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AN EXTREMAL LENGTH PROBLEM ON A BORDERED RIEMANN SURFACE(1)

BY

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ABSTRACT. Partition the contours of a compact bordered Riemann surface R' into four disjoint closed sets $\alpha_0, \alpha_1, \alpha_2$ and γ with α_0 and α_1 nonempty. Let F denote the family of all locally rectifiable 1-chains in $R'-\gamma$ which join α_0 to α_1. The extremal length problem on R' considers the existence of a real-valued harmonic function u on R' which is 0 on α_0, 1 on α_1, a constant on each component ν_k of α_2 with $\int_{\nu_k} {}^*du = 0$ and ${}^*du = 0$ along γ such that the extremal length of F is equal to the reciprocal of the Dirichlet integral of u, that is, $\lambda(F) = D_{R'}(u)^{-1}$.

Let $\bar{R}$ denote a bordered Riemann surface with a finite number of boundary components and S a compactification of $\bar{R}$ with the property that $\partial\bar{R} \subset S$. We consider the extremal length problem on $\bar{R}$ (as a subset of S) when α_0, α_1, and α_2 are relatively closed subarcs of $\partial\bar{R}$ and when α_0, α_1 and α_2 are closed subsets of $\partial S = (S - \bar{R}) \cup \partial\bar{R}$.

Introduction. A. Marden and B. Rodin [5, pp. 237–269] extended the extremal length problem on a compact bordered Riemann surface by considering the Kerékjártó-Stoilöw compactification R^* of an open Riemann surface R. They partitioned the ideal boundary $\beta = R^* - R$ of R into four disjoint sets $\alpha_0, \alpha_1, \alpha_2$ and γ with α_0 and α_1 nonempty and α_0, α_1 and $\alpha_0 \cup \alpha_1 \cup \alpha_2$ closed in R^*. They showed that if the extremal length of the family F^* of all locally rectifiable 1-chains in $R^* - \gamma$ which join α_0 to α_1 is finite, then there exists a real-valued harmonic function u on R which is 0 on α_0, 1 on α_1, a constant on each component τ_k of α_2 with the constant chosen so that $\int_{\tau_k} {}^*du = 0$ and ${}^*du = 0$ along γ (in the sense of limits along curves tending to the ideal boundary) such that $\lambda(F^*) = D_R(u)^{-1}$.

Let $\bar{R}$ denote a bordered Riemann surface with a finite number of boundary components and S a compactification of $\bar{R}$ with the property that $\partial\bar{R} \subset S$ (see Theorem 4). Set $\beta = S - \bar{R}$ and $\partial S = \beta \cup \partial\bar{R}$. Partition $\partial\bar{R}$ into the four

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sets $\alpha_0, \alpha_1, \alpha_2$ and γ, where α_0 and α_1 are nonempty, α_0, α_1 and α_2 are each composed of a finite number of relatively closed subarcs of $\partial\bar{R}$ and closed contours of $\bar{R}$, the closures on S of α_0, α_1 and α_2 are pairwise disjoint and $\gamma = \partial S - (\alpha_0 \cup \alpha_1 \cup \alpha_2)$. Let F denote the family of all locally rectifiable 1-chains in $S - \gamma$ which join α_0 to α_1. Theorem 5 solves the extremal length problem on $\bar{R}$.

Theorems 6 and 7 consider the extremal length problem on $\bar{R}$ when α_0, α_1 and α_2 are closed subsets of ∂S.

REMARK 1. Suppose that the closure on S of each component of $\alpha_0 \cup \alpha_1 \cup \alpha_2$ is a closed subset of $\bar{R}$. By considering the double of $\bar{R}$ across γ and using [5] we can solve the extremal length problem on $\bar{R}$. For this reason throughout this paper we shall assume that this is not the case.

REMARK 2. Suppose that $\bar{R}$ is any bordered Riemann surface and that the sets α_0, α_1 and α_2 are chosen so that they intersect a finite number of contours of $\bar{R}$. By doubling $\bar{R}$ across the remaining contours we obtain a bordered Riemann surface with a finite number of boundary components. Therefore, since we will only consider the case when $\alpha_0 \cup \alpha_1 \cup \alpha_2$ intersects a finite number of contours of $\bar{R}$, it suffices to consider the extremal length problem on a bordered Riemann surface with a finite number of boundary components.

1. **Preliminaries.** Throughout this paper we tacitly assume that all harmonic functions are real-valued and that all curves and chains are locally rectifiable.

Let $\bar{R}$ denote a bordered Riemann surface with a finite number of boundary components and border $\partial\bar{R}$. Each component of $\partial\bar{R}$, called a contour of $\bar{R}$, can be classified as either type I or type II, where a contour of $\bar{R}$ is said to be of type I if it is the homeomorphic image of a circle or type II if it is the homeomorphic image of the open unit interval.

Partition the contours of $\bar{R}$ into four sets $\alpha_0, \alpha_1, \alpha_2$ and γ, where α_0 and α_1 are nonempty. Define the family F of all locally rectifiable chains in $\bar{R} - \gamma$ joining α_0 to α_1 by $C \in F$ if and only if C is a continuous map from a union of closed intervals $[a_0, a_1] \cup [a_2, a_3] \cup \cdots \cup [a_{n-1}, a_n]$ into $\bar{R} - \gamma$ with $C(a_0) \in \alpha_0$, $C(a_n) \in \alpha_1$ and, for odd $j < n$, $C(a_j)$ and $C(a_{j+1})$ belong to the same component of α_2.

Traditionally the elements of F are called curves.

Let $\rho|dz|$ denote a metric (linear density) on $\bar{R}$ and H the family of all metrics $\rho|dz|$ on $\bar{R}$ such that $L(F, \rho|dz|) = \inf_{c\in F} \int_c \rho|dz|$ and $A(\rho|dz|, \bar{R}) = \iint_{\bar{R}} \rho^2\, dx\, dy$ are defined and not simultaneously 0 or $+\infty$. The least upper bound $\lambda(F)$ of $L^2(F, \rho|dz|)/A(\rho|dz|, \bar{R})$ over all $\rho|dz| \in H$ is called the extremal length of F. It is a conformal invariant ([2], [3], [4], [8] or [10]).

If F^* denotes the family of all curves on $\bar{R}$ which separate α_0 and α_1, then $\lambda(F^*)$ is called the conjugate extremal length of F. For the results in this paper $\lambda(F^*) = \lambda(F)^{-1}$ (see [8, pp. 124, 128]).

Let u denote a harmonic function on $\bar{R}$. Then $*du = -u_y\,dx + u_x\,dy$ is a linear differential on $\bar{R}$. We say that $*du = 0$ along a differentiable curve $z(t) = x(t) + iy(t)$ if $-u_y x_t + u_x y_t = 0$ for all t and that the flux of u across $z(t)$ is the integral of $*du$ over $z(t)$.

The following theorem will be needed (see [2, p. 147] or [8, p. 36]).

THEOREM 1. *Let G be a subdomain of a Riemann surface R, and suppose that a harmonic function u_Ω, defined in G, is assigned to every sufficiently large regular subdomain Ω of R. Then, if $D_G(u_\Omega - u_{\Omega'}) \longrightarrow 0$ as $\Omega, \Omega' \longrightarrow R$, there exists a harmonic function u in G with the following properties*:

(i) $D_G(u_\Omega - u) \longrightarrow 0$ *as* $\Omega \longrightarrow R$,

(ii) $D_G(u_\Omega) \longrightarrow D_G(u)$ *as* $\Omega \longrightarrow R$,

(iii) $u_\Omega(z) - u_\Omega(z_0) \longrightarrow u(z)$ *as* $\Omega \longrightarrow R$.

In (iii), z_0 is an arbitrary point in G, fixed in advance, and the convergence is uniform on every compact subset of G.

2. **Known results.** Partition the contours of a compact bordered Riemann surface T into four distinct sets $\alpha_0, \alpha_1, \alpha_2$ and γ with α_0 and α_1 nonempty. Further, if α_2 is nonempty, partition its contours into a finite number of distinct sets $\{\tau_k\}$.

Let u denote the unique harmonic function on T, see [2, p. 157], which is 0 on α_0, 1 on α_1, a constant on each part τ_k of α_2 with the constant chosen so that the flux of u across τ_k is zero and $*du = 0$ along γ. Let F denote the family of all curves in $T - \gamma$ which join α_0 to α_1. The following result is known [5, pp. 250–251].

THEOREM 2. $\lambda(F) = D_T(u)^{-1}$.

Marden and Rodin [5] used the Kerékjártó-Stoilöw compactification of a Riemann surface to extend this result.

Let R be a Riemann surface and R^* its Kerékjártó-Stoilöw compactification [2, pp. 81–87]. Partition the ideal boundary of R, $R^* - R$, into four disjoint sets $\alpha_0, \alpha_1, \alpha_2$ and γ with α_0 and α_1 nonempty and α_0, α_1 and $\alpha_0 \cup \alpha_1 \cup \alpha_2$ closed in R^*. Let F^* denote the family of curves in $R^* - \gamma$ which join α_0 to α_1. Marden and Rodin proved that if $\lambda(F^*) < \infty$, then there is a harmonic function u on R which is 0 on α_0, 1 on α_1, a constant on each component τ_k of α_2 with the flux of u across τ_k equal to 0 and $*du = 0$ along γ (in the sense of limits along curves tending to the ideal boundary) so that

THEOREM 3. $\lambda(F^*) = D_R(u)^{-1}$.

3. The compactification S. The compactification of $\bar{R}$ that we will use to generalize Theorem 3 is obtained by modifying Ahlfors and Sario's construction of the Kerékjártó-Stoilöw compactification of an open Riemann surface [2, pp. 82–87].

Let T be a topological space. A set $\beta \subset T$ is said to be nonseparating on T if every domain $G \subset T$ remains connected after removal of all points in $G \cap \beta$.

For convenience denote by Q, with or without subscripts, any domain on $\bar{R}$ which is not relatively compact, but has compact boundary. Note that Q is never void.

DEFINITION. An ideal boundary component of $\bar{R}$ is a nonvoid collection q of sets Q which satisfy the following conditions: if $Q_0 \in q$ and $Q \supset Q_0$, then $Q \in q$; if $Q_1, Q_2 \in q$ there exists a $Q_3 \in q$ with $Q_3 \subset Q_1 \cap Q_2$; the intersection of all closures of Q, $Q \in q$, is empty.

The proof of the following theorem is analogous to that of Ahlfors and Sario [2, pp. 82–87] and shall be omitted.

THEOREM 4. *There exists a unique compactification* S *of* $\bar{R}$ *such that* S *is a locally connected Hausdorff space,* $\beta = S - \bar{R}$ *is totally disconnected and* β *is nonseparating on* S.

Note that $\partial\bar{R}$ is a subset of S.

4. Pseudosymmetric exhaustion. In order to consider the possibility that $\alpha_0 \cup \alpha_1 \cup \alpha_2$ contains a component whose closure on S contains a point of $S - \bar{R}$ it will be necessary to construct a "nearly canonical" exhaustion $\{R_n\}$ of $\bar{R}$ such that R_1 intersects each contour of $\bar{R}$.

Let T denote the union of all type II contours of $\bar{R}$. Suppose that T is nonempty. We modify Ahlfors and Sario's proof of the existence of an exhaustion of an open Riemann surface [2, pp. 144–145] to construct a "nearly canonical" exhaustion of $\bar{R}$.

Cover $\bar{R}$ by a countable number of parameter discs and parameter half-discs V_{0n}, V'_{0n}, where $V_{0n} \subset \bar{R} - T$ and $V'_{0n} \cap T \neq 0$. Without loss of generality assume that each V'_{0j} intersects only one type II contour of $\bar{R}$, say T_j, and that $T_1, \cdots, T_q$ are the q distinct components of T.

Let Γ be a simple arc in $\bar{R} - \partial\bar{R}$ with $\Gamma \cap V'_{0j} \neq 0$ for $j = 1, \cdots, q$. Assume that $V_{01}, \cdots, V_{0m}, V'_{01}, \cdots, V'_{0q}$ cover Γ and pass to the double $D(\bar{R})$ of $\bar{R}$ at T.

Let Cl denote closure on $D(\bar{R})$. Following Ahlfors and Sario we construct a regular exhaustion $\{\mathrm{Cl}\, W_k^*\}$ of $D(\bar{R})$ such that each component H_{kj} of $\mathrm{Cl}\, W_{k+1}^* - W_k^*$ is a compact bordered Riemann surface with a finite number of analytic Jordan boundary curves $B_1, \cdots, B_r$ on the boundary of $\mathrm{Cl}\, W_k^*$ and certain others on the boundary of $\mathrm{Cl}\, W_{k+1}^*$.

For $m = 1, 2, \cdots, r-1$ join B_m and B_{m+1} by two distinct arcs bounding a narrow strip s_m so that $\mathrm{Cl}\, s_m \cap \mathrm{Cl}\, s_j = 0$ for $m \neq j$. Deleting $\bigcup s_m$ from H_{kj} we obtain a compact bordered Riemann surface H'_{kj} bounded by a piecewise analytic Jordan curve K_j and the boundary curves of H_{kj} in the boundary of $\mathrm{Cl}\, W_{k+1}^*$. There is a Jordan curve J_{kj} in H'_{kj} bounding with K_j a doubly-connected domain in H_{kj} [4, Lemma 8.1]. There is a Jordan curve Γ_{kj} in this domain which separates the boundary components of the domain. For each j add the subdomain of H_{kj} bounded by Γ_{kj} and $B_1, \cdots, B_m$ as well as these curves to $\mathrm{Cl}\, W_k^*$ to obtain a compact bordered Riemann surface W_k. The exhaustion $\{W_k\}$ is canonical.

Set $R_k = W_k \cap \bar{R}$. Then $\{R_n\}$ is a sequence of compact bordered Riemann surfaces on $\bar{R}$ with the following properties:

(i) $\bigcup R_n = \bar{R}$, $R_n \subset R_{n+1} \subset \bar{R}$, $\partial\bar{R} - T \subset \partial R_n$, and $\partial\bar{R} \cap t \neq 0$ for each component t of T;

(ii) ∂R_n is composed of analytic Jordan curves, open analytic Jordan arcs in $\bar{R} - \partial\bar{R}$ with end points on $\partial\bar{R}$ and subarcs of components of T;

(iii) each component of $\bar{R} - R_n$ is noncompact and its boundary consists of a single open analytic arc in $\bar{R} - \partial\bar{R}$ with end points on $\partial\bar{R}$;

(iv) ∂R_n and T are orthogonal at their points of intersection.

The exhaustion $\{R_n\}$ of $\bar{R}$ will be called a pseudosymmetric exhaustion of $\bar{R}$.

5. **First result.** We are ready to prove the following theorem.

THEOREM 5. *Let S denote the compactification of $\bar{R}$ defined above. Suppose that α_0, α_1 and α_2 are subsets of $\partial\bar{R}$ which satisfy the following conditions*:

(i) *α_0 and α_1 are nonempty*;

(ii) *α_0, α_1 and α_2 are composed of a finite number of relatively closed arcs and full contours of $\bar{R}$*;

(iii) *α_0, α_1 and α_2 have pairwise disjoint closures on S.*

Set $\beta = S - \bar{R}$, $\gamma = \beta \cup (\partial\bar{R} - (\alpha_0 \cup \alpha_1 \cup \alpha_2))$ and denote by F the family of all curves in $\bar{R} - \gamma$ which join α_0 to α_1. There exists a harmonic function u on $\bar{R}$ which is 0 on α_0, 1 on α_1, a constant on each component τ_k of α_2 with the constant chosen so that the flux of u across τ_k is 0 and $^*du = 0$ along γ such that $0 < \lambda(F) = D_{\bar{R}}(u)^{-1} < \infty$.

A precise meaning for $^*du = 0$ along $\gamma \cap \partial\bar{R}$ exists. On $\gamma - \partial\bar{R}$ the sense is of a limit along curves tending to each component of $\gamma - \partial\bar{R}$.

For the remainder of this paper Cl will represent the closure on S.

PROOF. Let $\{R_n\}$ denote a pseudosymmetric exhaustion of $\bar{R}$ and δ a component of $\alpha_0 \cup \alpha_1 \cup \alpha_2$. If $\mathrm{Cl}\,\delta \subset \partial\bar{R}$, assume that $\delta \subset \partial R_1$. If not, let σ denote the type II contour of $\bar{R}$ which intersects δ. If $\sigma - \delta \neq 0$, let f denote a homeomorphism of σ onto $\{t: 0 < t < 1\}$. There is a constant b $(0 < b < 1)$ such that $f^{-1}(\{t: b \leq t < 1\}) = \delta \cap \sigma$ or $f^{-1}(\{t: 0 < t \leq b\}) = \delta \cap \sigma$. Set $\epsilon = \min(b, \frac{1}{2})$. Assume that $f^{-1}(\{t: b - \epsilon < t < b + \epsilon\}) \subset \partial R_1$.

Set $\alpha_{jn} = \alpha_j \cap \partial R_n$ $(j = 0, 1$ and $2)$ and $\gamma_n = \partial R_n - (\alpha_{0n} \cup \alpha_{1n} \cup \alpha_{2n})$. For each n, let F_n denote the family of all curves in $R_n - \gamma_n$ which join α_{0n} to α_{1n} and let u_n denote the harmonic function on R_n satisfying Theorem 2. Since $F_n \subset F_{n+1}$ and $F = \bigcup F_n$, $\lambda(F) = \lim \lambda(F_n)$ and $\lambda(F_{n+1}) \leq \lambda(F_n) = D_{R_n}(u_n)^{-1} < \infty$. Thus, $0 \leq \lambda(F) = \lim D_{R_n}(u_n)^{-1} < \infty$.

There exists a finite collection of pairwise disjoint open analytic Jordan arcs $\{J_j\}_{j=1}^{A}$ in $\bar{R} - \partial\bar{R}$ with end points on $\partial\bar{R} \cap \gamma$ and analytic Jordan curves $\{J_j\}_{j=A+1}^{B}$ in $\bar{R} - \partial\bar{R}$ which, when taken together, separate $\alpha_0 \cup \alpha_2$ and α_1. Suppose that J_j $(j = 1, \cdots, B)$ separates the subset $\alpha_1(j)$ of α_1 and $\alpha_0 \cup \alpha_2 \cup (\alpha_1 - \alpha_1(j))$.

For $j = 1, \cdots, A$ let p_{1j} and p_{2j} denote the end points of J_j on $\partial\bar{R} \cap \gamma$. Take two open connected sets U_{1j} and U_{2j} on $\partial\bar{R}$ (relative to $\partial\bar{R}$) with $p_{1j} \in U_{1j} \subset \partial\bar{R} \cap \gamma$ and $p_{2j} \in U_{2j} \subset \partial\bar{R} \cap \gamma$. There exists an open analytic Jordan arc L_j in $\bar{R} - \partial\bar{R}$ with end points q_{1j} on $U_{1j} - \{p_{1j}\}$ and q_{2j} on $U_{2j} - \{p_{2j}\}$ so that L_j separates $\alpha_1(j)$ and J_j and so that J_j, the arc on U_{1j} joining p_{1j} and q_{1j}, the arc on U_{2j} joining p_{2j} and q_{2j}, and L_j bound a simply-connected domain D_j on $\bar{R} - \partial\bar{R}$; we can assume that the domains $D_1, \cdots, D_A$ have pairwise disjoint closures on $\bar{R}$. Hence, there is a conformal mapping f_j of $\mathrm{Cl}\,D_j$ onto $\{z = x + iy: 0 \leq x \leq a_j$ and $0 \leq y \leq b_j\}$ so that $f_j(p_{1j}) = 0$, $f_j(p_{2j}) = ib_j$, $f_j(q_{1j}) = a_j$ and $f_j(q_{2j}) = a_j + ib_j$ $(0 < a_j, b_j)$.

Considering a canonical exhaustion of $\bar{R} - \partial\bar{R}$ we conclude that for each $k = A + 1, \cdots, B$ there is an analytic Jordan curve L_k in $\bar{R} - \partial\bar{R}$ such that L_k separates $\alpha_1(k)$ and J_k, and such that J_k and L_k bound a doubly-connected domain D_k on $\bar{R} - \partial\bar{R}$. Let f_k denote a conformal mapping of $\mathrm{Cl}\,D_k$ onto $\{z: 1 < |z| < d_k\}$.

We can assume that $D_1, \cdots, D_B$ have pairwise disjoint closures on $\bar{R}$.

Let G denote the family of all curves in $D = \bigcup_{j=1}^{B} D_j$ defined by $l \in G$ if for some k, $l \subset \mathrm{Cl}\,D_k - \gamma$ and l joins L_k to J_k. Since each curve in F contains a curve in G, $\lambda(G) \leq \lambda(F)$. Also, $\lambda(G) > 0$. Thus, $\lambda(F) > 0$ and the sequence $\{D_{R_n}(u_n)\}$ converges to a finite limit.

Take $n < m$. Then

$$D_{R_n}(u_m, u_n) = \int_{\partial R_n} u_m \, {}^*du_n = \int_{\alpha_{1n}} {}^*du_n = D_{R_n}(u_n).$$

Hence,

$$0 < D_{R_n}(u_m - u_n) = D_{R_n}(u_m) - D_{R_n}(u_n) < D_{R_m}(u_m) - D_{R_n}(u_n) \quad \text{for } n < m. \tag{1}$$

Therefore,

$$D_{R_n}(u_m - u_n) \longrightarrow 0 \quad \text{as} \quad m, n \longrightarrow \infty \quad (m \geqslant n).$$

Let Q_N denote the double of R_N at $\alpha_{0N} - \mathrm{Cl}\,\gamma_N$. Extend each u_n, $n \geqslant N$, to a harmonic function u'_n on Q_N [2, p. 129]. From Theorem 1, with $z_0 \in \alpha_{0N} - \mathrm{Cl}\,\gamma_N$, there exists a harmonic function v_N on Q_N with the following properties:

(i) $D_{Q_N}(u'_n - v_N) \longrightarrow 0$ as $n \longrightarrow \infty$,

(ii) $D_{Q_N}(u'_n) \longrightarrow D_{Q_N}(v_N)$ as $n \longrightarrow \infty$,

(iii) $u'_n(z) \longrightarrow v_N(z)$ for $z \in Q_N$,

and the convergence is uniform on every compact subset of Q_N.

It is easily seen after a similar reflection across $\gamma \cup \gamma^*$ (γ^* is the reflection of γ in the first doubling process) that ${}^*dv_N = 0$ along $\gamma \cup \gamma^*$.

From (iii) it follows that v_N is the restriction to Q_N of a harmonic function v on Q, the double of $\bar{R}$ across $\alpha_0 - \mathrm{Cl}\,\gamma$.

We must show that

$$\lim D_{Q_N}(u'_N) = D_Q(v) = \lim D_{Q_N}(v_N).$$

This follows, since for $n \geqslant N$, $D_{Q_n}(u'_n) \geqslant D_{Q_N}(u'_n)$ and $D_{Q_N}(u'_N) < D_{Q_N}(u'_n)$

For the harmonic function $u = v|\bar{R}$,

(i) $u|R_N = v_N|R_N$,

(ii) $D_{R_n}(u_n - u) \longrightarrow 0$ as $n \longrightarrow \infty$,

(iii) $D_{R_n}(u_n) \longrightarrow D_{\bar{R}}(u)$ as $n \longrightarrow \infty$,

(iv) ${}^*du = 0$ along γ,

(v) $u_n(z) \longrightarrow u(z)$ for $n \longrightarrow \infty$ and $z \in \bar{R}$,

and the convergence is uniform on every compact subset of $\bar{R}$.

6. **Second result.** Theorem 6 will show that if α_0, α_1 and α_2 are certain closed subsets of ∂S, then the conclusion of Theorem 5 is valid.

THEOREM 6. *Let S denote the compactification of $\bar{R}$ defined above. Set $\beta = S - \bar{R}$ and $\partial S = \beta \cup \partial \bar{R}$. Suppose that α_0, α_1 and α_2 are closed subsets of ∂S which satisfy the following conditions*:

(6.1) α_0 *and* α_1 *each contain an arc on* ∂S *or a type* I *contour of* $\bar{R}$;

(6.2) α_0, α_1 *and* α_2 *are composed of points of* β, *type* I *contours of* $\bar{R}$ *and a finite number of arcs on* ∂S;

(6.3) α_0, α_1 *and* α_2 *are pairwise disjoint*;

(6.4) *For each* $\beta' \in (\alpha_0 \cup \alpha_1 \cup \alpha_2) \cap (\mathrm{Cl}(\partial\bar{R}) - \partial\bar{R})$ *there is a neighborhood of* β' *on* S *which fails to meet* $\beta - (\alpha_0 \cup \alpha_1 \cup \alpha_2)$.

Set $\gamma = \partial S - (\alpha_0 \cup \alpha_1 \cup \alpha_2)$ *and denote by* F *the family of all curves in* $S - \gamma$ *which join* α_0 *to* α_1. *Let* $\beta^* = (\mathrm{Cl}(\partial\bar{R}) - (\alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \partial\bar{R}))$. *There exists a continuous function* u *on* $S - \beta^*$ *which is harmonic on* $\bar{R}$, 0 *on* α_0, 1 *on* α_1, *a constant on each component* v_j *of* α_2 *with the flux of* u *across* v_j *equal to* 0 *and* ${}^*du = 0$ *along* γ *such that* $0 < \lambda(F) = D_{\bar{R}}(u)^{-1} < \infty$.

The conditions imposed upon $u|(S - \bar{R})$ are understood to be as limits in a sense to be made precise below.

PROOF. Without loss of generality assume that $\alpha_0 \cup \alpha_1 \cup \alpha_2$ contains a component δ' with $\delta' - \beta \neq 0$ and $\delta' \cap \beta \neq 0$. Let $\{R_n\}$ denote a pseudosymmetric exhaustion of $\bar{R}$ and δ any component of $\alpha_0 \cup \alpha_1 \cup \alpha_2$ that intersects $\partial\bar{R}$. If $\delta \subset \partial\bar{R}$, assume that $\delta \subset \partial R_1$. If not, as we traverse $\partial\bar{R}$ in the positive sense we traverse the type II contours $\sigma_1, \cdots, \sigma_q$ of $\bar{R}$ which intersect δ.

Consider the case where $\sigma_j - \delta \neq 0$ for $j = 1$ or q. Without loss of generality assume that $\sigma_q - \delta \neq 0$ and let f_k $(k = 1$ and $q)$ denote a homeomorphism of σ_k onto $\{t\colon 0 < t < 1\}$. If $\sigma_1 - \delta \neq 0$, there is a constant a_1 $(0 < a_1 < 1)$ such that $f_1^{-1}(\{t\colon a_1 \leqslant t < 1\}) = \delta \cap \sigma_1$; if not, set $a_1 = ½$. Let a_q $(0 < a_q < 1)$ denote the constant such that $f_q^{-1}(\{t\colon 0 < t \leqslant a_q\}) = \delta \cap \sigma_q$ and set $\epsilon = \min(a_1, 1 - a_1, a_q, 1 - a_q)$. Assume that $f_k^{-1}(\{t\colon a_k - \epsilon < t < a_k + \epsilon\}) \subset \partial R_1$ for $k = 1$ and q.

Set $\Gamma_n = \mathrm{Cl}(\partial R_n - \partial\bar{R})$. Each component Γ'_n of $\Gamma_n - \partial\bar{R}$ is either an open analytic Jordan arc in $\bar{R} - \partial\bar{R}$ with end points on two distinct type II contours of $\bar{R}$ or a type I contour of R_n in $\bar{R} - \partial\bar{R}$. If Γ'_n is a type I contour of R_n, let $C'(\Gamma'_n)$ denote the component of $S - \Gamma'_n$ with $C'(\Gamma'_n) \cap \partial\bar{R} = 0$. Without loss of generality assume that each $C'(\Gamma'_n)$ intersects at most one of the following sets: $\alpha_0, \alpha_1, \alpha_2$. If $C'(\Gamma'_n)$ intersects α_k $(k = 0, 1$ or $2)$ write $C(\Gamma'_n, k)$ for $C'(\Gamma'_n)$, otherwise $C(\Gamma'_n)$.

Suppose that Γ'_n is not a type I contour of R_n and that our exhaustion satisfies the following remark, which is valid for some subsequence; let $A'(\Gamma'_n)$ denote the component of $S - \Gamma'_n$ which satisfies either $A'(\Gamma'_n) \cap (\alpha_0 \cup \alpha_1 \cup \alpha_2) \cap \mathrm{Cl}(\partial\bar{R}) = 0$ or for some $k = 0, 1$ or 2, $A'(\Gamma'_n) \cap \alpha_k \cap \mathrm{Cl}(\partial\bar{R}) \neq 0$ and $A'(\Gamma'_n) \cap \alpha_j \cap \mathrm{Cl}(\partial\bar{R}) = 0$ for $j \neq k$. Write $A(\Gamma'_n, k)$ for $A(\Gamma'_n)$ if the latter occurs and $B(\Gamma'_n)$ if the former.

We can assume that $A(\Gamma'_n, k) \cap (\bigcup_{j=0}^2 \alpha_j - \alpha_k) = 0$ and $B(\Gamma'_n) \cap (\bigcup_{j=0}^2 \alpha_j) = 0$. Further, from (6.4), we can assume that each $A(\Gamma'_n, k)$ fails to meet $\gamma - \partial\bar{R}$.

We have associated with each component Γ'_n of Γ_n some $A(\Gamma'_n, k), B(\Gamma'_n)$, $C(\Gamma'_n)$ or $C(\Gamma'_n, k)$ domain on S. For each $k = 0, 1$ and 2 let $\Gamma_n(k)$ denote the union of all components Γ'_n of Γ_n which are associated with some $A(\Gamma'_n, k)$ domain and let $\Gamma_{Cn}(k)$ denote the union of all components Γ'_n of Γ_n which are associated with some $C(\Gamma'_n, k)$ domain. Take $n \geq j \geq 1$. Set $\Gamma_{Ajn}(k) = \{\Gamma' : \Gamma'$ is a component of $\Gamma_{Cn}(k)$ and $\Gamma' \subset A(\Gamma'_j, k)$ for some component Γ'_j of $\Gamma_j(k)\}$ and $\Gamma_{\gamma j} = \Gamma_j - \bigcup_{L=0}^2 \Gamma_j(L)$. Let $M_{\gamma j}$ denote the totality of all $B(\Gamma'_j), C(\Gamma'_j)$, $C(\Gamma'_j, 0), C(\Gamma'_j, 1)$ and $C(\Gamma'_j, 2)$ domains, where Γ'_j is a component of $\Gamma_{\gamma j}$.

For $k = 0, 1$ and 2 set

$S_{jn} = R_n - M_{\gamma j}$, $\alpha_{kn}(A) = \alpha_k \cap \partial S_{jn} \cap \partial\bar{R}$ $(= \alpha_k \cap \partial R_n \cap \partial\bar{R})$, $\alpha_{kjn} = \alpha_{kn}(A) \cup \Gamma_n(k) \cup \Gamma_{Cj}(k) \cup \Gamma_{Ajn}(k)$, and

$\gamma_{jn} = \partial S_{jn} - \bigcup_{L=0}^2 \alpha_{Ljn}$.

Set $T_j = S - M_{\gamma j}$, $\alpha_{kj} = (\alpha_k \cap T_j) \cup \Gamma_{Cj}(k)$ and

$\gamma_j = (T_j \cap \gamma) \cup (\Gamma_{\gamma j} - \bigcup_{L=0}^2 \Gamma_{Cj}(L))$. In the sense of limits along curves tending to points of $\alpha_k \cap \beta \cap T_j$ and $\gamma \cap \beta \cap T_j$, $\alpha_{kj} = \lim_{n\to\infty} \alpha_{kjn}$, $\gamma_j = \lim_{n\to\infty} \gamma_{jn}$ and $T_j = \lim_{n\to\infty} S_{jn}$.

For fixed j, let F_{jn} denote the family of all curves in $S_{jn} - \gamma_{jn}$ which join α_{0jn} to α_{1jn} and let u_{jn} denote the harmonic function on the compact bordered Riemann surface S_{jn} satisfying Theorem 2. Let F_j denote the family of all curves in $T_j - \gamma_j$ which join α_{0j} to α_{1j}. Clearly,

$$\lambda(F_j) \geq \lambda(F_{jn+1}) \geq \lambda(F_{jn}) \geq \lambda(F_{j1}) > 0.$$

Also, since the family G_j of all curves in $T_j - (\beta \cup \gamma_j)$ which join $\alpha_{0j} - \beta$ to $\alpha_{1j} - \beta$ is a subset of F_j, from Theorem 5, $\lambda(F_j) \leq \lambda(G_j) < \infty$. Hence,

$$0 < \lim_{n\to\infty} \lambda(F_{jn}) \leq \lambda(F_j) < \infty.$$

To establish the inequality $\lim_{n\to\infty} \lambda(F_{jn}) \geq \lambda(F_j)$ we follow the procedure of Marden and Rodin [5, pp. 151–156] by defining a family F_{j0} in T_j such that $\lambda(F_{j0}) = \lambda(F_j)$ and $\lambda(F_{j0}) = \lim_{n\to\infty} \lambda(F_{jn})$.

Take $l_N \in F_{jN}$ and $n \leq N$ so that l_N intersects

$$\bigcup_{L=0}^{2} ((\alpha_{LN}(A) - \alpha_{Ln}(A)) \cup \Gamma_N(L) \cup \Gamma_{AjN}(L)).$$

We will define a type of restriction of l_N to S_{jn}, denoted by $l_N \| S_{jn}$, with the property that $l_N \| S_{jn} \in F_{jn}$. Assume that l_N is parametrized so that $l_N(0) \in \alpha_{0jN}$ and $l_N(1) \in \alpha_{1jN}$. There is a greatest t such that $l_N(t) \in \alpha_{0jn}$; call it t_1. Suppose that l_N intersects $\alpha_{2jn} - \alpha_{2n}(A)$. Let t_2 be the smallest t such that $l_N(t) \in \alpha_{2jn} - \alpha_{2n}(A)$ and denote by C_2 the component of $\alpha_{2jn} - \alpha_{2n}(A)$ which contains $l_N(t_2)$. Let t_3 be the greatest t such that $l_N(t) \in C_2$. Continue in this manner defining points t_{2k}, t_{2k+1} such that t_{2k} is the smallest t and t_{2k+1} the largest t such that $l_N(t)$ lies on a component C_{2k} of $\alpha_{2jn} - \alpha_{2n}(A)$ until we define t_{m-2}, t_{m-1} ($m \geqslant 4$ and even) such that $l_N|\{t: t_{m-1} < t \leqslant 1\} \cap (\alpha_{2jn} - \alpha_{2jN}) = 0$. Letting t_m be the smallest t such that $l_N(t) \in \alpha_{1jn}$ and $C_{(m+2)/2}$ the component of α_{1jn} which contains $l_N(t_m)$, we obtain an even number of stopping times $0 \leqslant t_1 < \cdots < t_m \leqslant 1$, a sequence of stopping points $l_N(t_1), \cdots, l_N(t_m)$ and a component sequence $C_1, \cdots, C_{(m+2)/2}$ of distinct components of $\alpha_{0jn} \cup \alpha_{1jn} \cup (\alpha_{2jn} - \alpha_{2n}(A))$.

Suppose that l_N does not intersect $\alpha_{2jn} - \alpha_{2n}(A)$. Let t_2 be the smallest t such that $l_N(t) \in \alpha_{1jn}$ and let C_2 denote the component of α_{1jn} which contains $l_N(t_2)$. We obtain the stopping times $0 \leqslant t_1 < t_2 \leqslant 1$, stopping points $l_N(t_1), l_N(t_2)$ and a component sequence C_1, C_2. Necessarily, by our assumption on l_N, $t_1 = 0$ and $t_2 = 1$ is impossible.

In each case define $l_N \| S_{jn}$ to be the restriction of l_N to $[t_1, t_2] \cup [t_3, t_4] \cup \cdots \cup [t_{m-1}, t_m]$.

A 1-chain l_0 on T_j will belong to F_{j0} if either $l_0 = l$ for some $l \in F_j$ or if l_0 is a continuous map of an open dense subset of $[0, 1]$ into T_j such that:

(i) If t_0 is not in the domain $\operatorname{dom} l_0$ of l_0 and $0 < t_0 < 1$, then there exist sequences $\{r_k\}$, $\{s_k\}$ in $\operatorname{dom} l_0$ such that $r_k \uparrow t_0$, $s_k \downarrow t_0$, and a point $\beta_0 \in \beta \cap T_j$ with $l_0(r_k) \longrightarrow \beta_0$, $l_0(s_k) \longrightarrow \beta_0$. If $t_0 = 0$ (respectively 1) we require only a sequence $\{s_k\}$ (respectively $\{r_k\}$).

(ii) There exists a subsequence $\{S_{jn(k)}\}$ of $\{S_{jn}\}$ such that $l_0 \| S_{jn(k)} \equiv (l_0 | S_{jn(k)}) \| S_{jn(k)} \in F_{jn(k)}$ for all $k \geqslant 1$.

(iii) If $t \in \operatorname{dom} l_0$ then there exists a p such that $t \in \operatorname{dom} l_0 \| S_{jn(k)}$ for all $k \geqslant p$.

We will now show that $\lambda(F_{j0}) = \lambda(F_j)$. If a 1-chain $l_0 \in F_{j0}$ can be extended to $\{t: 0 < t < 1\}$ with values in T_j, then the extension will be in F_j. Let $\{\xi_n(p)\}_{p=1}^{V_n}$ denote the components of $\bigcup_{L=0}^{2} \Gamma_n(L)$. To each $\xi_n(p)$ we associate a simply-connected domain D_{np} in S_{jn} bounded by $\xi_n(p)$ and an open analytic Jordan arc τ_{np} in $S_{jn} - \partial S_{jn}$ with end points on $\partial \bar{R} \cap S_{jn}$, and a homeomorphism f_{np} of $\operatorname{Cl} D_{np}$ onto $\{z = x + iy: 0 \leqslant x \leqslant a_{np}, 0 \leqslant y \leqslant b_{np}\}$, $0 < a_{np}, b_{np}$, which is conformal on $D_{np} - \partial \bar{R}$ with $f_{np}(\xi_n(p)) = \{x: 0 \leqslant x \leqslant a_{np}\}$, $f_{np}(\tau_{np}) = \{x + ib_{np}: 0 < x < a_{np}\}$ and $f_{np}(\partial \bar{R} \cap D_{np}) = \{iy: 0 < y < b_{np}\} \cup \{a_{np} + iy: 0 < y < b_{np}\}$.

Let $\{\xi_n(p)\}_{p=V_n+1}^{W_n}$ denote the components of $\bigcup_{L=0}^{2} \Gamma_{Ajn}(L)$. To each $\xi_n(p)$, $p = V_n + 1, \cdots, W_n$, we associate an annular domain D_{np} in S_{jn} bounded by $\xi_n(p)$ and an analytic Jordan curve τ_{np} in $S_{jn} - \partial S_{jn}$, and a homeomorphism f_{np} of $\mathrm{Cl}\, D_{np}$ onto $\{z\colon a_{np} \leqslant |z| \leqslant b_{np}\}$, $0 < a_{np} < b_{np}$, which is conformal on D_{np} with $f_{np}(\xi_n(p)) = \{z\colon |z| = a_{np}\}$ and $f_{np}(\tau_{np}) = \{z\colon |z| = b_{np}\}$. We can assume that the domains D_{np}, $p = 1, \cdots, W_n$, have pairwise disjoint closures on S_{jn}.

We shall show that if no D_{np} is crossed infinitely many times by l_0 then l_0 can be extended continuously to $\{t\colon 0 < t < 1\}$. Choose $t_0 \notin \mathrm{dom}\, l_0$ with $0 < t_0 < 1$. Let $\{r_m\}$, $\{s_m\}$ and $\beta_1 \in \beta \cap T_j$ be as above. It suffices to show that $l_0(t) \longrightarrow \beta_1$ for $t \longrightarrow t_0$ and $t \in \mathrm{dom}\, l_0$.

Let G denote a neighborhood of β_1 on T_j. We want a neighborhood of t_0 whose l_0-image is in a "small" neighborhood of G. To do this we assume that G is a component of $T_j - S_{jN}$ for some N. Let D_{Np} be one of the domains described above for $\xi_N(p) = \mathrm{Cl}\, G - G$. For n sufficiently large, $l_0(r_n), l_0(s_n) \in G$.

If $l_0(t) \not\rightarrow \beta_1$ as $t \longrightarrow t$ $(t \in \mathrm{dom}\, l_0)$ for some domains G and D_{Np}, there would exist a sequence $\{v_k\}$ in $\mathrm{dom}\, l_0$ with $v_k \longrightarrow t_0$ and $l_0(v_q) \notin G \cup D_{Np}$ for each $q \geqslant 1$. A subsequence of $\{v_k\}$ is monotone, say $v_k \uparrow t_0$. Choose $r_{k(1)}$ and $v_{k(1)} > r_{k(1)}$ and $S_{jm(1)} \supset S_{jN}$ so that $r_{k(1)}, v_{k(1)} \in \mathrm{dom}\, l_0 \| S_{jm(1)}$. Since $l_0 \| S_{jm(1)} \in F_{jm(1)}$, there is a crossing of D_{Np} within $\{t\colon r_{k(1)} < t < v_{k(1)}\}$. Next choose $r_{k(2)}, v_{k(2)}, S_{jm(2)}$ so that $r_{k(1)} < v_{k(1)} < r_{k(2)} < v_{k(2)}$ and $r_{k(2)}, v_{k(2)} \in \mathrm{dom}\, l_0 \| S_{jm(2)}$. There must be a crossing of D_{Np} within $\{t\colon r_{k(2)} < t < v_{k(2)}\}$. In this way we see that l_0 crosses D_{Np} infinitely many times. The cases $t_0 = 0$ and $t_0 = 1$, if they arise, are handled analogously.

Let $F_{j0}(n)$ denote the family of all curves in F_{j0} which cross some D_{np} infinitely many times. Clearly, $\lambda(F_{j0}(n)) = \infty$.

Let F'_{j0} denote the family of all curves in F_{j0} which cannot be continuously extended to $\{t\colon 0 < t < 1\}$. Clearly, $F'_{j0} \subset \bigcup_{n \geqslant j} F_{j0}(n)$. So $\lambda(F'_{j0}) = \infty$.

Now, $F_{j0} = F'_{j0} \cup (F_{j0} - F'_{j0})$. So $\lambda(F_{j0}) = \lambda(F_{j0} - F'_{j0})$ and $\lambda(F_{j0}) = \lambda(F_j)$.

We are now ready to show that $\lim_{n\to\infty} \lambda(F_{jn}) \geqslant \lambda(F_{j0})$. Choose any $x < \lambda(F_{j0})$ and any metric $\rho|dz|$ on T_j such that $L^2(F_{j0}, \rho|dz|) > x$ and $A(\rho|dz|, T_j) = 1$. To show that $\lim_{n\to\infty} \lambda(F_{jn}) \geqslant \lambda(F_{j0})$ it suffices to show that

$$\lim_{n\to\infty} L^2(F_{jn}, \rho|dz|) \geqslant x.$$

If that failed to hold there would exist a subsequence along which $L^2(F_{jn}, \rho|dz|)$ had a limit $y < x$. If we can show that to each $\epsilon > 0$ there exists a $l(\epsilon) \in F_{j0}$ satisfying $(\int_{l(\epsilon)} \rho|dz|)^2 \leqslant y + 7\epsilon$, then we have the desired contradiction,

$$y < x < L^2(F_{j0}, \rho|dz|) \leqslant \left(\int_{l(\epsilon)} \rho|dz|\right)^2 \leqslant y + 7\epsilon.$$

Fix $\epsilon > 0$. By passage to a subsequence of $\{F_{jn}\}$ we may assume that

$$|L^2(F_{jn}, \rho\,|dz|) - y| < \epsilon/2^n \qquad (n \geqslant 1).$$

Whenever a subsequence of $\{F_{jn}\}$ or $\{l_n: l_n \in F_{jn}\}$ is extracted and the notation is unchanged we tacitly agree that $\{S_{jn}\}$ shall refer to the corresponding subsequence of $\{n\}$.

Choose $l_n \in F_{jn}$ so that

$$\left|\left(\int_{l_n} \rho\,|dz|\right)^2 - y\right| < \epsilon/2^n \qquad (n \geqslant 1).$$

From the sequence $\{l_n\}$, possibly after some modifications, we will extract a subsequence which will be used to construct $l(\epsilon)$.

If there is an M such that

$$\{l_M(t) \cap \partial S_{jM}\colon 0 \leqslant t \leqslant 1\} \subset (\alpha_0 \cup \alpha_1 \cup \alpha_2) \cap \partial\bar{R},$$

take $l(\epsilon) = l_M \in F_{j0}$; the assertion follows. Therefore, assume that this is not the case. We first find a subsequence $\{l_{n(k)}\}$ of $\{l_n\}$ such that all $l_{n(k)} \| S_{jN}$, $n(k) \geqslant N$, have the same component sequence on ∂S_{jN}. Since there are only a finite number of possible component sequences on ∂S_{jj} we may select a first subsequence of $\{l_n\}$, all elements of which have the same component sequence on ∂S_{jj}. By induction we obtain for each $N \geqslant j$ a subsequence of the preceding one, all elements of which follow a common component sequence on ∂S_{jN}. Using the diagonal process we obtain a subsequence with the desired property. Continue to designate this subsequence by $\{l_n\}$.

Now modify each l_n so that not only will all $l_n \| S_{jN}$, $n \geqslant N$, follow the same component sequence, but also $l_n \| S_{jn-1}$ and $l_{n-1} \| S_{jn-1}$ will have the same sequence of stopping points on ∂S_{jn-1}. To accomplish this use the diagonal process to find a preliminary sequence, again denoted by $\{l_n\}$, with the following property. Suppose that l_N has m stopping points on ∂S_{jN}. Then for each $k \leqslant m$ the kth stopping point ζ_n of $l_n \| S_{jN}$, $n \geqslant N$, gives rise to a convergent sequence of points $\{\zeta_n\}$ on a component of $\bigcup_{L=0}^{2} (\alpha_{LjN} - \alpha_{LN}(A))$. Around the limit point of this sequence put a topological disc the circumference of which has very small ρ-length.

For each N there are as many discs on ∂S_{jN} as there are stopping points for $l_n \| S_{jN}$, $n \geqslant N$. Choose the circumference of these discs small enough so that their total ρ-length is less than $2^{-N}\epsilon$. By the diagonal process we achieve a situation in which each stopping point of $l_n \| S_{jN}$ on ∂S_{jn} is inside the appropriate disc for all $n \geqslant N$. For each disc take a point in the intersection of its circumference and the corresponding component of the component sequence; call such a point a distinguished stopping point. A modification of l_n will mean the result of replacing part of its path inside a disc by a path on the circumference of the disc. Modify l_N so that all its stopping points are distinguished; in general, modify l_n so that the stopping points of $l_n \| S_{jn-1}$ on ∂S_{jn-1} and $l_n \| S_{jn}$ on ∂S_{jn} are distinguished. Denote the modified sequence again by $\{l_n\}$. This modification has increased the ρ-length of l_n by at most $2^{-n+1}\epsilon + 2^{-n}\epsilon$. For the present sequence $\{l_n\}$,

$$\left|\left(\int_{l_M} \rho |dz|\right)^2 - y\right| < \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n-1}} + \frac{\epsilon}{2^n} = \frac{4\epsilon}{2^n}.$$

Clearly, each new l_n is in F_{jn}.

By induction, for each n we reparametrize l_n so that $\operatorname{dom} l_n \| S_{jn-1} = \operatorname{dom} l_{n-1}$. So $\operatorname{dom} l_{n-1}$ consists of a finite number of closed intervals $[t_1, t_2] \cup [t_3, t_4] \cup \cdots \cup [t_{m-1}, t_m]$ and $l_{n-1}(t_k) = l_n(t_k)$ for $1 \leqslant k \leqslant m$.

We now construct $l(\epsilon)$. On $\operatorname{dom} l_1$ set $l(\epsilon) = l_1$. In general, if $l(\epsilon)$ has been defined on $\operatorname{dom} l_{n-1}$ set $l(\epsilon) = l_n$ on $\operatorname{dom} l_n - \operatorname{dom} l_{n-1}$. So $l(\epsilon)$ is a curve on T_j. Its domain is an open subset of $[0, 1]$ which, by reparametrization, may be assumed to be dense.

Note, since $l_n | \operatorname{dom} l_{n-1} = l_n \| S_{jn-1}$,

$$\left(\int_{l_n \| S_{jn-1}} \rho |dz|\right)^2 \geqslant L^2(F_{jn-1}, \rho |dz|) > y - \epsilon/2^{n-1}.$$

Also, $(\int_{l_n} \rho |dz|)^2 < y + 4\epsilon/2^n$. Hence,

$$\left(\int_{l_n | (\operatorname{dom} l_n - \operatorname{dom} l_{n-1})} \rho |dz|\right)^2 < \frac{6\epsilon}{2^n}.$$

Therefore, by Schwarz's inequality,

$$\left(\int_{l(\epsilon)} \rho |dz|\right)^2 < \left(\int_{l_1} \rho |dz|\right)^2 + \sum_{n=1}^{\infty} \frac{6\epsilon}{2^n} < y + 7\epsilon.$$

Clearly, see [5, p. 225], $l(\epsilon) \in F_{j0}$. Hence,

$$0 < \lambda(F_j) = \lim_{n \to \infty} D_{S_{jn}}(u_{jn})^{-1} < \infty.$$

Take $m \leqslant n$. We will show that $D_{S_{jm}}(u_{jm} - u_{jn}) \longrightarrow 0$ as $m, n \longrightarrow \infty$. Note, since $\gamma_{jm} \subset \gamma_{jn}$, u_{jm} is a constant on each component Γ' of $\Gamma_{Cj}(2)$ and the flux of u_{jm} along Γ' is 0,

$$D_{S_{jm}}(u_{jm}, u_{jn}) = \int_{\alpha_{1jm}} {}^*du_{jn} + \int_{\alpha_{2jm}} u_{jm} \, {}^*du_{jn}$$

$$= \int_{\alpha_{1jm}} {}^*du_{jn} + \int_{\alpha_{2m}(A) + \Gamma_m(2) + \Gamma_{Ajm}(2)} u_{jm} \, {}^*du_{jn}.$$

Set $k = 1$ and 2. For each component δ of $\Gamma_p(k)$ let $A(\delta, k)$ denote the domain on S associated with δ. Define $\Lambda_{kp}(1)$ to be the union of all components δ of $\Gamma_p(k)$ for which $A(\delta, k) \cap \gamma \cap \partial\bar{R} = 0$, $\Lambda_{kp}(2)$ to be the union of all components δ of $\Gamma_p(k)$ for which $(A(\delta, k) \cap \partial S) - (\gamma \cup \beta)$ is an open arc on $\partial\bar{R}$ and set $\Lambda_{kp}(3) = \Gamma_p(k) - (\Lambda_{kp}(1) \cup \Lambda_{kp}(2))$.

For simplicity we shall write $\Gamma' \in \Lambda_{kp}(m)$ or $\Gamma' \in \Gamma_{Ajp}(k)$ whenever we mean that Γ' is a component of that set.

For each δ in $\Lambda_{kp}(1)$ let $\alpha_{kp}(1, \delta)$ and $\alpha_{kp}(2, \delta)$ denote the two components of $\alpha_{kp}(A)$ for which $\alpha_{kp}(1, \delta) \cup \delta \cup \alpha_{kp}(2, \delta)$ is connected. For each δ in $\Lambda_{kp}(2)$ let $\alpha_{kp}(\delta)$ denote the component of $\alpha_{kp}(A)$ and $\gamma_{jp}(\delta)$ the component of γ_{jp} for which $\alpha_{kp}(\delta) \cup \delta \cup \gamma_{jp}(\delta)$ is connected. Finally, for each δ in $\Lambda_{kp}(3)$ let $\gamma_{jp}(1, \delta)$ and $\gamma_{jp}(2, \delta)$ denote the components of γ_{jp} for which $\gamma_{jp}(1, \delta) \cup \delta \cup \gamma_{jp}(2, \delta)$ is connected.

Let E_q, $q = 1, \cdots, P$, denote the components of $S_{jn} - (S_{jm} - \partial S_{jm})$. Set $\tau_q = \{\Gamma'_n \in \Gamma_{Ajn}(k): \Gamma'_n \subset E_q\}$, $\Gamma_{Ajmn}(k) = \bigcup_{q=1}^{P} \tau_q$ and for each $\Gamma' \in \Gamma_{Ajm}(k)$, let $\Gamma_\sigma(\Gamma')$, $\sigma = 1, \cdots, \omega$, denote the finite number of components of $\Gamma_{Ajn}(k) - \Gamma_{Ajmn}(k)$ such that $\{\Gamma', \Gamma_1(\Gamma'), \cdots, \Gamma_\omega(\Gamma')\}$ bound a compact bordered Riemann surface E on S_{jn}.

Orient each subarc of ∂S_{jp}, $p \geqslant 1$, so that $S_{jp} - \partial S_{jp}$ lies locally to the left. Then

$$\int_{\Gamma'} {}^*du_{jn} = \sum_{\sigma=1}^{\omega} \int_{\Gamma_\sigma(\Gamma')} {}^*du_{jn} \quad \text{for } \Gamma' \in \Gamma_{Ajm}(k), m \leqslant n.$$

Thus,

$$\int_{\Gamma_{Ajm}(1)} {}^*du_{jn} = \int_{\Gamma_{Ajn}(1) - \Gamma_{Ajmn}(1)} {}^*du_{jn} \tag{2}$$

and

$$\int_{\Gamma_{Ajm}(2)} {}^*du_{jn} = 0. \tag{3}$$

We now consider the following three cases:

A. Suppose that $\delta_m(q) \in \Lambda_{km}(1) \cap \partial E_q$. Then E_q has positively oriented border

$$\sum_{L=1}^{2} (\alpha_{kn}(L, \delta_n(q)) - \alpha_{km}(L, \delta_m(q))) + \delta_n(q) - \delta_m(q) + \tau_q,$$

where $\delta_n(q) \in \Lambda_{kn}(1) \cap \partial E_q$. Thus,

$$\int_{\delta_m(q)} {}^*du_{jn} = \sum_{L=1}^{2} \int_{\alpha_{kn}(L, \delta_n(q)) - \alpha_{km}(L, \delta_m(q))} {}^*du_{jn} + \int_{\delta_n(q) + \tau_q} {}^*du_{jn}. \tag{4}$$

B. If $\delta_m(q) \in \Lambda_{km}(2) \cap \partial E_q$, then

$$\partial E_q = \alpha_{kn}(\delta_n(q)) - \alpha_{km}(\delta_m(q)) + \delta_{nq} + \gamma_{jn}(\delta_n(q)) - \gamma_{jm}(\delta_m(q)) - \delta_m(q) + \tau_q,$$

where $\delta_n(q) \in \Lambda_{kn}(2) \cap \partial E_q$. Thus,

$$\text{(5)} \qquad \int_{\delta_m(q)} {}^*du_{jn} = \int_{\alpha_{kn}(\delta_n(q)) - \alpha_{km}(\delta_m(q))} {}^*du_{jn} + \int_{\delta_n(q)+\tau_q} {}^*du_{jn}.$$

C. If $\delta_m(q) \in \Lambda_{km}(3) \cap \partial E_q$, then

$$\partial E_q = \sum_{L=1}^{2} (\gamma_{jn}(L, \delta_n(q)) - \gamma_{jm}(L, \delta_m(q))) + \delta_n(q) - \delta_m(q) + \tau_q,$$

where $\delta_n(q) \in \Lambda_{kn}(3) \cap \partial E_q$. Thus,

$$\text{(6)} \qquad \int_{\delta_m(q)} {}^*du_{jn} = \int_{\delta_n(q)+\tau_q} {}^*du_{jn}.$$

Take $k = 1$. From (4), (5) and (6) we conclude that

$$\int_{\Gamma_m(1)} {}^*du_{jn} = \int_{\alpha_{1n}(A) - \alpha_{1m}(A)} {}^*du_{jn} + \int_{\Gamma_n(1) + \Gamma_{Ajmn}(1)} {}^*du_{jn}.$$

Thus, using (2), $\Gamma_{Ajmn}(1) \subset \Gamma_{Ajn}(1)$ and $\alpha_{1m}(A) \subset \alpha_{1n}(A)$,

$$\int_{\alpha_{1jm}} {}^*du_{jn} = \int_{\alpha_{1jn}} {}^*du_{jn} = D_{S_{jn}}(u_{jn}).$$

We wish to show that

$$\int_{\alpha_{2m}(A) + \Gamma_m(2) + \Gamma_{Ajm}(2)} u_{jm} {}^*du_{jn} = 0.$$

Let α_2^* denote the noncompact components of α_2 on $\mathrm{Cl}(\partial \bar{R})$ and set $\alpha_{2m}(A)^* = \alpha_2^* \cap \alpha_{2m}(A)$. Clearly, $\int_{\alpha_{2m}(A) - \alpha_{2m}(A)^*} u_{jm} {}^*du_{jn} = 0$. Also, since u_{jm} is a constant on each component of $\Gamma_{Ajm}(2)$, from (3), $\int_{\Gamma_{Ajm}(2)} u_{jm} {}^*du_{jn} = 0$. Hence, we need only show that

$$\int_{\alpha_{2m}(A)^* + \Gamma_m(2)} u_{jm} {}^*du_{jn} = 0.$$

For each $q = 1, \cdots, P$, ∂E_q can be decomposed as in A, B or C Fix any q. We consider the three decompositions of ∂E_q.

A. Suppose that $\alpha_{2n}(1, \delta_n(q))$ and $\alpha_{2n}(2, \delta_n(q))$ fail to meet any $A(2, \delta_m(r))$, $r \neq q$. Then $\alpha_{2m}(1, \delta_m(q))$, $\alpha_{2m}(2, \delta_m(q)) \in \alpha_{2m}(A)^*$, $\alpha_2(1, \delta_n(q)) \cup \Gamma_n(q) \cup \alpha_{2n}(2, \delta_n(q))$ is a component of α_{2jn} and u_{jm} is a constant on

$$\xi_m = \alpha_{2m}(1, \delta_m(q)) \cup \delta_m(q) \cup \alpha_{2m}(2, \delta_m(q)).$$

Hence, $\int_{\xi_m} u_{jm} {}^*du_{jn} = 0$.

Suppose that there is an $r \neq q$ so that $\alpha_{2n}(1, \delta_n(q)) \cap A(2, \delta_m(r)) \neq 0$. Necessarily, $\partial E_r \cap \partial \bar{R}$ is composed of sets of the form $\alpha_{2n}(1, \delta_n(r))$ and $\alpha_{2n}(2, \delta_n(r))$ or $\alpha_{2n}(\delta_n(r))$ and $\gamma_{jn}(\delta_n(r))$. For simplicity assume that the former occurs, that $\alpha_{2n}(1, \delta_n(q)) = \alpha_{2n}(1, \delta_n(r))$ and that $\alpha_{2n}(2, \delta_n(r)) \cap A(2, \delta_m(t)) = 0$ for $t \neq r$. As above, we conclude that $\int_{\xi_m} u_{jm} {}^*du_{jn} = 0$, where

$$\xi_m = \alpha_{2m}(1, \delta_m(q)) \cup \delta_m(q) \cup \alpha_{2m}(2, \delta_m(q)) \cup \delta_m(r) \cup \alpha_{2m}(2, \delta_m(r)).$$

Cases B and C are handled analogously.

We have shown that $D_{S_{jm}}(u_{jn}, u_{jn}) = D_{S_{jn}}(u_{jn})$ for $n \geqslant m$. Thus, $D_{S_{jm}}(u_{jm} - u_{jn}) < D_{S_{jm}}(u_{jm}) - D_{S_{jn}}(u_{jn})$. Hence, since the sequence $\{D_{S_{jp}}(u_{jp})\}$ has a finite limit, $D_{S_{jn}}(u_{jm} - u_{jn}) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

To show that the functions u_{jn} converge to a limit function u_j on T_j we proceed as in the proof of Theorem 5 by considering the double Q_N of S_{jN} at $(\alpha_{0N}(A) - \mathrm{Cl}(\gamma_{jN})) \cup \Gamma_{Cj}(0)$ and the extensions u'_{jn} of u_{jn} $(n \geqslant N)$ on Q_N. There exists a harmonic function v'_{jN} on Q_N with the following properties:

$D_{Q_N}(u'_{jn} - v'_{jN}) \longrightarrow 0$ as $n \longrightarrow \infty$,

$D_{Q_N}(u'_{jn}) \longrightarrow D_{Q_N}(v'_{jN})$ as $n \longrightarrow \infty$,

$u'_{jn}(z) \longrightarrow v'_{jN}(z)$ as $n \longrightarrow \infty$ for $z \in Q_N$,

and the convergence is uniform on every compact subset of Q_N.

Let $D(\bar{R})$ denote the double of T_j across $(\alpha_0 \cup \Gamma_{Cj}(0)) - (\beta \cup \mathrm{Cl}\,\gamma)$. Then v'_{jN} is the restriction to Q_N of a continuous function v_j on $D(\bar{R}) \cup (T_j \cap (\beta - \beta^*))$ which is harmonic on $D(\bar{R})$.

To show that $\lim D_{Q_N}(u'_{jN}) = \lim D_{Q_N}(v'_{jN})$ it suffices to show that $\lim D_{S_{jN}}(u_{jN}) = \lim D_{S_{jN}}(v_{jN})$, $v_{jN} = v'_{jN} | S_{jN}$.

Note that for $n \geqslant N$,

$$D_{S_{jN}}(u_{jN} - u_{jn}) = D_{S_{jN}}(u_{jN}) - 2D_{S_{jn}}(u_{jn}) + D_{S_{jN}}(u_{jn})$$

tends to 0 as n and N tend to ∞. Thus,

$$\lim_{n\to\infty} D_{S_{jN}}(u_{jN} - u_{jn}) = D_{S_{jN}}(u_{jN}) - 2\lim_{n\to\infty} D_{S_{jn}}(u_{jn}) + D_{S_{jN}}(v_{jN})$$

and

$$0 = -\lim_{n\to\infty} D_{S_{jN}}(u_{jN}) + \lim_{n\to\infty} D_{S_{jN}}(v_{jN}).$$

We now conclude that the continuous function $u_j = v'_j | (T_j - \beta^*)$ on T_j satisfies:

$u_j|(T_j - \beta)$ is harmonic,
$u_j|S_{jN} = v'_{jN}|S_{jN}$,
$D_{S_{jn}}(u_{jn} - u_j) \longrightarrow 0$ as $n \longrightarrow \infty$,
$D_{S_{jn}}(u_{jn}) \longrightarrow D_{T_j}(u_j) > 0$ as $n \longrightarrow \infty$,
${}^*du = 0$ along γ_j,
$u_{jn}(z) \longrightarrow u_j(z)$ as $n \longrightarrow \infty$ for $z \in T_j - \beta$, and the convergence is uniform on every compact subset of T_j.

We have shown that

$$0 < \lambda(F_j) = D_{T_j}(u_j)^{-1} < \infty.$$

Suppose that $\bigcup_{k=0}^{2}\Gamma_{Cj}(k) = 0$ for all j. Then $F_j \subset F_{j+1}$ and $F = \bigcup F_j$. It follows easily that

$$0 < \lambda(F) = \lim \lambda(F_j) = \lim D_{T_j}(u_j)^{-1} < \infty.$$

If there exist a $j(0) \geqslant 1$ and a $k(0) = 0, 1$ or 2 such that $\Gamma_{Cj(0)}(k(0)) \neq 0$, then $\Gamma_{Cj}(k(0)) \neq 0$ for all $j \geqslant j(0)$. Hence,

$$\lambda(F) \geqslant \lambda(F_{j+1}) \geqslant \lambda(F_j) > 0.$$

Thus, since the family H of all curves in $S - (\beta \cup \gamma)$ which join $\alpha_0 - \beta$ to $\alpha_1 - \beta$ is a subset of F, $0 < \lim \lambda(F_n) \leqslant \lambda(F) < \infty$.

The opposite inequality is established as before.

Note, for $m \leqslant n$,

$$D_{T_m}(u_m, u_n) = \lim_{j\to\infty} D_{S_{mj}}(u_{mj}, u_{nj}) = \lim_{j\to\infty} D_{S_{nj}}(u_{nj}) = D_{T_n}(u_n).$$

Hence, $D_{T_m}(u_m - u_n) < D_{T_m}(u_m) - D_{T_n}(u_n)$, and $D_{T_m}(u_m - u_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

We conclude that there exists a continuous function u on $S - \beta^*$ which is harmonic on $\bar{R}$, 0 on α_0, 1 on α_1, a constant on each component ν of α_2 with the flux of u along ν equal to 0 and ${}^*du = 0$ along γ such that $0 < \lambda(F) = D_{\bar{R}}(u)^{-1} < \infty$.

7. Final result. It is also possible to assume that α_0, α_1 and α_2 each contain perfect nowhere dense subsets of $\mathrm{Cl}\,(\partial \bar{R})$.

Suppose that the type II contour σ_1 of $\bar{R}$ intersects a perfect nowhere dense subset of α_k ($k = 0, 1$ or 2) and that as we traverse the component of ∂S which contains σ_1 in the positive sense from σ_1 we traverse the type II contours $\sigma_1, \sigma_2, \cdots, \sigma_n$ of $\bar{R}$. We shall define the M-perfect nowhere dense subset of α_k which intersects σ_1.

For each $j = 1, \cdots, n$ let δ_j denote the union of all perfect nowhere dense subsets of α_k on σ_j and set $\delta_{n+1} = \delta_1$. If $\delta_1 \cap \delta_2 \neq 0$ (respectively $\delta_1 \cap \delta_n \neq 0$) let $s(r)$ denote the largest integer such that $\delta_j \cap \delta_{j+1} \neq 0$ $(\delta_{n+2-j} \cap \delta_{n+1-j} \neq 0)$ for $j = 1, \cdots, s-1$ $(j = 1, \cdots, r)$. Define the M-perfect nowhere dense subset δ of α_k which intersects σ_1 as follows:

$\delta = \delta_1$ if $\delta_1 \cap \delta_2 = 0$ and $\delta_1 \cap \delta_n = 0$;

$\delta = \bigcup_{j=1}^{s} \delta_j$ if $\delta_1 \cap \delta_2 \neq 0$ and $\delta_1 \cap \delta_n = 0$;

$\delta = \bigcup_{j=0}^{r} \delta_{n-j+1}$ if $\delta_1 \cap \delta_2 = 0$ and $\delta_1 \cap \delta_n \neq 0$;

$\delta = \bigcup_{j=1}^{s} \delta_j \cup \bigcup_{j=1}^{r} \delta_{n-j+1}$ if $\delta_1 \cap \delta_2 \neq 0$ and $\delta_1 \cap \delta_n \neq 0$.

If a type I contour σ of $\bar{R}$ contains a perfect nowhere dense subset of α_k we define the M-perfect nowhere dense subset of α_k which intersects σ to be the union of all perfect nowhere dense subsets of α_k on σ.

Clearly, two M-perfect nowhere dense subsets of α_k are either equal or disjoint.

Let τ_k and τ_l denote M-perfect nowhere dense subsets of α_k and α_l, respectively, $l \neq k$ and $l, k = 0, 1$ or 2. Suppose that τ_k and τ_l intersect the contour Σ' of $\bar{R}$. If Σ' is a type I contour of $\bar{R}$, take any point p on γ that is "between" τ_k and τ_l, that is, take any point p on γ such that as we traverse Σ' in the positive (respectively negative) sense from p we traverse a subset of τ_k (respectively τ_l) before we traverse a subset of τ_l (respectively τ_k). Set $\Sigma = \Sigma' - \{p\}$ if Σ' is type I and $\Sigma = \Sigma'$ otherwise.

There is a homeomorphism H of Σ onto $\{t: 0 < t < 1\}$. Assume that $H(\tau_k)$ is the Cantor set on $T_a = \{t: a \leqslant t \leqslant b\}$ (possibly minus one or both end points) and that $H(\tau_l)$ is the Cantor set on $\{t: c \leqslant t \leqslant d\}$ (possibly minus one or both end points), $0 \leqslant a < b \leqslant 1$ and $0 \leqslant c < d \leqslant 1$. Assume that $a < c$. We shall say that τ_k and τ_l intersect if $a < c < b$ and that τ_k and τ_l are disjoint if $b < c$. If τ_k and τ_l intersect, as we traverse T_a in the positive sense from a to b we meet $\tau_k(1), \tau_l(1), \tau_k(2), \tau_l(2), \cdots, \tau_l(n)$, $\tau_k(n+1)$, where $\tau_k(j)$ is a subset of τ_k, $\tau_l(j)$ is a subset of τ_l and $1 \leqslant n < \infty$. Call n the number of times τ_l and τ_k intersect on Σ.

Note, if Σ' is a type I contour of $\bar{R}$ and p_1 and p_2 are any two points of γ "between" τ_k and τ_l, then the number of times τ_k and τ_l intersect on $\Sigma' - \{p_1\}$ is equal to the number of times τ_k and τ_l intersect on $\Sigma' - \{p_2\}$.

Let $\{\tau_L(j)\}_{j=1}^{J(L)}$ denote the M-perfect nowhere dense subsets of α_L, $L = 0, 1$ and 2. Suppose that the subsets τ' and τ'' of $\tau_k(j)$ intersect a component K of ∂S. We shall say that τ' and τ'' are separated by $A = \bigcup_{L=0}^{2} \bigcup_{j=1}^{J(L)} (\tau_L(j) - \tau_k(j))$ if as we traverse K in both the positive and negative sense from τ' we traverse subsets of A before traversing τ''.

For each $j = 1, \cdots, J(k)$ decompose $\tau_k(j)$ into a finite number of subsets $\{\tau_k(j, s)\}_{s=1}^{S(j)}$ so that $\tau_k(j, s)$ and A are disjoint and so that for $s \neq t$,

$\tau_k(j, s)$ and $\tau_k(j, t)$ are separated by A. We shall mean when we say that "α_k contains a finite number of perfect nowhere dense subsets of $\mathrm{Cl}(\partial\bar{R})$" that α_k is the finite union of the sets $\tau_k(j, s), s = 1, \cdots, S(j)$ and $j = 1, \cdots, J(k)$.

We now state our general result.

THEOREM 7. *Let S denote the compactification of $\bar{R}$ defined above. Let $\beta = S - \bar{R}$ and $\partial S = \beta \cup \partial\bar{R}$. Suppose that α_0, α_1 and α_2 are closed subsets of ∂S which satisfy* (6.1), (6.2), (6.3) *and*

(7.1) *α_0, α_1 and α_2 are each composed of points of β, type* I *contours of $\bar{R}$ and a finite number of arcs and perfect nowhere dense sets on ∂S.*

Set $\gamma = \partial S - (\alpha_0 \cup \alpha_1 \cup \alpha_2)$ and denote by F the family of all curves in $S - \gamma$ which join α_0 to α_1. Let $\beta^* = \mathrm{Cl}(\partial\bar{R}) - (\alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \partial\bar{R})$. Then there exists a continuous function u on $S - \beta^*$ which is harmonic on $\bar{R}$, 0 on α_0, 1 on α_1, a constant on each component v_j of α_2 with the flux of u across v_j equal to zero and $^*du = 0$ along γ such that $0 < \lambda(F) = D_{\bar{R}}(u)^{-1} < \infty$.

The proof of this result is similar to that of Theorem 6 and will be omitted.

BIBLIOGRAPHY

1. L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, 2nd ed., McGraw-Hill, New York, 1966. MR 32 #5844.

2. L. V. Ahlfors and L. Sario, *Riemann surfaces*, Princeton Math. Ser., no 26, Princeton Univ. Press, Princeton, N. J., 1960. MR 22 #5729.

3. J. A. Jenkins, *Univalent functions and conformal mapping*, 2nd ed., Springer-Verlag, Berlin and New York, 1965.

4. ———, *Lecture notes at Washington University*, 1968–1969.

5. A. Marden and B. Rodin, *Extremal and conjugate extremal distance on open Riemann surfaces with applications to circular-radial slit mappings*, Acta Math. 115 (1966), 237–269. MR 34 #2862.

6. C. D. Minda, *Extremal length and harmonic functions on Riemann surfaces*, Trans. Amer. Math. Soc. 171 (1972), 1–22.

7. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR 13, 640.

8. B. Rodin and L. Sario, *Principal functions*, Van Nostrand, Princeton N. J., 1968. MR 37 #5378.

9. L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Die Grundlehren der math. Wissenschaften, Band 164, Springer-Verlag, Berlin and New York, 1970. MR 41 #8660.

10. L. Sario and K. Oikawa, *Capacity functions*, Die Grundlehren der math. Wissenschaften, Band 149, Springer-Verlag, Berlin and New York, 1969. MR 40 #7441.

11. G. Springer, *Introduction to Riemann surfaces*, Addison-Wesley, Reading, Mass., 1957. MR 19, 1169.

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