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ASYMPTOTIC VALUES OF MODULUS 1 OF BLASCHKE PRODUCTS

BY

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ABSTRACT. A sufficient condition is found for each subproduct of a Blaschke product to have an asymptotic value of modulus 1 along a prescribed arc of a specified type in the unit disc. The condition obtained is found to be necessary in the case of further restrictions of the arc, and the two results give rise to a necessary and sufficient condition for the existence of T_γ-limits of modulus 1 for Blaschke products.

1. **Introduction.** A sequence $\{a_n\}$ of complex numbers in the unit disc D is a Blaschke sequence if $\sum_{n=1}^{\infty} (1 - |a_n|)$ converges. If $\{a_n\}$ is a Blaschke sequence of nonzero numbers, the associated Blaschke product is defined by the formula

$$B(z, \{a_n\}) = \prod_{n=1}^{\infty} b(z, a_n) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{(a_n - z)}{(1 - z\bar{a}_n)}.$$

Without loss of generality we always deal with Blaschke products of this type.

It is well known that $B(z, \{a_n\})$ is regular in D and that the radial limit $\lim_{r\to 1-0} B(re^{i\theta}, \{a_n\})$ exists and has modulus 1 for almost every θ in $(-\pi, \pi]$.

Cargo [1] has generalised the concept of radial limit as follows. For a given number ζ on $\bar{D}\backslash D$ let

$$R(m, \zeta, \gamma) = \{z: 1 - |z| \geqslant m|\arg(\bar{\zeta}z)|^\gamma, 0 < |z| < 1\},$$

where $\arg(\bar{\zeta}z)$ denotes a number in $(-\pi, \pi]$, and m and γ are positive. The function $B(z, \{a_n\})$ is said to have a T_γ-limit L at ζ if and only if

$$\lim_{z\to\zeta, z\in R(m,\zeta,\gamma)} B(z, \{a_n\}) \tag{1.1}$$

exists and is equal to L for each positive number m. The following theorem is due to Cargo.

THEOREM A. *If $\{a_n\}$ is a Blaschke sequence, $\zeta \in \bar{D}\backslash D$, $\gamma > 1$, and*

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(1.2) $$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{\zeta} a_n|^{\gamma}} < \infty$$

then $B(z, \{a_n\})$ *and all its subproducts have* T_γ*-limits of modulus* 1 *at* ζ.

Frostman [2] had earlier proved that, when $\gamma = 1$, (1.2) is a necessary and sufficient condition for $B(z, \{a_n\})$ and all its subproducts to have radial limits of modulus 1 at ζ; a theorem of Lindelöf [3, p. 301] shows that this, in turn, is a necessary and sufficient condition for $B(z, \{a_n\})$ and all its subproducts to have T_1-limits of modulus 1 at ζ. When $\gamma > 1$ the condition (1.2) is not necessary for $B(z, \{a_n\})$ and all its subproducts to have a T_γ-limit at the point ζ, as is seen from the following result proved by Linden and Somadasa [4] when $t > \frac{1}{2}(\gamma + 1)$ and by Protas [5] when $t > 1$.

THEOREM B. *Let* $\gamma > 1$. *If* $\zeta \in \overline{D}\backslash D$ *then, for each* t *in* $(1, \infty)$, *there exists a Blaschke product* $B(z, \{a_n\})$ *such that* $B(z, \{a_n\})$ *and each of its subproducts has a* T_γ*-limit of modulus* 1 *at* ζ *while*

(1.3) $$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-\bar{\zeta} a_n|^{t}} = \infty.$$

The purpose of this paper is to obtain necessary and sufficient conditions for a Blaschke product and its subproducts to have limits of modulus 1 along arcs in D which have just one endpoint on $\overline{D}\backslash D$. Without loss of generality we will suppose henceforth that this endpoint is the point 1, the amendments needed to the statements and proofs of our results to deal with other endpoints being obvious. Although we bear in mind the case of the T_γ-limit the main result can be obtained for more general types of limit as follows.

THEOREM 1. (i) *Let* $0 < \theta_0 < \pi$, *and let* r: $(0, \theta_0) \longrightarrow (0, 1)$ *define an arc* $\Gamma = \{r(\theta)e^{i\theta}: \theta \in (0, \theta_0)\}$ *in* D *such that* $\lim_{\theta\to 0} r(\theta) = 1$. *If* $\{r_n e^{i\theta_n}\}$ *is a Blaschke sequence such that*

(1.4) $$\sum_{n=1}^{\infty} \frac{1-r_n}{|1-r_n e^{i\theta_n}|} < \infty,$$

(1.5) $$\lim_{\theta\to 0} F(\theta) = 0,$$

where

$$F(\theta) = \sum_{\frac{1}{2}\theta < \theta_n < 2\theta} \frac{1-r_n}{1-r(\theta)+|\theta-\theta_n|},$$

then

(1.6) $$\lim_{z \to 1, z \in \Gamma} B(z, \{b_n\})$$

exists and has modulus 1 *for each subsequence* $\{b_n\}$ *of* $\{r_n e^{i\theta_n}\}$.

(ii) *If the arc* Γ *is defined as in* (i), *and* r *satisfies a Lipschitz condition*

(1.7) $$|r(\theta) - r(\theta')| \leq K|\theta - \theta'|, \qquad \theta, \theta' \in (0, \theta_0),$$

for some constant K, *then the existence of the limit* (1.6) *with modulus* 1 *for each subsequence* $\{b_n\}$ *of* $\{r_n e^{i\theta_n}\}$ *implies that both* (1.4) *and* (1.5) *hold.*

From this we readily deduce the following theorem for T_γ-limits by putting $r(\theta) = 1 - \theta^\gamma$ in Theorem 1, using an analogous result with

$$r(\theta) = 1 - |\theta|^\gamma \quad \text{when } \theta \in (-\theta_0, 0),$$

and making application of a theorem of Lindelöf [3, p. 303].

THEOREM 2. *If* $\gamma > 1$ *a necessary and sufficient condition for* $B(z, \{r_n e^{i\theta_n}\})$ *and all its subproducts to have a* T_γ*-limit of modulus* 1 *at the point* 1 *is that* (1.4) *holds and*

(1.8) $$\sum_{\frac{1}{2}\theta < |\theta_n| < 2\theta} \frac{1 - r_n}{\theta^\gamma + |\theta - |\theta_n||} \longrightarrow 0 \quad \text{as } \theta \longrightarrow 0 + 0.$$

It is clear that Cargo's Theorem A can be obtained readily as a corollary of Theorem 2, and we shall see also that Theorem B can be sharpened as follows.

COROLLARY 1. *For each point* ζ *in* $\overline{D} \backslash D$ *there is a Blaschke sequence* $\{a_n\}$ *for which* (1.3) *holds for each* t *in* $(1, \infty)$ *while each subproduct of* $B(z, \{a_n\})$ *has a* T_γ*-limit of modulus* 1 *at* ζ *for each* γ *in* $(1, \infty)$.

We also note, without proof, the following immediate corollary which relates the existence of T_γ-limits to the existence of limits for particular regions of the type $R(m, \zeta, \gamma)$.

COROLLARY 2. *Let* $\gamma > 1$. *If the limit* (1.1) *exists and has modulus* 1 *for* $\{a_n\}$ *and all its subsequences for one positive value of* m *then the limit exists and has modulus* 1 *for all positive values of* m.

Finally we note the following result for Blaschke products with zeros restricted to the lower half of the unit disc.

COROLLARY 3. *Let* $\{a_n\}$ *be a Blaschke sequence with each of its members contained in* $D \cap \{z\colon \mathfrak{I}z < 0\}$, *and let* Γ *be any path in* $D \cap \{z\colon \mathfrak{I}z > 0\}$ *the points of whose closure meets* $\overline{D}$ *only in the point* 1. *Then the following are equivalent.*

(i) $B(z, \{a_n\})$ *and all its subproducts have radial limits of modulus* 1 *at the point* 1,

(ii) *the limit* (1.6) *exists and has modulus* 1 *for each subsequence* $\{b_n\}$ *of* $\{a_n\}$.

The proof that (ii) implies (i) is well known [3, p. 301]. The proof that (i) implies (ii) is immediate since $F(\theta)$ is identically zero and, as we have noted earlier, condition (i) implies (1.4).

The methods employed here give rise to certain sufficient conditions relating to the limits of derivatives of Blaschke products along the arcs Γ of the type considered in Theorem 1(i). The results in this case are not as complete as in the case of Theorem 1, and their relevance to other known theorems is discussed in §4.

2. **The proof of Theorem 1.** The proof of Theorem 1 is based on the following lemma, which shows the effect of conditions of the type (1.6) on certain terms of the Blaschke product.

LEMMA 1. *Let* $\{a_n\}$ *be a nonzero Blaschke sequence such that*

$$\sum_{n=1}^{\infty} \frac{1-|a_n|}{|1-a_n|} < \infty. \tag{2.1}$$

Let r *satisfy the conditions of Theorem* 1(i). *Then if* $\epsilon > 0$ *and* $a_n = r_n e^{i\theta_n}$ $(-\pi < \theta_n \leq \pi)$ *there is a positive number* φ *such that*

$$\left| \prod_{n=1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n) - \prod_{n=1}^{\infty} b(1, a_n) \right| < e^{8\epsilon} - 1 + 2\epsilon,$$

when $z = r(\theta)e^{i\theta}$, $0 < \theta < \varphi$.

The condition (2.1) implies the existence of $\prod_{n=1}^{\infty} b(1, a_n)$ and the existence of a natural number N such that

$$\sum_{n=N+1}^{\infty} \frac{1-|a_n|}{|1-a_n|} < \epsilon, \tag{2.2}$$

$$\left| \prod_{n=1}^{N} b(1, a_n) - \prod_{n=1}^{\infty} b(1, a_n) \right| < \epsilon. \tag{2.3}$$

There exists also a positive number φ_0, less than $\frac{1}{2}\pi$, such that

$$\left| \prod_{n=1}^{N} b(z, a_n) - \prod_{n=1}^{N} b(1, a_n) \right| < \epsilon, \qquad z = r(\theta)e^{i\theta},\ \tfrac{1}{2} < r(\theta) < 1,$$

when $0 < \theta < \varphi_0$.

We consider further the product

$$P(z) = \prod_{n=N+1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n)$$

$$= \prod_{n=N+1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} \left\{1 - \frac{(1-|a_n|)(1+ze^{-i\theta_n})}{1-z\bar{a}_n}\right\}.$$

For each term in this product we have $\frac{1}{2}|\theta_n| \leqslant |\theta - \theta_n| < 3\pi/2$, so that

$$|1 - z\bar{a}_n|^2 = (1 - rr_n)^2 + 4rr_n \sin^2 \tfrac{1}{2}(\theta - \theta_n) > (1 - r_n)^2 + r_n \sin^2 \tfrac{1}{4}\theta_n$$

$$> (1 - r_n)^2 + \tfrac{1}{4}r_n \sin^2 \tfrac{1}{2}\theta_n > \frac{1}{16}|1 - a_n|^2,$$

where $z = r(\theta)e^{i\theta} = re^{i\theta}$. Hence

$$\sum_{n=N+1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} \left|\frac{(1-|a_n|)(1+ze^{-i\theta_n})}{1-z\bar{a}_n}\right| < 8 \sum_{n=N+1}^{\infty} \frac{1-|a_n|}{|1-a_n|} < 8\epsilon,$$

from which we obtain $|P(z) - 1| < e^{8\epsilon} - 1$.

We now put $\varphi = \min(\varphi_0, \frac{1}{2}\theta_j; j = 1, 2, \cdots, N, \theta_j > 0)$. Then if $0 < \theta < \varphi$ and $\theta_n \in (\frac{1}{2}\theta, 2\theta)$ we have $0 < \theta_n < 2\theta < 2\varphi$, that is, $n > N$. Hence the inequality $0 < \theta < \varphi$ implies

$$\left|\prod_{n=1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n) - \prod_{n=1}^{\infty} b(1, a_n)\right|$$

$$\leqslant \left|\prod_{n=1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n) - \prod_{n=1}^{N} b(z, a_n)\right|$$

$$+ \left|\prod_{n=1}^{N} b(z, a_n) - \prod_{n=1}^{N} b(1, a_n)\right| + \left|\prod_{n=1}^{N} b(1, a_n) - \prod_{n=1}^{\infty} b(1, a_n)\right|$$

$$< |P(z) - 1| + 2\epsilon < e^{8\epsilon} - 1 + 2\epsilon,$$

as stated. Thus we have proved Lemma 1.

We now consider the sufficiency of the conditions (1.4) and (1.5) for the existence of the limit (1.6) when Γ satisfies the conditions of Theorem 1(i). Lemma 1 shows that we need only show that

$$P_1(z) = \prod_{n=1,\theta_n \in (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n)$$

has limit 1 as $\theta \longrightarrow 0$.

Without loss of generality we suppose that $\frac{1}{2} < r_n < 1$. If $\theta_n \in (\frac{1}{2}\theta, 2\theta)$ then, when $r = r(\theta) > \frac{1}{2}$,

$$|1 - z\bar{a}_n|^2 = (1 - rr_n)^2 + 4rr_n \sin^2 \tfrac{1}{2}(\theta - \theta_n)$$

$$> (1 - r)^2 + ((\theta - \theta_n)/\pi)^2$$

$$> \tfrac{1}{2}\pi^{-2} (1 - r + |\theta - \theta_n|)^2.$$

Hence, bearing in mind the relation (1.5), we can find a positive number φ' such that

$$\sum_{n=1,\theta_n \in (\frac{1}{2}\theta, 2\theta)}^{\infty} |1 - b(z, a_n)| < 2\sqrt{2}\pi \sum_{\frac{1}{2}\theta < \theta_n < 2\theta} \frac{1 - r_n}{1 - r + |\theta - \theta_n|} < \epsilon$$

when $0 < \theta < \varphi'$, and this leads to $|P_1(z) - 1| < e^{\epsilon} - 1$. Combining this with the result of Lemma 1, and using the notation of the proof of that lemma, we finally obtain

$$\left|\prod_{n=1}^{\infty} b(z, a_n) - \prod_{n=1}^{\infty} b(1, a_n)\right| < \left|\prod_{n=1}^{\infty} b(z, a_n) - \prod_{n=1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n)\right|$$

$$+ \left|\prod_{n=1,\theta_n \notin (\frac{1}{2}\theta, 2\theta)}^{\infty} b(z, a_n) - \prod_{n=1}^{\infty} b(1, a_n)\right|$$

$$< |P_1(z) - 1| + 2\epsilon + e^{8\epsilon} - 1 < e^{\epsilon} + 2\epsilon + e^{8\epsilon} - 2,$$

when $0 < \theta < \min(\varphi, \varphi')$. Thus we have that the conditions (1.4) and (1.5) are sufficient for the limit (1.6) to exist and to be equal to $B(1, \{a_n\})$. The proof given applies to any subproduct of $B(z, \{a_n\})$, so that Theorem 1(i) has been proved.

We next prove Theorem 1(i) by considering the necessity of the conditions (1.4) and (1.5) for the limit (1.6) to exist and have modulus 1 for each subsequence $\{b_n\}$ of $\{a_n\}$ when Γ satisfies the further restriction indicated. If (1.4) does not hold a well-known theorem of Frostman [2] shows that some subproduct of $B(z, \{a_n\})$ fails to have a radial limit of modulus 1 at 1. Consequently an application of Lindelöf's theorem [3, p. 301] shows that such a subproduct does not have a limit of modulus 1 as $z \to 1$ along the arc $\Gamma = \{r(\theta)e^{i\theta}: \theta \in (0, \theta_0)\}$.

Hence we examine the situation in which (1.4) holds but (1.5) does not. If we put $a_n = r_n e^{i\theta_n}$ and

$$G(\theta) = \sum_{\frac{1}{2}\theta < \theta_n < 2\theta} \frac{(1 - r_n)(1 - r(\theta))}{(1 - r(\theta) + |\theta - \theta_n|)^2},$$

$$H(\theta) = \sum_{\frac{1}{2}\theta < \theta_n < 2\theta} \frac{(1 - r_n)|\theta - \theta_n|}{(1 - r(\theta) + |\theta - \theta_n|)^2},$$

we have that either $G(\theta)$ or $H(\theta)$ does not have a zero limit as $\theta \longrightarrow 0$. Let us consider first the former case.

Under our assumption there is a positive number p such that $G(\theta) > p$ for a set E of positive numbers θ with 0 as an accumulation point. Now

$$|B(re^{i\theta}, \{a_n\})|^2 \leqslant \prod_{\frac{1}{2}\theta<\theta_n<2\theta} \left\{1 - \frac{(1-r^2)(1-r_n^2)}{(1-rr_n)^2 + 4rr_n \sin^2 \frac{1}{2}(\theta-\theta_n)}\right\}.$$

But the condition (1.7) implies that $|r(\theta_n) - r(\theta)| \leqslant K|\theta - \theta_n|$, while Lindelöf's theorem, together with the assumption that the limit (1.6) is nonzero implies that $r_n > r(\theta_n)$, except for a finite number of integers n. With the exception of these integers, we deduce that

$$1 - r_n < 1 - r(\theta_n) < 1 - r(\theta) + K|\theta - \theta_n|, \tag{2.4}$$

from which we obtain

$$(1-rr_n)^2 + 4rr_n \sin^2 \tfrac{1}{2}(\theta - \theta_n) < (1 - r + 1 - r_n + |\theta - \theta_n|)^2$$

$$< (2+K)^2 (1 - r + |\theta - \theta_n|)^2.$$

Hence, for sufficiently small values of θ in E, we have

$$|B(re^{i\theta}, \{a_n\})|^2 \leqslant \prod_{\frac{1}{2}\theta<\theta_n<2\theta} \exp\left\{- \frac{(1-r)(1-r_n)}{(2+K)^2(1-r+|\theta-\theta_n|)^2}\right\} < \exp \frac{-p}{(2+K)^2}$$

so that the limit (1.6), if it exists, cannot have modulus 1. This implies that

$$\lim_{\theta\to 0} G(\theta) = 0. \tag{2.5}$$

Finally we consider the case in which (1.6) and (2.5) hold, but $H(\theta)$ does not have a zero limit as $\theta \longrightarrow 0$. In this case we construct a subproduct of $B(z, \{a_n\})$ which does not have the appropriate limit.

If $H(\theta)$ does not have a zero limit as $\theta \longrightarrow 0$ then either

$$\sum_{\theta<\theta_n<2\theta} \frac{(1-r_n)(\theta_n - \theta)}{(1 - r + \theta_n - \theta)^2} \tag{2.6}$$

or

$$\sum_{\frac{1}{2}\theta<\theta_n<\theta} \frac{(1-r_n)(\theta - \theta_n)}{(1 - r + \theta - \theta_n)^2} \tag{2.7}$$

does not have a zero limit as $\theta \longrightarrow 0$. We suppose, without loss of generality, that it is the former.

By the condition (2.5), and consideration of the terms $G(\theta_n)$, we have

$$\lim_{n\to\infty, \theta_n>0} \frac{1-r_n}{1-r(\theta_n)} = 0,$$

and we can use (2.4) to obtain

(2.8) $$\lim_{n\to\infty,\theta_n>0} \frac{1-r_n}{1-r(\theta)+|\theta-\theta_n|} = 0,$$

the limit being uniform with respect to θ on some interval $(0, t)$ of positive length. Since we also have

$$\frac{|\theta_n-\theta|}{1-r(\theta)+|\theta_n-\theta|} < 1,$$

there exists a positive constant q and a subsequence $\{\rho_n e^{i\varphi_n}\}$ of $\{a_n\}$ such that

(2.9) $$q < \sum_{\theta<\varphi_n<2\theta} \frac{(1-\rho_n)(\varphi_n-\theta)}{(1-r(\theta)+\varphi_n-\theta)^2} < 2q < \tfrac{1}{2}\pi$$

for a set E' of positive values θ having 0 as an accumulation point. The set E' will contain a sequence $\{\psi_k\}$ such that $0 < \psi_{k+1} < \frac{1}{4}\psi_k$ for $k = 1, 2, \cdots$, and we define a subset $\{d_n\}$ of $\{a_n\}$ to consist of the set $\bigcup_{k=1}^{\infty} \{\rho_n e^{i\varphi_n}: \psi_k < \varphi_n < 2\psi_k\}$.

The condition (1.4) implies that if $B(z, \{d_n\})$ has a limit as $z \longrightarrow 1$ along Γ then this limit is $B(1, \{d_n\})$. However, by Lemma 1, we can show that $B(z, \{d_n\})$ does not have this limit by verifying that the argument of

(2.10) $$\prod_{\arg d_n\in(\frac{1}{2}\theta,2\theta)} b(z, d_n)$$

does not have 0 as a limit (modulo 2π) as $\theta \longrightarrow 0$ through values in the sequence $\{\psi_k\}$.

Let $d_n = \tau_n e^{i\xi_n}$ for $n = 1, 2, \cdots$. It is well known [2] that the argument of the product (2.10) is

$$S(\theta) = \sum_{\frac{1}{2}\theta<\xi_n<2\theta} \arcsin \frac{r\sin(\xi_n-\theta)(1-\tau_n^2)}{|z-d_n||1-z\bar{d}_n|}.$$

By our choice of the sequence $\{d_n\}$ we have

$$S(\psi_k) = \sum_{\psi_k<\xi_n<2\psi_k} \arcsin \frac{r\sin(\xi_n-\psi_k)(1-\tau_n^2)}{|z_k-d_n||1-z_k\bar{d}_n|}.$$

The condition (2.8) shows that

$$|z_k-d_n| \sim |1-z_k\bar{d}_n| \sim |1-r(\psi_k)+i(\xi_n-\psi_k)|$$

as $k \longrightarrow \infty$, for all ξ_n in the relevant ranges $(\psi_k, 2\psi_k)$. Hence, for sufficiently large values of k, we have

$$S(\psi_k) = (2+\delta(k)) \sum_{\psi_k<\xi_n<2\psi_k} \arcsin \frac{\sin(\xi_n-\psi_k)(1-\tau_n)}{|1-r(\psi_k)+i(\xi_n-\psi_k)|^2},$$

where $\lim_{k\to\infty}\delta(k) = 0$. Since

$$\frac{1}{\sqrt{2}}\,(1 - r(\psi_k) + \xi_n - \psi_k) < |1 - r(\psi_k) + i(\xi_n - \psi_k)| < 1 - r(\psi_k) + \xi_n - \psi_k$$

for associated values of n and k, the inequalities (2.9) show that $S(\psi_k)$ lies between q and $8q$ for sufficiently large values of k. But we have supposed that $0 < q < \pi/4$. Hence $\lim_{k\to\infty}\arg B(r(\psi_k)e^{i\psi_k}, \{d_n\})$ is not equal to $\arg B(1, \{d_n\})$. This completes the proof of Theorem 1.

3. **The proof of Corollary 1.** For the proof of Corollary 1 we define

$$a_n = \left(1 - \frac{1}{n(\log n)^2(\log\log n)^2}\right)\exp\left(\frac{i}{\log n}\right), \qquad n = 5, 6, 7, \cdots,$$

and show that $B(z, \{a_n\})$ and all its subproducts have a T_γ-limit of modulus 1 at the point 1 for each number γ greater than 1, while (1.3) is valid with $\zeta = 1$ for all t greater than 1.

We begin by noting that

$$\frac{1 - |a_n|}{|1 - a_n|^t} \sim \frac{(\log n)^{t-2}}{n(\log\log n)^2}$$

as $n \to \infty$, so that $\sum_{n=5}^{\infty}(1 - |a_n|)/|1 - a_n|^t$ diverges if and only if $t > 1$. It remains to verify (1.8) and it will clearly be sufficient to show that

$$\sum_{\frac{1}{2}\theta<\theta_n<2\theta} \frac{1 - |a_n|}{\theta^\gamma + |\theta - \theta_n|} \longrightarrow 0 \tag{3.1}$$

as $\theta \to 0 + 0$ for each number γ greater than 1. Since this latter sum is an increasing function of γ when $\theta \in (0, 1)$ we may suppose, without loss of generality, that $\gamma > 2$.

For a given positive number θ we choose N so that $\log N \leqslant \theta^{-1} < \log(N + 1)$. Thus $\frac{1}{2}\theta < \theta_n < 2\theta$ implies that $N^{\frac{1}{2}} < n < (N + 1)^2$.

Now, if $N_1 = [N + N(\log N)^{2-\gamma}], N_2 = [N - N(\log N)^{2-\gamma}]$, we obtain

$$\begin{aligned} S(N_2, N_1 - 1) = \sum_{n=N_2}^{N_1-1} \frac{1 - |a_n|}{\theta^\gamma + |\theta - \theta_n|} &< \frac{(N_1 - N_2)(\log(N + 1))^\gamma}{N_2(\log N_2)^2(\log\log N_2)^2} \\ &= O\left(\frac{1}{(\log\log N)^2}\right) \end{aligned}$$

as $N \to \infty$. Using this notation we also obtain

$$S(N_1, (N+1)^2) < \sum_{n=N_1}^{(N+1)^2} \frac{\log n \log(N+1)}{n(\log n)^2 (\log\log n)^2 \log(n/(N+1))}$$

$$= O((\log\log N)^{-2}) \int_{N_1-1}^{(N+1)^2} \frac{dx}{x\log(x/(N+1))}$$

$$= O((\log\log N)^{-1})$$

as $N \longrightarrow \infty$, by application of the method of the Maclaurin integral test, noting that the latter integral is

$$\log\log(N+1) - \log\log\left(\frac{N_1 - 1}{N+1}\right) \sim (\gamma - 1)\log\log N.$$

In the same way we obtain

$$S([\sqrt{N}], N_2 - 1) = O((\log\log N)^{-2}) \sum_{n=[\sqrt{N}]}^{N_2-1} \frac{1}{n \log N/n} \quad \text{as } N \longrightarrow \infty.$$

The general term in the latter sum increases with n when $n < N/e$ and decreases when $n > N/e$. Hence we can apply the method of the Maclaurin integral test in each of the cases $\sqrt{N} \leqslant n < N/e$ and $N/e < n \leqslant (N+1)^2$ to obtain $S([\sqrt{N}], N_2 - 1) = O((\log\log N)^{-1}$ as $N \longrightarrow \infty$.

Since the sum on the left-hand side of (3.1) is bounded by

$$S([\sqrt{N}], N_2 - 1) + S(N_2, N_1 - 1) + S(N_1, (N+1)^2)$$

the relation (3.1) follows immediately, thus completing the proof of Corollary 1.

4. **Limits of derivatives of Blaschke products.** The work of Protas [5] has led to the following two theorems.

THEOREM C. *Let* $\{a_n\}$ *be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|}{|1 - a_n|^t} < \infty \tag{4.1}$$

for some $t \geqslant 1$. *If* k *is an integer such that* $0 \leqslant k < t - 1$ *then the function* $B^{(k)}(z, \{b_n\})$ *has a* $T_{t/(k+1)}$*-limit at* 1 *for each subsequence* $\{b_n\}$ *of* $\{a_n\}$.

THEOREM D. *Let* $t \geqslant 1$, *and let* k *be any nonnegative integer such that* $0 \leqslant k \leqslant t - 1$. *Then for each number* σ *greater than* $t/(k+1)$ *there is a Blaschke product* $B(z, \{a_n\})$ *such that* (4.1) *holds but* $B^{(k)}(z, \{a_n\})$ *does not have a* T_σ*-limit at* 1.

In the next theorem we note a result which is an immediate generalisation of the sufficiency part of Theorem 1, and which leads to Theorem 4 in which we find that the sufficiency condition of Theorem C can be weakened. We then deduce the existence of a Blaschke sequence $\{a_n\}$ for which $B^{(j)}(z, \{a_n\})$ has a T_γ-limit for $j = 1, 2, \cdots, k$ while (4.1) is false for certain values of t less than $(k+1)\gamma$.

THEOREM 3. *Let* Γ *be an arc defined as in Theorem* 1, *and let* k *be a positive integer. Let* $\{r_n e^{i\theta_n}\}$ *be a Blaschke sequence such that*

$$\sum_{n=1}^{\infty} \frac{1 - r_n}{|1 - r_n e^{i\theta_n}|^{k+1}} < \infty, \tag{4.2}$$

$$\lim_{\theta \to 0} \sum_{\frac{1}{2}\theta < \theta_n < 2\theta} \frac{1 - r_n}{\{1 - r(\theta) + |\theta - \theta_n|\}^{k+1}} = 0. \tag{4.3}$$

Then $\lim_{z \to 1, z \in \Gamma} B^{(k)}(z, \{b_n\})$ *exists for each subsequence* $\{b_n\}$ *of* $\{r_n e^{i\theta_n}\}$.

THEOREM 4. *Let* $\{r_n e^{i\theta_n}\}$ *be a Blaschke sequence such that* (4.2) *holds and*

$$\lim_{\theta \to 0} \sum_{\frac{1}{2}\theta < |\theta_n| < 2\theta} \frac{1 - r_n}{\{\theta^\gamma + |\theta - |\theta_n||\}^{k+1}} = 0 \tag{4.4}$$

for some positive integer k. *Then* $B^{(k)}(z, \{a_n\})$ *has a* T_γ*-limit at* 1.

Defining

$$B_m(z) = B(z, \{a_n\}) \left(\frac{1 - \bar{a}_m z}{a_m - z} \right),$$

we note [5] that

$$B^{(k)}(z, \{a_n\}) = -\sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{m=1}^{\infty} B_m^{(k-j-1)}(z) \frac{(j+1)!\, \bar{a}_m^j (1 - |a_m|^2)}{(1 - \bar{a}_m z)^{j+2}}.$$

The method used to prove Theorem 1(i) can then be applied to prove Theorem 3, using the principle of mathematical induction with respect to k. Theorem 4 follows immediately in the same way that Theorem 2 follows from Theorem 1.

We conclude by noting the following deduction from Theorem 4.

THEOREM 5. *Let* k *be a positive integer and let* $\gamma > 1$. *Then there is a Blaschke sequence* $\{a_n\} = \{r_n e^{i\theta_n}\}$ *for which* $B^{(j)}(z, \{a_n\})$ *has a* T_γ*-limit at the point* 1 *for* $j = 1, 2, \cdots, k$ *while* (4.1) *is false for each* $t \geq 1 + \gamma k$.

Let

$$a_n = \left(1 - \frac{1}{n^\alpha \log n}\right) e^{in^{-\beta}}, \qquad n = 2, 3, \cdots,$$

where, to begin with, we assume that

(4.5) $$\alpha > \max(1, \beta), \qquad 0 < \beta < \frac{1}{\gamma - 1}.$$

Then $(1 - |a_n|)/|1 - a_n|^t \sim n^{\beta t - \alpha}/\log n$, as $n \longrightarrow \infty$, so that (4.1) is false if and only if

(4.6) $$\beta t \geqslant \alpha - 1,$$

that is (4.2) holds with $a_n = r_n e^{i\theta_n}$ if and only if

(4.7) $$k + 1 < (\alpha - 1)/\beta.$$

For small positive values of θ we define N so that $(N + 1)^{-\beta} < \theta \leqslant N^{-\beta}$. We also define $p = 1 + \beta - \beta\gamma$, which implies that $0 < p < 1$. Then for some constant C we have

$$\sum_{|n-N| \leqslant N^p} \frac{1 - r_n}{\{\theta^\gamma + |\theta - \theta_n|\}^{k+1}} < \frac{CN^{p+\beta\gamma(k+1)-\alpha}}{\log N} = \frac{CN^{1+\beta+\beta\gamma k-\alpha}}{\log N},$$

while the remaining terms of (4.4) are bounded by

$$\sum_{N^p < |n-N| \leqslant CN, n \in N} \frac{CN^{(\beta+1)(k+1)-\alpha}}{|N - n|^{k+1} \log N} < \frac{CN^{(\beta+1)(k+1)-pk-\alpha}}{\log N} = \frac{CN^{1+\beta+\beta\gamma k-\alpha}}{\log N}.$$

Hence the sum (4.4) has limit 0 as $\theta \longrightarrow 0$ if $(\alpha - 1)/\beta \geqslant 1 + \gamma k$. Given k and γ we find α and β to satisfy (4.5), (4.7) and the equality $(\alpha - 1)/\beta = 1 + \gamma k$. Then (4.6) holds whenever $t \geqslant 1 + \gamma k$, and the proof of Theorem 5 is complete.

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