theta(v_1) = v_2$ for some $\theta \in G$. Then $\tau(v_2) = \tau\theta(v_1) = \theta'\tau(v_1)$ for some $\theta' \in \mathcal{F}$. Since θ' fixes G-orbits, there is a $\theta'' \in G$ such that $\theta'(\tau(v_1)) = \theta''(\tau(v_1))$ and this completes the argument that $\tau \in \mathcal{S}$.

Consider this situation for the action $GL(V) \times G_p(\bigwedge^2 V) \to G_p(\bigwedge^2 V)$ where $p \leqslant \dim \bigwedge^2 V/2$, and let $\mathcal{S}, \mathcal{F}$ be the orbit shuffling and fixing groups respectively. From the preceding discussion, the determination of the possible duality theories satisfying (I), (II), (III) and (V) of [3] for this particular V and p is precisely the problem of determining $\mathcal{F}$ and $\mathcal{S}$, each coset of $\mathcal{F}$ in $\mathcal{S}$ giving rise to a distinct theory in the following way. Let $\rho \in \mathcal{S}$, let

$$D\colon\ G_p(\bigwedge^2 V) \to G_{m-p}(\bigwedge^2(V^*)), \qquad D\colon\ G_{m-p}(\bigwedge^2(V^*)) \to G_p(\bigwedge^2 V)$$

($m = \dim \bigwedge^2 V$) be the Scheuneman-Gauger duality (see §1 of [3]). Then as ρ runs over a set of distinct coset representatives of $\mathcal{F}$ in $\mathcal{S}$, the pairs

$$D \circ \rho\colon\ G_p(\wedge^2 V) \to G_{m-p}(\wedge^2(V^*)), \qquad \rho^{-1} \circ D\colon\ G_{m-p}(\wedge^2(V^*)) \to G_p(\wedge^2 V)$$

exhaust the possible theories for this V and p. If two such theories for a ρ_1 and ρ_2 were to agree, that is,

$$V^* \oplus \wedge^2(V^*)/D(\rho_1(S)) \cong V^* \oplus \wedge^2(V^*)/D(\rho_2(S)) \qquad \text{for all } S \in G_p(\wedge^2 V),$$

then by a routine calculation (Theorem 1 of §1 of [3]) $\rho_1^{-1} \circ \rho_2 \in \mathcal{F}$, $\rho_1 \mathcal{F} = \rho_2 \mathcal{F}$ which is impossible.

So let V be a g-dimensional k-vector space and let $1 \leqslant p \leqslant \binom{g}{2}/2$. Consider the diagram

$$\begin{array}{ccccc}
 & & GL(\wedge^p \wedge^2 V) & & \\
 & & | & & \\
\operatorname{Aut} G_p(\wedge^2 V) & \xleftarrow{\ \pi\ } & \operatorname{Stab} D_p(\wedge^2 V) & \xleftarrow{\ \wedge^p\ } & GL(\wedge^2 V) \\
| & & | & & | \\
S & \longleftarrow & S_1 & \longleftarrow & S_2 \\
| & & | & & | \\
\mathcal{F} & \longleftarrow & \mathcal{F}_1 & \longleftarrow & \mathcal{F}_2 \\
| & & | & & | \\
\pi \wedge^p \wedge^2 GL(V) & \longleftarrow & \wedge^p \wedge^2 GL(V) & \longleftarrow & \wedge^2 GL(V)
\end{array}$$

where π is the natural map, Stab $D_p(\wedge^2 V)$ is the stabilizer of the set $D_p(\wedge^2 V)$ of decomposable p-vectors of $\wedge^p \wedge^2 V$, $S_1 = \pi^{-1}(S)$, $\mathcal{F}_1 = \pi^{-1}(\mathcal{F})$, $S_2 = (\wedge^p)^{-1}(S_1)$, $\mathcal{F}_2 = (\wedge^p)^{-1}(\mathcal{F}_1)$ and all other symbols have the same significance as in §§1 and 2 of [3]. For the sake of simpler notation I have written $\mathcal{F}_2$, S_2 instead of something like $\mathcal{F}_{2,p}$, $S_{2,p}$ which would indicate more explicitly that a priori there is no reason to believe the $\mathcal{F}_2$'s and S_2's for different p's are related. By Proposition 3 of [3], π is surjective. By Westwick's theorem (p. 1127 of [6, Theorem and the preceding remark]), $\wedge^p GL(\wedge^2 V) = \operatorname{Stab} D_p(\wedge^2 V)$ when $p < \dim \wedge^2 V/2$, and when $p = \dim \wedge^2 V/2$ then $(\operatorname{Stab} D_p(\wedge^2 V) : \wedge^p GL(\wedge^2 V)) = 2$. The determination of $\mathcal{F}$ and S is thus nearly always the same as determining $\mathcal{F}_2$ and S_2. $S_2(\mathcal{F}_2)$ is the group of linear transformations on $\wedge^2 V$ shuffling (fixing) $GL(V)$-orbits of p-dimensional subspaces.

2. **Determination of the orbit shuffling and fixing groups when** $\dim V \neq 4$. Let $g = \dim V$ and suppose $g \neq 4$. We show first that $\mathcal{F}_2 = \wedge^2 GL(V)$ relying heavily on Westwick's theorem [6, p. 1127] and the following algebro-geometric lemma whose proof was suggested to me by John Fogarty.

LEMMA 1. *Let W be an n-dimensional k-vector space and let $D \subset W$ be a d-dimensional homogeneous affine subset (not necessarily irreducible). For any*

$x \in W - D$ and any p satisfying $1 \leqslant p \leqslant n - d$, there is a p-dimensional subspace S containing x such that $S \cap D = \{0\}$.

PROOF. By the dimension of D we mean the maximal dimension of its irreducible components. Go by induction on n. Let H_x be the set of all hyperplanes of W containing x, a closed irreducible subset of the Grassmannian of all hyperplanes of W. Write $D = D_1 \cup \cdots \cup D_m$ where the D_i are the irreducible components of D. For each i it is easy to find a hyperplane containing x but not all of D_i. Thus if we call $H_{x,i}$ the collection of all hyperplanes through x but not containing D_i, we see that $H_{x,i}$ is a nonempty open subset of H_x. Hence $\bigcap_i H_{x,i}$ is nonempty, that is, there is a hyperplane W_1 through x which contains none of the D_i. Then consider W_1 and $D' = D \cap W_1$, and observe that $\dim D' \leqslant d - 1$ since each component of D is reduced by the intersection with W_1. Apply the induction hypothesis to complete the proof.

Since the decomposable 2-vectors in $\bigwedge^2 V$ are a homogeneous affine subvariety of $\bigwedge^2 V$ of dimension $2(g-2)+1$, we get the

COROLLARY 2. *Let x be a nondecomposable vector in $\bigwedge^2 V$ and suppose $\dim V = g$. For any p satisfying $1 \leqslant p \leqslant \binom{g}{2} - 2g + 3$ there is a p-dimensional subspace S containing x but no nonzero decomposable vectors.*

PROPOSITION 3. *If $1 \leqslant p \leqslant \binom{g}{2} - 2g + 3$ and $g \neq 4$ then $F_2 = \bigwedge^2 GL(V)$.*

PROOF. Since $g \neq 4$, by Westwick's theorem [6, p. 1127] $\bigwedge^2 GL(V)$ is the stabilizer in $GL(\bigwedge^2 V)$ of the set of all decomposable 2-vectors. Let $\rho \in F_2$, that is, ρ stabilizes each $GL(V)$-orbit of p-dimensional subspaces. Suppose however that ρ does not stabilize the set of decomposable 2-vectors in $\bigwedge^2 V$. In particular, let x be a decomposable vector in $\bigwedge^2 V$ such that $\rho(x)$ is nondecomposable. By Corollary 2, we can find a p-dimensional subspace S containing $\rho(x)$ but no nonzero decomposable. Now $\rho^{-1}(S)$ is a p-dimensional subspace containing x and ρ stabilizes $GL(V)$-orbits so there is a $\theta \in GL(V)$ with $\bigwedge^2(\theta)(\rho^{-1}(S)) = S$. This is clearly impossible since $\bigwedge^2(\theta)$ takes decomposables to decomposables and S has no nonzero decomposables. Hence ρ stabilizes the set of decomposable vectors; this stabilizer is $\bigwedge^2 GL(V)$.

Since we only need to pin down F_2 when $1 \leqslant p \leqslant \binom{g}{2}/2$, we need to check when $\binom{g}{2} - 2g + 3 \geqslant [\binom{g}{2}/2]$ (i.e. greatest integer). An easy computation reveals this is the case whenever $g \geqslant 7$. Excluding $g = 4$, the values of g, p for which F_2 is not determined by Proposition 3 are: $g = 3, p = 1, g = 5, p = 4$ or 5, and $g = 6, p = 7$.

I am forced to handle these remaining cases by a combination of ad hoc arguments which only serve to point out that if one was suitably good at Grassmannian geometry, a single proof for all cases could probably be given.

If $\dim V = g = 3$, then $\dim \bigwedge^2 V = 3$ and $GL(\bigwedge^2 V) = \bigwedge^2 GL(V)$. Thus $F_2 = \bigwedge^2 GL(V)$ when $g = 3, p = 1$.

Notice that if $\dim V = g$, there are subspaces S of $\bigwedge^2 V$ of dimension up to $(g-1)$ which consist entirely of decomposable vectors. Namely, they are subspaces of a subspace of the type $v \wedge V$ where v runs over V. A subspace $v \wedge V$ will be called an M_δ subspace.

So suppose $g = 5$ and $p = 4$, and let $\rho \in F_2$. If ρ does not preserve decomposables there is an $x \wedge y$ such that $\rho(x \wedge y)$ is not decomposable. By the paragraph above we can pick an M_δ subspace S containing $x \wedge y$ and $\dim S = 4$. Since $\rho \in F_2$ there is a $\theta \in GL(V)$ with $\bigwedge^2(\theta)(S) = \rho(S)$. This is clearly impossible since $\bigwedge^2(\theta)$ takes M_δ's to M_δ's and $\rho(S)$ is obviously not an M_δ. Hence ρ preserves decomposables. As before, this forces $F_2 = \bigwedge^2 GL(V)$.

The remaining cases are handled using the following result.

LEMMA 4. *Let $S \subset \bigwedge^2 V$ be an s-dimensional subspace of codimension at least two and containing no M_δ subspace. If $2(g-2) - s \geqslant 2$ where $g = \dim V$, then there is an $(s+1)$-dimensional subspace T containing S, and having no M_δ subspace.*

PROOF. An M_δ is $(g-1)$-dimensional. Let v be any vector in $\bigwedge^2 V - S$. If $S + \langle v \rangle$ has no M_δ subspace we are done. So suppose $S + \langle v \rangle \supseteq z \wedge V$ for some $z \in V$. Since $z \wedge V \not\subseteq S$ we know that $(z \wedge V) \cap S$ is $(g-2)$-dimensional. Let $w \in V$ be such that $z \wedge w \notin S$.

Now let T be any other $(s+1)$-dimensional space containing S and suppose it has an M_δ subspace $y \wedge V$. Then $(y \wedge V) \cap S$ is again $(g-2)$-dimensional, and since $2(g-2) - s \geqslant 2$, we have $(y \wedge V) \cap (z \wedge V) \supseteq (y \wedge V) \cap (z \wedge V) \cap S$ is at least 2-dimensional. That is, for some independent sets $\{z', z''\}, \{y', y''\}$ in V, we have $z \wedge z' = y \wedge y', z \wedge z'' = y \wedge y''$. In terms of subspaces of V these equations read $\langle z, z' \rangle = \langle y, y' \rangle, \langle z, z'' \rangle = \langle y, y'' \rangle$. Intersecting these results, we see that $\langle z \rangle = \langle y \rangle$ and $z \wedge V = y \wedge V$. In other words, the only M_δ in any $(s+1)$-dimensional subspace T containing S, is $z \wedge V$. Since $\operatorname{cod} S \geqslant 2$ we can enlarge S to an $(s+1)$-dimensional space T such that $z \wedge w \notin T$. Hence $z \wedge V \not\subseteq T$, and by preceding remarks this is sufficient to guarantee that T has no M_δ subspace.

Now suppose $g = 5, p = 5$, or $g = 6, p = 7$. Let $\rho \in F_2$. If ρ does not preserve decomposables there is an M_δ subspace $z \wedge V$ such that $\rho(z \wedge V)$ is not an M_δ subspace (in fact, since $\rho(z \wedge V)$ is $(g-1)$-dimensional, it contains no M_δ subspace). By one or two applications of Lemma 4 we can enlarge $\rho(z \wedge V)$ to a p-dimensional subspace T containing no M_δ subspace. Set

$S = \rho^{-1}(T)$ and note that $z \wedge V \subset S$. Since $\rho \in F_2$, there is a $\theta \in GL(V)$ satisfying $\wedge^2(\theta)(S) = T$. But $\wedge^2(\theta)$ takes M_δ's to M_δ's and T contains no M_δ's. This contradiction shows ρ stabilizes the set of decomposable vectors and as before $F_2 = \wedge^2 GL(V)$. We have thus proven part of

PROPOSITION 5. *When* $g \neq 4$, $F_2 = \wedge^2 GL(V) = S_2$.

PROOF. The preceding lemma and discussion show that $F_2 = \wedge^2 GL(V)$. Every element of S_2 normalizes $F_2 = \wedge^2 GL(V)$. Considering the action of $\wedge^2 GL(V)$ on $\wedge^2 V$ this says that S_2 shuffles $\wedge^2 GL(V)$-orbits. There are exactly $[g/2] + 1$ such orbits (see §5 of [2]); $\{0\}$, $O_1, \cdots, O_{[g/2]}$, and O_k consists of those vectors which can be written as a sum of k-decomposables but not less than k. (Under the isomorphism $\wedge^2 V \simeq \mathrm{Alt}(V^*)$ (§4 of [2]) the elements of O_k correspond to forms of rank $2k$ on V^*.) For each $i < j$, $O_i \subset \mathrm{Cl}(O_j)$ where Cl denotes Zariski-closure. Thus, as varieties, the various O_i have different dimensions. So if $\sigma \in S_2$ shuffles orbits, by a dimension argument it fixes them. In particular it fixes O_1—the set of decomposable vectors. Westwick's result then forces $S_2 = \wedge^2 GL(V)$.

We are now prepared to prove

THEOREM 6. *Suppose* $g \neq 4$. *If* $p < \dim \wedge^2 V/2$ *then* $S = F = \pi\wedge^p\wedge^2(GL(V))$. *If* $p = \dim \wedge^2 V/2$ *then* $(S : \pi\wedge^p\wedge^2(GL(V))) \leq 2$, *hence* $(S : F) \leq 2$.

PROOF. If $p < \dim \wedge^2 V/2$ then both π and $\wedge^p$ in the diagram at the end of §1 are surjective. So the result follows from Proposition 5. If $\dim \wedge^2 V/2 = p$, then $(\mathrm{Stab}\, D_p(\wedge^2 V) : \wedge^p GL(\wedge^2 V)) = 2$ by Westwick's result. Hence, since $\wedge^p(S_2) = S_1 \cap (\wedge^p GL(\wedge^2 V))$, we get $(S_1 : \wedge^p S_2) \leq 2$. But $\wedge^p(S_2) = \wedge^p\wedge^2 GL(V)$, so $(S_1 : \wedge^p\wedge^2 GL(V)) \leq 2$. Now π is always surjective, so in addition $(S : \pi\wedge^p\wedge^2 GL(V)) \leq 2$, and obviously $(S : F) \leq 2$.

3. Determination of the orbit shuffling and fixing groups when $\dim V = 4$. When $\dim V = 4$, $\dim \wedge^2 V = 6$ and $p = 1, 2$, or 3.

For the cases $p = 1$ and $p = 2$ we rely on the classification of the algebras N such that $\mathrm{cod}\, N^2 = 4$ and $\dim N^2 = 5$ and 4 respectively (§5 and Theorem 7.12 of [2]).

There are exactly two orbits in $G_1(\wedge^2 V)$ when $\dim V = 4$, the orbit O_1 of $\langle x_1 \wedge x_2 \rangle$ and the orbit O_2 of $\langle x_1 \wedge x_2 + x_3 \wedge x_4 \rangle$ where $x_1, \cdots, x_4$ is any basis of V. Now $O_1 \subset \mathrm{Cl}(O_2)$ since the nondecomposable vectors in $\wedge^2 V$ are open. Thus $\dim O_1 < \dim O_2$. (The closure of an orbit O is the union of O and orbits of strictly smaller dimension.) Hence any automorphism of $G_1(\wedge^2 V)$ which shuffles orbits must fix them.

There are exactly three orbits in $G_2(\bigwedge^2 V)$ when $\dim V = 4$, the orbit $\mathcal{O}_1$ of $\langle x_1 \wedge x_2, x_3 \wedge x_4\rangle$, the orbit $\mathcal{O}_2$ of $\langle x_1 \wedge x_4 + x_2 \wedge x_3, x_2 \wedge x_4\rangle$ and the orbit $\mathcal{O}_3$ of $\langle x_2 \wedge x_4, x_3 \wedge x_4\rangle$ where $x_1, \cdots, x_4$ is any basis of V. The dimensions of these three orbits are pairwise distinct. This can be shown by computing the ranks of the Jacobian at $1_{GL(V)}$ of the maps $\rho_i : GL(V) \to \bigwedge^2\bigwedge^2 V$ where $\rho_i(g) = \bigwedge^2(\bigwedge^2(g))(v_i)$ and $v_1 = (x_1 \wedge x_2) \wedge (x_3 \wedge x_4)$, $v_2 = (x_1 \wedge x_4 + x_2 \wedge x_3) \wedge (x_2 \wedge x_4)$, $v_3 = (x_2 \wedge x_4) \wedge (x_3 \wedge x_4)$ since $\dim \mathcal{O}_i = (\operatorname{rank} J\rho_i(1)) - 1$. Since these computations are tiresome at best, I will indicate here a proof which is satisfactory when $k = \mathbf{C}$—the field of complex numbers. In this case any variety is also endowed with a strong topology (Chapter 1, §10 of [4]) and for any constructible subset of a variety, its closure in the strong topology is the same as its Zariski-closure. In particular, this holds for the orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ since they are locally closed subvarieties [2, Lemma 7.7] of $G_2(\bigwedge^2 V)$. Holding x_2, x_3, x_4 fixed and letting x_1 approach x_4, we see that $\mathcal{O}_3$ is contained in the strong closure of $\mathcal{O}_1$ hence $\mathcal{O}_3 \subset \mathrm{Cl}(\mathcal{O}_1)$ where Cl is Zariski-closure. Similarly, since $\langle x_2 \wedge x_3, x_2 \wedge x_4\rangle$ is a representative of $\mathcal{O}_3$, fixing x_2, x_3, x_4 and letting x_1 approach x_4 we see that $\mathcal{O}_3$ is contained in the strong closure of $\mathcal{O}_2$, hence $\mathcal{O}_3 \subset \mathrm{Cl}(\mathcal{O}_2)$. Thus $\dim \mathcal{O}_3 < \dim \mathcal{O}_1$, $\dim \mathcal{O}_2$. Also $G_2(\bigwedge^2 V) = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 = \mathrm{Cl}(\mathcal{O}_1) \cup \mathrm{Cl}(\mathcal{O}_2)$ is irreducible in the Zariski topology, so $G_2(\bigwedge^2 V) = \mathrm{Cl}(\mathcal{O}_1)$ or $\mathrm{Cl}(\mathcal{O}_2)$. That is, either $\mathcal{O}_1 \subset \mathrm{Cl}(\mathcal{O}_2)$ or $\mathcal{O}_2 \subset \mathrm{Cl}(\mathcal{O}_1)$. In either case $\dim \mathcal{O}_2 \neq \dim \mathcal{O}_1$. Thus, since the orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ have pairwise distinct dimensions, any automorphism of $G_2(\bigwedge^2 V)$ which shuffles orbits must fix them.

We have thus seen that for $g = 4$ and $p = 1$ or 2 the groups $\mathcal{S}$ and $\mathcal{F}$ are identical. For the case $g = 4$, $p = 3$ we consider again the diagram of groups at the end of §1.

Since $\dim V = 4$, $\bigwedge^2 GL(V)$ is of index two in the stabilizer of the set of decomposable two-vectors—Stab D_2. So suppose $\rho \in \mathcal{F}_2$ and ρ does not preserve decomposables. Repeating the arguments of the case $g = 5$, $p = 4$ of §2 we arrive at a contradiction. So ρ does stabilize decomposables. Hence $(\mathcal{F}_2 : \bigwedge^2 GL(V)) \leqslant 2$. But $\bigwedge^2 GL(V)$ is connected, so it is the identity component of $\mathcal{F}_2$. Now $\mathcal{S}_2$ normalizes $\mathcal{F}_2$, hence its identity component $\bigwedge^2 GL(V)$. Repeating the arguments of Proposition 5, we see that $\mathcal{S}_2$ preserves decomposables, thus $(\mathcal{S}_2 : \bigwedge^2 GL(V)) \leqslant 2$. Since $3 = \binom{4}{2}/2$, $\bigwedge^3 GL(\bigwedge^2 V)$ is of index 2 in Stab $D_3(\bigwedge^2 V)$, and thus $(\mathcal{S}_1 : \bigwedge^p \mathcal{S}_2) \leqslant 2$. Hence

$$(\mathcal{S}_1 : \textstyle\bigwedge^p \bigwedge^2 GL(V)) = (\mathcal{S}_1 : \bigwedge^p \mathcal{S}_2)(\bigwedge^p \mathcal{S}_2 : \bigwedge^p \bigwedge^2 GL(V))$$
$$\leqslant (\mathcal{S}_1 : \textstyle\bigwedge^p \mathcal{S}_2)(\mathcal{S}_2 : \bigwedge^2 GL(V)) \leqslant 2 \cdot 2 = 4.$$

Now π is always surjective so $(S : \pi\wedge^p\wedge^2 GL(V)) \leqslant 4$. By the first and second main theorems of [3], however, $(F : \pi\wedge^p\wedge^2 GL(V)) \geqslant 2$. That is, the automorphism ρ of the first main theorem when $\dim V = 4$, $p = 3$ centralizes $\pi\wedge^p\wedge^2 GL(V)$, hence shuffles orbits. But in the second main theorem it is shown ρ fixes orbits. Thus $\rho \in F - \pi\wedge^p\wedge^2 GL(V)$. Hence $(S : F) \leqslant 2$. We have thus proved

THEOREM 7. *Suppose* $g = \dim V = 4$. *When* $p = 1$ *or* 2 *then* $S = F$, *and when* $p = 3 = \binom{g}{2}/2$ *then* $(S : F) < 2$.

4. The uniqueness theorem. To establish the uniqueness it remains only to show $S = F$ when $p = \binom{g}{2}/2$. We know $(S : F) \leqslant 2$ in these cases, and in fact $(S : F) = (S_1 : F_1)$ since π is surjective and its kernel belongs to F_1. Now S_1 normalizes F_1, hence its identity component $\wedge^p\wedge^2 GL(V)$. Likewise, since $\wedge^p\wedge^2 SL(V)$ is the commutator subgroup of $\wedge^p\wedge^2 GL(V)$, S_1 normalizes it as well.

Suppose that $S_1 \neq F_1$ and pick any s in S_1. Now s stabilizes $\wedge^p\wedge^2 SL(V)$ by conjugation, so it stabilizes the corresponding Lie algebra $d(\wedge^p\wedge^2) sl(V)$ by conjugation ($d(\wedge^p\wedge^2)$ is the differential of $\wedge^p\wedge^2$), inducing on it an automorphism. Now $d(\wedge^p\wedge^2)$ is an isomorphism of $sl(V)$ onto $d(\wedge^p\wedge^2) sl(V)$, and $sl(V)$ has basically two types of automorphisms. The first are conjugations by elements of $GL(V)$, the second are compositions of a conjugation with the operation of taking a matrix to its negative transpose.

If the conjugation of s on $d(\wedge^p\wedge^2) sl(V)$ is an automorphism of the first type, then multiplying s by a suitable element of $\wedge^p\wedge^2 GL(V)$ (only its coset matters) we can presume it centralizes $d(\wedge^p\wedge^2) sl(V)$, hence $d(\wedge^p\wedge^2) gl(V)$. The associative algebra spanned by $d(\wedge^p\wedge^2) gl(V)$ contains $\wedge^p\wedge^2 GL(V)$, so s centralizes it. By the first main theorem of [3] (second main theorem), when $g \neq 4$ ($g = 4$), this contradicts $(S_1 : F_1) = 2$.

So suppose the conjugation of s on $d(\wedge^p\wedge^2) sl(V)$ is of the second type. Multiplying s by a suitable element of $\wedge^p\wedge^2 GL(V)$, and identifying $d(\wedge^p\wedge^2) sl(V)$ with $sl(V)$, we can assume conjugation by s induces $-\mathrm{Id}$ on the Cartan subalgebra H of diagonal matrices of trace zero. Thus, as a linear transformation on $\wedge^p\wedge^2 V$, s must take any H-weight space to an H-weight space whose weight is the negative of the original one. Since the groups S_1 and $\wedge^p\wedge^2 GL(V)$ have identical Lie algebras, and since the groups and their algebras stabilize the same subspaces of $\wedge^p\wedge^2 V$, s stabilizes the simple $sl(V)$-submodules. By the preceding remark, each of these simple submodules would have to be self-contragredient. If $\lambda_1, \cdots, \lambda_l$ are the fundamental weights of $sl(V)$ ($\dim V = l + 1$), then the module contragredient to the simple $sl(V)$-module of highest weight $\Sigma_i m_i\lambda_i$ is the one with highest weight $\Sigma_i m_{l+1-i}\lambda_i$. Now applying the results of §3 of [3], one can pick out simple submodules of $\bigwedge^p \bigwedge^2 V$ which are not self-contragredient. Thus this second type of automorphism cannot occur.

As a result $S_1 = F_1$ for all g, p and we have completed the proof of the

MAIN THEOREM. *There is only one algebraic duality theory.*

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